

JANET FLIKKEMA

Intertwining Operators and Endomorphism Algebras for Representations of Covers of Reductive p -adic Groups



Institute for Mathematics,
Astrophysics and Particle Physics

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Intertwining Operators and Endomorphism Algebras for Representations of Covers of Reductive p -adic Groups

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Chapter 1

Introduction

This thesis concerns itself with smooth complex representations of finite covers of reductive p -adic groups. More precisely, we study intertwining operators for such representations and use them to describe Bernstein blocks for principal series representations of Brylinski-Deligne covers. In this introduction, we give basic explanations of these notions; they are covered in more detail in Chapter 2 and the references provided there. Here, our focus is to state the main results of this thesis, put them into context, and refer to where each result is proved in the text.

Let us elaborate on the definitions that are mentioned in the first two sentences of this introduction. A reductive p -adic group G' is the group of F -points of a reductive algebraic group over a non-archimedean local field F , whose residue field has characteristic p . For example, one can think of F being the field $\mathbb{Q}_p$ of p -adic numbers. The structure of the group G' comes from its structure as a linear algebraic group, as well as its topological structure induced by the (p -adic) metric on F . A finite cover $G \rightarrow G'$ is an extension of G' by a finite abelian group A , such that A lies in the center of G and such that the projection $G \rightarrow G'$ is a topological covering map. This makes G into a totally disconnected locally compact group and we can study its smooth complex representations, i.e. its complex representations such that every vector is fixed by some compact open subgroup. Many methods in the representation theory of reductive p -adic groups generalize to covering groups, for example: parabolic induction, supercuspidal representations, unramified characters [FP22]. Also the Bernstein decomposition generalizes, namely the category $\text{Rep}(G)$ of smooth complex representations of G decomposes into smaller categories called Bernstein blocks. These Bernstein blocks are equivalent to module categories of certain endomorphism algebras. So to describe Bernstein blocks, one can try to describe the corresponding endomorphism algebras. This was done in [Sol22] for reductive groups without the use of type theory and the methods should generalize to covers of reductive groups, although not automatically.

In Chapter 4 of this thesis, we study an important ingredient of the methods in [Sol22] for covering groups, namely the Harish-Chandra μ -function. It is

defined using intertwining operators that go from one parabolically induced representation to another; they were studied in [Wal03] and generalized to covering groups in [Li12]. More precisely, let P be a parabolic subgroup of G ; it is the inverse image of a parabolic subgroup P' of G' . In G' , the group P' has a Levi decomposition $P' = M'U_{P'}$, where $U_{P'}$ is the unipotent radical of P' and the Levi subgroup M' is a reductive group. The Levi decomposition of P' lifts to a Levi decomposition $P = MU_P$ in G , where M is the inverse image of M' in G and $U_P \cong U_{P'}$. This is because covering groups split over unipotent subgroups, which means that for every unipotent subgroup U' of G' , its inverse image in G is isomorphic to the product $A \times U'$ as a group. Consequently, $U := \{1\} \times U' \cong U'$ is a unipotent subgroup of G . This important result due to [MW94] is essentially the reason why parabolic induction and Harish-Chandra's intertwining operators also work for covering groups. Let $Q = MU_Q$ be another parabolic subgroup of G with the same Levi subgroup M . We have functors of (normalized) parabolic induction

$$I_P^G : \text{Rep}(M) \rightarrow \text{Rep}(G), \quad I_Q^G : \text{Rep}(M) \rightarrow \text{Rep}(G).$$

Let π be an irreducible representation of M . The intertwining operator

$$J_{Q|P}(\pi) : I_P^G(\pi) \rightarrow I_Q^G(\pi)$$

is defined using an integral over $(U_P \cap U_Q) \setminus U_Q$, see §2.2.3 for details. Our first important result is about an adjointness property of such intertwining operators. For a smooth complex representation π , we can take its complex conjugate representation $\bar{\pi}$, as well as its contragredient representation $\pi^\vee$, which means taking smooth vectors in the dual representation. Doing both, we obtain the Hermitian dual $\bar{\pi}^\vee$ of π . There is a pairing between π and $\bar{\pi}^\vee$, giving rise to a pairing between $I_P^G(\pi)$ and $I_P^G(\bar{\pi}^\vee)$, and similarly a pairing between $I_Q^G(\pi)$ and $I_Q^G(\bar{\pi}^\vee)$. With respect to these pairings, we prove the following result.

Theorem A. *The intertwining operators $J_{Q|P}(\pi)$ and $J_{P|Q}(\bar{\pi}^\vee)$ are adjoint, in the sense that for $f_1 \in I_P^G(\pi)$, $f_2 \in I_Q^G(\bar{\pi}^\vee)$ we have*

$$\langle J_{Q|P}(\pi)f_1, f_2 \rangle = \langle f_1, J_{P|Q}(\bar{\pi}^\vee)f_2 \rangle.$$

There is a similar result for the contragredient representation $\pi^\vee$ instead of $\bar{\pi}^\vee$. For the proofs of these statements, see §4.3. The results were already stated without proof in [Wal03], but since they play an important role in our proof of Theorem B, we wanted to give an explicit proof.

Now let $\bar{P}$ be the parabolic subgroup of G with Levi subgroup M , opposite to P . Let $X_{\text{nr}}(M)$ be the complex torus of unramified characters of M , i.e. the complex one-dimensional representations of M that are trivial on all compact subgroups of M . Let σ be an irreducible supercuspidal representation of M , where supercuspidal means that σ does not occur in any representation parabolically induced from a representation of a smaller Levi subgroup. The Harish-Chandra μ -function is a rational function on the complex algebraic variety $\sigma \cdot X_{\text{nr}}(M)$. Up to multiplication by a scalar, it is given

by $\mu(\sigma \otimes \chi) = j(\sigma \otimes \chi)^{-1}$, where

$$j(\sigma \otimes \chi) := J_{P|\bar{P}}(\sigma \otimes \chi) \circ J_{\bar{P}|P}(\sigma \otimes \chi) : I_P^G(\sigma \otimes \chi) \rightarrow I_{\bar{P}}^G(\sigma \otimes \chi)$$

is a rational function in $\chi \in X_{\text{nr}}(M)$ that does not depend on the choice of parabolic subgroup P . The μ -function decomposes as a product $\mu = \prod_{\alpha} \mu_{\alpha}$, where α runs over the positive roots in the reduced root system of (G, M) and where each μ_{α} is computed in a Levi subgroup M_{α} of G that contains M as a maximal Levi subgroup. It can be shown that each μ_{α} then becomes a rational function in only one complex variable $X_{\alpha} \in \mathbb{C}^{\times}$, with the following explicit description.

Theorem B. *In the above context, there exists a unitary $\sigma_0 \in \sigma \cdot X_{\text{nr}}(M)$ such that for every α , μ_{α} has the form*

$$\begin{aligned} \mu_{\alpha}(\sigma_0 \otimes \chi) &= \mu_{\alpha}(\sigma_0, X_{\alpha}) \\ &= c_{\alpha} \cdot \frac{(1 - X_{\alpha})(1 - X_{\alpha}^{-1})}{(1 - q_{\alpha}X_{\alpha})(1 - q_{\alpha}X_{\alpha}^{-1})} \frac{(1 + X_{\alpha})(1 + X_{\alpha}^{-1})}{(1 + q_{\alpha*}X_{\alpha})(1 + q_{\alpha*}X_{\alpha}^{-1})}, \end{aligned} \tag{1.0.1}$$

where $c_{\alpha} \in \mathbb{R}_{>0}$ and $q_{\alpha}, q_{\alpha*} \in \mathbb{R}$ with $q_{\alpha} \geq q_{\alpha*} \geq 1$.

This result was already known for reductive groups due to Silberger [Sil80]. In §4.4, we give a proof using different methods that generalize to covering groups. The proof uses the adjointness relation from Theorem A, as well as techniques with hermitian and unitary representations.

Together with other results that are already known for covering groups, the generalization of the formula for the μ -function to covering groups implies that the results in [Sol22] should hold for covering groups. In Chapter 5, we apply those results to describe principal series Bernstein blocks of (Brylinski-Deligne) covers of split reductive groups. For principal series representations of covers of split reductive groups, we prove in §5.3.2, specifically Corollary 5.3.7, that the poles of the μ -function can only have the following values.

Proposition C. *Let $G \rightarrow G'$ be a cover of a split group G' , let T' be a maximal split torus of G' and T its inverse image in G . Let σ be an irreducible representation of T and let $\sigma_0 \in \sigma \cdot X_{\text{nr}}(T)$ be as in Theorem B. Then for every α , the poles $q_{\alpha}, q_{\alpha*}$ of $\mu_{\alpha}(\sigma_0, X_{\alpha})$ are given by one of the following three options:*

- $q_{\alpha} = q_{\alpha*} = 1$, in which case μ_{α} is constant,
- $q_{\alpha} = q_F$ and $q_{\alpha*} = 1$, with q_F the cardinality of the residue field of F ,
- $q_{\alpha} = q_{\alpha*} = q_F^{1/2}$.

Let us now move on to the main result of Chapter 5. This result holds specifically for principal series representations of *Brylinski-Deligne* covers of split groups, so let us elaborate a bit more on such groups. The finite covers of split simply connected groups can be classified due to the work of Steinberg,

Moore and Matsumoto. Namely, let G'_{sc} be a split simply connected group, then all central extensions of G'_{sc} arise as a pushforward of a universal central extension [Moo68, §1]

$$1 \rightarrow \pi_1(G'_{\text{sc}}) \rightarrow \tilde{G}_{\text{sc}} \rightarrow G'_{\text{sc}} \rightarrow 1.$$

Assume that G'_{sc} is moreover simple, then $\pi_1(G'_{\text{sc}}) = K_2(F)$ is the second degree algebraic K -theory group of F . To obtain a topological cover, one has to push forward the extension of G'_{sc} by $K_2(F)$ along a locally constant symbol; the Hilbert symbol $K_2(F) \rightarrow \mu(F)$ is a universal one. Here, $\mu(F)$ is the finite cyclic group of roots of unity of F . So all covers of G'_{sc} arise as a pushforward of a universal cover

$$1 \rightarrow \mu(F) \rightarrow G_{\text{univ}} \rightarrow G'_{\text{sc}} \rightarrow 1.$$

The theory of Brylinski and Deligne was developed as a generalization of this, see §2.1.4 for details. For split reductive groups G' , the theory provides us with central extensions

$$1 \rightarrow K_2(F) \rightarrow \tilde{G} \rightarrow G' \rightarrow 1,$$

that we can push forward along a Hilbert symbol to obtain covering groups. The advantage of the results of Brylinski and Deligne is that these covering groups have a nice combinatorial description in terms of Weyl-group invariant bilinear forms on the cocharacter lattice of a maximal torus of G' . However, the Brylinski-Deligne covering groups do not give us *all* covering groups of G' . Namely, in general there does not exist a universal central extension of G' . Nevertheless, if we study a certain problem concerning covers of split groups and manage to reduce it to covers of split simply connected groups, then we know all such groups are described by Brylinski-Deligne covers, so we can still take advantage of the available results. This is actually what was done to prove Proposition C.

Before stating the main result of Chapter 5, let us also say something more about the Bernstein decomposition. We already mentioned that the category $\text{Rep}(G)$ can be decomposed into smaller categories, called Bernstein blocks. These Bernstein blocks are denoted by $\text{Rep}_{\mathfrak{s}}(G)$ and they are associated to equivalence classes $\mathfrak{s}$ of supercuspidal representations of Levi subgroups of G . Let us now state the last result; it is quite technical, so for details and more context please consult Chapters 2 and 5.

Theorem D. *Let $G \rightarrow G'$ be a Brylinski-Deligne cover of a split group G' , let T' be a maximal torus of G' and T its inverse image in G . Let σ be an irreducible representation of T and $\mathfrak{s}_T, \mathfrak{s}_G$, its inertial equivalence classes for T and G , respectively. Let σ_1 be an irreducible subrepresentation of $\sigma|_{T_{\text{cpt}}}$, where T_{cpt} is the maximal compact subgroup of T and let $\text{ind}_{T_{\text{cpt}}}^T : \text{Rep}(T_{\text{cpt}}) \rightarrow \text{Rep}(T)$ be the functor of compact induction. Then there are equivalences of categories*

$$\text{Rep}_{\mathfrak{s}_T}(T) \simeq \text{End}_T(\text{ind}_{T_{\text{cpt}}}^T(\sigma_1)) - \text{Mod} \text{ and}$$

$$\text{Rep}_{\mathfrak{s}_G}(G) \simeq \text{End}_G(I_P^G(\text{ind}_{T_{\text{cpt}}}^T(\sigma_1))) - \text{Mod}.$$

The endomorphism algebras have the following structure.

- (i) $\text{End}_T(\text{ind}_{T_{\text{cpt}}}^T(\sigma_1)) \cong \mathbb{C}[\mathcal{O}]$ is the ring of regular functions on the complex algebraic torus $\mathcal{O} = \sigma X_{\text{nr}}(T) \cong X_{\text{nr}}(Z(T))$.
- (ii) $\text{End}_G(I_P^G(\text{ind}_{T_{\text{cpt}}}^T(\sigma_1))) \cong \mathcal{H}(\mathcal{O}, G) \rtimes \mathbb{C}[R(\mathcal{O}), \natural]$ where $\mathcal{H}(\mathcal{O}, G)$ is an affine Hecke algebra with known parameters containing $\mathbb{C}[\mathcal{O}]$ as a subalgebra, $R(\mathcal{O})$ is a finite group and $\natural$ is a cocycle that may be nontrivial.

The structure of the Hecke algebra $\mathcal{H}(\mathcal{O}, G)$ is determined by $\mathbb{C}[\mathcal{O}]$ and the roots α such that μ_α is not constant. For those roots α , the poles $q_\alpha, q_{\alpha*}$ of μ_α give the Hecke algebra parameters of $\mathcal{H}(\mathcal{O}, G)$. By Proposition C, these parameters are given by either $q_\alpha = q_F$ and $q_{\alpha*} = 1$, or $q_\alpha = q_{\alpha*} = q_F^{1/2}$. More details are given in Chapter 5. There, we also show with an example that the result does not hold for covers that are not Brylinski-Deligne, and with another example that the cocycle $\natural$ may be nontrivial.

Chapter 2

Background

2.1 Reductive p -adic groups and their covering groups

In this section, we discuss the “group” part of the preliminaries, namely non-archimedean local fields, reductive p -adic groups, covers of such groups and Brylinski-Deligne covers.

2.1.1 Non-archimedean local fields and the Hilbert symbol

In this section, we fix some notation regarding the local fields F that we consider in this thesis and we prove some results about the Hilbert symbol on F . Let p be a prime number. A non-archimedean local field F of residue characteristic p is one of the following (see [Mil20, Remark 7.49]):

- (characteristic zero) a finite extension of the field of p -adic numbers $\mathbb{Q}_p$, or
- (characteristic p) the field $\mathbb{F}_q((t))$ of Laurent series over a finite field $\mathbb{F}_q$, where q is a power of p .

A non-archimedean local field F is a discrete valuation field, i.e. it has a discrete valuation $v_F : F \rightarrow \mathbb{Z} \cup \{\infty\}$ satisfying:

- $v_F(ab) = v_F(a) + v_F(b)$ for all $a, b \in F$;
- $v_F(a + b) \geq \min(v_F(a), v_F(b))$;
- $v_F(a) = \infty$ if and only if $a = 0$.

The ring of integers of F is

$$\mathcal{O}_F := \{a \in F : v_F(a) \geq 0\},$$

the group of units of $\mathcal{O}_F$ is

$$\mathcal{O}_F^\times := \{a \in F : v_F(a) = 0\}$$

and the maximal ideal of $\mathcal{O}_F$ is

$$\mathfrak{p}_F := \{a \in F : v_F(a) > 0\}.$$

The residue field of F is $k_F := \mathcal{O}_F/\mathfrak{p}_F$ and the residue characteristic of F is $\text{char}(k_F) = p$. We fix a uniformizer ϖ_F of F , i.e. ϖ_F is an element in $\mathfrak{p}_F$ such that $v_F(\varpi_F) = 1$. Then $\mathfrak{p}_F = \varpi_F \mathcal{O}_F$ and any nonzero element in F can be uniquely written as $a = \varpi_F^{v_F(a)} u$ where $u \in \mathcal{O}_F^\times$.

The cardinality of the residue field of F is $q_F := |k_F|$. The norm on F is given by

$$|a|_F = q_F^{-v_F(a)}.$$

The norm induces a metric on F and F is complete with respect to this metric. Moreover, the metric gives F the structure of a totally disconnected locally compact (tdlc) topological group (with respect to addition). A neighbourhood basis around 0 is given by the open compact subgroups $\mathfrak{p}_F^n$.

The tdlc group F with addition has a Haar measure dx , which we assume to be normalized such that the open compact subgroup $\mathcal{O}_F$ has volume 1. Then for $n \in \mathbb{Z}$, the open compact subgroup $\mathfrak{p}_F^n = \varpi_F^n \mathcal{O}_F$ has volume q_F^{-n} . For $a \in F^\times$ and $f : F \rightarrow \mathbb{C}$ a locally constant function with compact support, we have that

$$\int_F f(ax) dx = |a|_F^{-1} \int_F f(x) dx.$$

In a later section, we will discuss Brylinski-Deligne covering groups of reductive p -adic groups. They are constructed using the Hilbert symbol on F (see [Wei95] for the definition). Let n be a positive integer such that p does not divide n . If F contains all n -th roots of unity, i.e. the set

$$\mu_n(F) := \{a \in F : a^n = 1\}$$

has n elements, then the n -th Hilbert symbol on F is a map

$$(-, -)_n : F^\times \times F^\times \rightarrow \mu_n(F).$$

We list some properties of the Hilbert symbol that will be the most important to us:

1. $(ab, c)_n = (a, c)_n (b, c)_n$ and $(a, bc)_n = (a, b)_n (a, c)_n$ for all $a, b, c \in F^\times$;
2. $(a, b)_n = (b, a)_n^{-1}$ for all $a, b \in F^\times$;
3. $(a, b)_n = 1$ for all $b \in F^\times$ if and only if $a \in F^{\times n}$;
4. $(a, b)_n = 1$ if and only if a is in the image of the norm map for the field extension $F(\sqrt[n]{b})/F$;
5. $(a, b) = 1$ if $a + b = 1$.

The Hilbert symbol is an example of a *Steinberg symbol*, which is by definition a map $(,) : F^\times \times F^\times \rightarrow A$ where A is an abelian group, such that $(,)$ satisfies properties 1 and 5 from the list above. The Steinberg symbols are related to the algebraic K -theory of F as follows. By a theorem of Matsumoto [Mil71, Theorem 11.1],

$$K_2(F) \cong F^\times \otimes F^\times / \langle a \otimes (1-a) \mid 1 \neq a \in F^\times \rangle.$$

This group is generated by the symbols $\{x, y\}$, where $x, y \in F^\times$, see [Mil71, §9]. We call $\{, \} : F^\times \times F^\times \rightarrow K_2(F)$ the *universal symbol*. The universal symbol gives rise to all Steinberg symbols in the following way. Given a Steinberg symbol $(,)$ with values in A , there exists exactly one homomorphism $K_2(F) \rightarrow A$ that sends $\{x, y\} \in K_2(F)$ to $(x, y) \in A$ for all $x, y \in F^\times$ [Mil71, Corollary 11.3]. So for example, n -th Hilbert symbol on F gives a map $(-, -)_n : K_2(F) \rightarrow \mu_n(F)$. This map will be used later in the construction of Brylinski-Deligne covering groups.

Using the properties of the Hilbert symbol listed above, we can prove some more properties that will be useful to us in later sections.

Lemma 2.1.1. *Suppose that F contains all n -th roots of unity and let $(-, -)_n$ be the n -th Hilbert symbol on F . Then the following three statements hold:*

- (i) *For all $a, b \in F^{\times n} \mathcal{O}_F^\times$, $(a, b)_n = 1$.*
- (ii) *Let $x_F \in \mathcal{O}_F^\times$ such that its image $\overline{x_F}$ in $k_F^\times$ generates $k_F^\times$. Then for all $a \notin F^{\times n} \mathcal{O}_F^\times$, $(a, x_F)_n \neq 1$.*
- (iii) *Let x_F as in (ii) and let ϖ_F be a uniformizer of F . Then $(\varpi_F, x_F)_n$ is a primitive n -th root of unity.*

Proof. By property 4 of the Hilbert symbol, $(a, b)_n = 1$ if and only if a is the norm of an element in $F(\sqrt[n]{b})$. If $b \in F^{\times n}$, then $(a, b) = 1$ for all $a \in F^\times$. So let $b \in \mathcal{O}_F^\times$, then to prove (i), it remains to show that $(a, b)_n = 1$ for all $a \in \mathcal{O}_F^\times$. Since F contains all n -th roots of unity, $F \subset F(\sqrt[n]{b})$ is a Galois extension, with Galois group isomorphic to $\mathbb{Z}/d\mathbb{Z}$, where $d \mid n$. Let ζ_d be a primitive d -th root of unity, and for $m \in \mathbb{Z}/d\mathbb{Z}$, write $\sigma_m : \sqrt[n]{b} \mapsto \zeta_d^m \sqrt[n]{b}$. Then the norm map on $F(\sqrt[n]{b})$ is given by $N(a) = \prod_m \sigma_m(a)$. Since $b \in \mathcal{O}_F^\times$, the norm map restricts to a group homomorphism $N : \mathcal{O}_F^\times(\sqrt[n]{b}) \rightarrow \mathcal{O}_F^\times$. Let ϖ_F be a uniformizer in F , and write $k_F = \mathcal{O}_F/(\varpi_F)$. Then $N : k_F^\times(\sqrt[n]{b}) \rightarrow k_F^\times$ is surjective since the norm map is always surjective for finite fields. By Hensel's lemma, since $p \nmid n$, this lifts to a surjection $N : \mathcal{O}_F^\times(\sqrt[n]{b}) \rightarrow \mathcal{O}_F^\times$. So if $a \in \mathcal{O}_F^\times$, then $(a, b)_n = 1$. This proves (i).

Let $x_F \in \mathcal{O}_F^\times$ such that its image $\overline{x_F}$ in $k_F^\times$ generates $k_F^\times$. Then since $p \nmid n$, x_F is not a d -th power for every $1 < d \mid n$. For this x_F , we have the Galois extension $F \subset F(\sqrt[n]{x_F})$ of degree n , with norm map N as above. Since $v_F(x_F) = 0$, $F \subset F(\sqrt[n]{x_F})$ is an unramified field extension. Hence, the uniformizing element ϖ_F of F is also a uniformizing element of $F(\sqrt[n]{x_F})$, and it satisfies $N(\varpi_F) = \varpi_F^n$. Thus

$$\begin{aligned} N(F^\times(\sqrt[n]{x_F})) &= N(\langle \mathcal{O}_F^\times(\sqrt[n]{x_F}), \varpi_F \rangle) = \langle N(\mathcal{O}_F^\times(\sqrt[n]{x_F})), N(\varpi_F) \rangle \\ &= \langle \mathcal{O}_F^\times, \varpi_F^n \rangle = F^{\times n} \mathcal{O}_F^\times. \end{aligned}$$

This shows that $(a, x_F)_n \neq 1$ whenever $a \notin F^{\times n} \mathcal{O}_F^\times$, which proves (ii).

Let x_F as in (ii). The group $F^\times / (F^{\times n} \mathcal{O}_F^\times)$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, with generator ϖ_F . By (ii), $(\varpi_F^\ell, x_F)_n = (\varpi_F, x_F)_n^\ell \neq 1$ for every $0 < \ell < n$. It follows that $(\varpi_F, x_F)_n$ is a primitive n -th root of unity. $\square$

Corollary 2.1.2. *The multiplicative subgroup $F^{\times n} \mathcal{O}_F^\times \subset F^\times$ is a maximal subgroup of $F^\times$ such that $(a, b)_n = 1$ for all a, b in this subgroup.* $\square$

We conclude this section by proving some more results about the Hilbert symbol. These results will be useful later, when we will consider some concrete examples of Brylinski-Deligne covering groups. We have that (as sets)

$$F^\times \cong \mathbb{Z} \times \mathcal{O}_F^\times \cong \mathbb{Z} \times (1 + \mathfrak{p}_F) \times k_F^\times = \langle \varpi_F \rangle \times (1 + \mathfrak{p}_F) \times \langle \overline{x_F} \rangle.$$

Lemma 2.1.3. *Let $y \in 1 + \mathfrak{p}_F$. Then $(y, b)_n = 1$ for all $b \in F^\times$.*

Proof. Recall that by assumption, F contains all n -th roots of unity. Let ζ_n be a primitive n -th root of unity in F ; it projects to a nontrivial element $\overline{\zeta_n}$ in $k_F^\times$. Since $\overline{\zeta_n}^n = 1$, the order of $k_F^\times$ is divisible by n , i.e. $n \mid (q_F - 1)$. Modulo $\mathfrak{p}_F$, we have that $\overline{y} = 1 = \overline{x_F}^{q_F - 1}$. By Hensel's Lemma, this lifts to $y = x_F^{q_F - 1}$ for some representative $x_F \in \mathcal{O}_F^\times$ of $\overline{x_F}$. Then

$$(y, b)_n = (x_F^{q_F - 1}, b)_n = (x_F, b)_{n^{q_F - 1}} = 1,$$

since n divides $q_F - 1$. $\square$

So to know all values of $(-, -)_n$ explicitly, it suffices to know the values of $(\varpi_F, \varpi_F)_n$, $(\varpi_F, x_F)_n$ and $(x_F, x_F)_n$. The following result follows directly from Lemma 2.1.1.

Lemma 2.1.4. *Let ϖ_F be a uniformizer of F and let $x_F \in \mathcal{O}_F^\times$ such that its image in $k_F^\times$ generates $k_F^\times$. Then $(x_F, x_F)_n = 1$ and $(\varpi_F, x_F)_n$ is a primitive n -th root of unity.* $\square$

Unfortunately, we can not use Lemma 2.1.1 to determine the value of $(\varpi_F, \varpi_F)_n$. In fact, unlike the results we have proved so far, the answer to this actually depends on the field F . We will first consider the case $n = 2$; in this case, the answer depends on whether or not F contains a primitive fourth root of unity. The quadratic Hilbert symbol $(-, -)_2 : F^\times \times F^\times \rightarrow \{\pm 1\}$ is given by

$$(a, b)_2 = \begin{cases} 1 & \text{if } 1 = \alpha^2 a + \beta^2 b \text{ for some } \alpha, \beta \in F, \\ -1 & \text{otherwise.} \end{cases}$$

Lemma 2.1.5. *Assume that p is odd. The equation*

$$1 = (\alpha^2 + \beta^2) \varpi_F$$

has a solution $\alpha, \beta \in F$ if and only if F contains a primitive 4-th root of unity. In other words, $(\varpi_F, \varpi_F)_2 = 1$ if and only if F contains a primitive 4-th root of unity.

Proof. If $\zeta_4 \in F$ is a primitive fourth root of unity, then a solution is given by

$$\alpha = \frac{\varpi_F^{-1} + \zeta_4/2}{\zeta_4 + 1}, \quad \beta = \frac{\zeta_4 \varpi_F^{-1} + 1/2}{\zeta_4 + 1}.$$

Conversely, suppose there is a solution

$$1 = (\alpha^2 + \beta^2) \varpi_F.$$

Then we must have $v_F(\alpha) = v_F(\beta) = n \leq -1$ and $v_F(\alpha^2 + \beta^2) = -1$. Write

$$\alpha = \sum_{i=n}^{\infty} \alpha_i \varpi_F^i, \quad \beta = \sum_{i=n}^{\infty} \beta_i \varpi_F^i, \quad (\alpha^2 + \beta^2) = \sum_{i,j=n}^{\infty} (\alpha_i \alpha_j + \beta_i \beta_j) \varpi_F^{i+j},$$

where $\alpha_i, \beta_i \in k_F$. Since we need $v_F(\alpha^2 + \beta^2) = -1$ (an odd number), it follows that $\alpha_n^2 + \beta_n^2 = 0 \pmod{\mathfrak{p}_F}$, where $\alpha_n, \beta_n \in k_F^\times$. If we rewrite the equation in the algebraic closure of k_F , which contains a primitive 4-th root of unity ζ_4 , we find

$$(\alpha_n + \zeta_4 \beta_n)(\alpha_n - \zeta_4 \beta_n) = 0,$$

from which it follows that $\zeta_4 = \pm \alpha_n / \beta_n \in k_F^\times$. By Hensel's lemma, this solution lifts to a primitive 4-th root of unity in F . $\square$

For example, the field $\mathbb{Q}_p$ contains a primitive 4-th root of unity if and only if $p \equiv 1 \pmod{4}$. A generalization of Lemma 2.1.5 is the following.

Lemma 2.1.6. *Suppose that F contains all n -th roots of unity and let ϖ_F be a uniformizer of F . Then*

$$(\varpi_F, \varpi_F)_n = \pm 1.$$

If n is odd, then $(\varpi_F, \varpi_F)_n = 1$. If n is even, then $(\varpi_F, \varpi_F)_n = 1$ if and only if F contains a primitive $2n$ -th root of unity.

Proof. Consider the field extension $F(\sqrt[n]{\varpi_F})$ over F . Let ζ_n be a primitive n -th root of unity in F . Then

$$N(\sqrt[n]{\varpi_F}) = \prod_{0 \leq m < n} (\zeta_n^m \sqrt[n]{\varpi_F}) = \varpi_F \prod_{0 \leq m < n} \zeta_n^m = \begin{cases} \varpi_F & \text{if } n \text{ is odd} \\ -\varpi_F & \text{if } n \text{ is even.} \end{cases}$$

So if n is odd, then $(\varpi_F, \varpi_F)_n = 1$. If n is even, then $(\varpi_F, -\varpi_F)_n = 1$, hence

$$1 = (\varpi_F, -\varpi_F)_n = (\varpi_F, \varpi_F)_n (\varpi_F, -1)_n.$$

We have that

$$(\varpi_F, -1)_n^2 = (\varpi_F, 1)_n = 1,$$

so $(\varpi_F, -1)_n = \pm 1$ and also $(\varpi_F, \varpi_F)_n = (\varpi_F, -1)_n^{-1} = \pm 1$. We have $(\varpi_F, \varpi_F)_n = 1$ if and only if $(\varpi_F, -1)_n = 1$, i.e. if and only if -1 is in the image of the norm map. This is true if and only if F contains a primitive $2n$ -th root of unity. $\square$

2.1.2 Reductive p -adic groups

This section serves as a summary of the relevant structure theory of reductive p -adic groups and to fix notation. For the (many) definitions that are mentioned without explanation, we refer to the cited references for further details.

Definition 2.1.7. A *reductive p -adic group* is the group $G := \mathcal{G}(F)$ of F -rational points of a connected reductive linear algebraic group $\mathcal{G}$ over F , where F is a non-archimedean local field of residue characteristic p . The topology on G comes from the topology on F , making G into a tdlc topological group.

Structure of $\mathcal{G}$

Connected reductive linear algebraic groups $\mathcal{G}$ over F are classified by their root data, see [Spr09]. We give a short summary here. Let $\mathcal{G}$ be a connected reductive linear algebraic group defined over F . The group $\mathcal{G}$ has maximal tori defined over F . Let $\mathcal{S}$ be a maximal F -split torus of $\mathcal{G}$ and let $\mathcal{T}$ be a maximal torus of $\mathcal{G}$ defined over F and containing $\mathcal{S}$. Let $X = X^*(\mathcal{S})$ denote the F -characters of $\mathcal{S}$, $Y = X_*(\mathcal{S})$ the F -cocharacters of $\mathcal{S}$ and denote the pairing between them by $\langle \cdot, \cdot \rangle$. The roots $\Phi \subset X$ are the weights of the adjoint representation of $\mathcal{S}$ on the Lie algebra of $\mathcal{G}$ and there is a corresponding set of coroots

$$\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\} \subset Y \text{ with } \alpha^\vee \text{ determined by } \langle \alpha, \alpha^\vee \rangle = 2.$$

Together they form the root datum $(X, \Phi, Y, \Phi^\vee)$ of $\mathcal{G}$ with respect to $\mathcal{S}$. For each $\alpha \in \Phi$, there is a unipotent F -split root subgroup $\mathcal{U}_\alpha$ of $\mathcal{G}$. The group $\mathcal{G}$ is generated by the F -subgroups $Z_{\mathcal{G}}(\mathcal{S})$ (the centralizer of $\mathcal{S}$ in $\mathcal{G}$) and the $\mathcal{U}_\alpha$ for $\alpha \in \Phi$. Since $\mathcal{G}$ is not necessarily split, its root system Φ with respect to $\mathcal{S}$ may be non-reduced. Namely, for $\alpha \in \Phi$, the possible multiples of α that also lie in Φ are $\alpha/2$ and 2α (but never both). If α and 2α are both roots, then $\mathcal{U}_{2\alpha} \subset \mathcal{U}_\alpha$.

The parabolic subgroups and Levi subgroups of $\mathcal{G}$ are characterized as follows. Let $\mathcal{P}$ be a parabolic F -subgroup of $\mathcal{G}$. Let $\mathcal{M} \times \mathcal{U}_{\mathcal{P}}$ be a Levi decomposition of $\mathcal{P}$. The unipotent radical $\mathcal{U}_{\mathcal{P}}$ of $\mathcal{P}$ is defined over F and F -split. The Levi subgroup $\mathcal{M}$ is reductive and we may choose it such that it is defined over F . Fix a maximal F -split torus $\mathcal{A}_{\mathcal{M}}$ in the center $Z(\mathcal{M})$, then $\mathcal{M} = Z_{\mathcal{G}}(\mathcal{A}_{\mathcal{M}})$. The Levi subgroup $\mathcal{M}$ contains a maximal F -split torus of $\mathcal{G}$ containing $\mathcal{A}_{\mathcal{M}}$, say $\mathcal{S}$. Let $\Sigma(\mathcal{M})$ denote the weights of the adjoint representation of $\mathcal{A}_{\mathcal{M}}$ on the Lie algebra of $\mathcal{G}$. We can view $\Sigma(\mathcal{M})$ as a subset of Φ and $\mathcal{M}$ is generated by $Z_{\mathcal{G}}(\mathcal{S})$ and the root subgroups $\mathcal{U}_\alpha$ for $\alpha \in \Phi \setminus \Sigma(\mathcal{M})$. The parabolic subgroup $\mathcal{P}$ determines a positive system of roots in $\Sigma(\mathcal{M})$, which we denote by $\Sigma(\mathcal{P})$. Then $\mathcal{U}_{\mathcal{P}}$ is generated by the root subgroups $\mathcal{U}_\alpha$ with $\alpha \in \Sigma(\mathcal{P})$. The parabolic subgroup opposite to $\mathcal{P}$ is $\overline{\mathcal{P}} := \mathcal{M} \times \mathcal{U}_{\overline{\mathcal{P}}}$ where $\mathcal{U}_{\overline{\mathcal{P}}}$ is generated by the root subgroups $\mathcal{U}_\alpha$ with $\alpha \in \Sigma(\overline{\mathcal{P}}) = -\Sigma(\mathcal{P}) = \Sigma(\mathcal{M}) \setminus \Sigma(\mathcal{P})$. For a parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$ with Levi decomposition $\mathcal{M} \times \mathcal{U}_{\mathcal{P}}$, we denote by $P = MU_P$ the corresponding groups of F -points.

Suppose that $\mathcal{G}$ is split over F , i.e. it contains a maximal torus that is split over F . So using the notation from before, we have $\mathcal{T} = \mathcal{S}$. A choice of Borel subgroup $\mathcal{B}$ (in the split case, this is a minimal parabolic subgroup) of $\mathcal{G}$ containing $\mathcal{T}$ determines a basis Δ of Φ and dually a basis $\Delta^\vee$ of $\Phi^\vee$. The roots in Δ (resp. coroots in $\Delta^\vee$) are called the *simple roots* (resp. *simple coroots*). We obtain the *based root datum* of $\mathcal{G}$,

$$(X, \Phi, \Delta, Y, \Phi^\vee, \Delta^\vee),$$

and the root datum of $\mathcal{G}$ determines $\mathcal{G}$ up to isomorphism. Conversely, every based root datum comes from a split reductive group and the root data are classified using Dynkin diagrams. Since $\mathcal{G}$ is split, the root system Φ is reduced. The group $\mathcal{G}$ is generated by $\mathcal{T}$ and the one-dimensional unipotent root subgroups $\mathcal{U}_\alpha$ for $\alpha \in \Delta$. Passing to F -points, $G = \mathcal{G}(F)$ is generated by $T = \mathcal{T}(F) \cong F^{\times r}$ and the root subgroups $U_\alpha = \mathcal{U}_\alpha(F) \cong F^+$.

Now let $\mathcal{G}$ be a connected reductive group over F that is not necessarily split over F . It is split over the separable closure F_s of F . So $\mathcal{G}$ is defined over $F \subset F_s$ and split over F_s . With respect to $\mathcal{T}$ and over F_s , we have a based root datum

$$(X_s, \Phi_s, \Delta_s, Y_s, \Phi_s^\vee, \Delta_s^\vee)$$

of $\mathcal{G}$. The F -split torus $\mathcal{S}$ in $\mathcal{T}$ determines a subset Δ_0 of Δ_s , consisting of the simple roots whose restriction to $\mathcal{S}$ is trivial. The Galois group $\Gamma := \text{Gal}(F_s/F)$ acts on Δ_s via a continuous homomorphism $\tau : \Gamma \rightarrow S(\Delta_s)$, where $S(\Delta_s)$ denotes the permutation group of Δ_s . The triple $(\Delta_s, \Delta_0, \tau)$ is called the *index* of $\mathcal{G}$. Up to isomorphism, $\mathcal{G}$ is determined by its *admissible triple*, consisting of:

- the *indexed root datum* $(X_s, \Phi_s, \Delta_s, Y_s, \Phi_s^\vee, \Delta_s^\vee, \Delta_0, \tau)$ of $\mathcal{G}$ together with
- the centralizer $Z_{\mathcal{G}_{\text{der}}}(\mathcal{S})$ of $\mathcal{S}$ in the derived subgroup $\mathcal{G}_{\text{der}}$ of $\mathcal{G}$, and
- the connected component $\mathcal{C} = Z(\mathcal{G})^0$ of the center of $\mathcal{G}$.

Conversely, every *representable* admissible triple comes from a reductive F -group, but we will not mention the representability conditions here. The subgroups $\mathcal{G}_{\text{der}}$, $Z_{\mathcal{G}_{\text{der}}}(\mathcal{S})$ and $\mathcal{C}$ of $\mathcal{G}$ are all defined over F . The derived group $\mathcal{G}_{\text{der}}$ is semisimple and $\mathcal{C}$ is a torus. There is a central isogeny $\mathcal{G}_{\text{der}} \times \mathcal{C} \rightarrow \mathcal{G}$ inducing a map on F -rational points $G_{\text{der}} \times C \rightarrow G$, so the structure of G is determined by the structure of G_{der} and C .

Compact subgroups of $G = \mathcal{G}(F)$

Now that we have summarized the structure of $\mathcal{G}$, we pass to the structure of its group of F -points $G = \mathcal{G}(F)$. The compact open subgroups of G are described using Bruhat-Tits theory, using valuated root data and the Bruhat-Tits building of $\mathcal{G}$, see [KP23] and [Tit79]. Let us give a short summary.

Let $\mathcal{G}$ be a connected reductive F -group and let $\mathcal{S}$ be a maximal split torus. Let Φ be the (possibly non-reduced) root system of $(\mathcal{G}, \mathcal{S})$. Each root $\alpha \in \Phi$ corresponds to a root subgroup $U_\alpha = \mathcal{U}_\alpha(F)$ of $G = \mathcal{G}(F)$. A *valuation* of the

root datum of $(\mathcal{G}, \mathcal{S})$ is a collection $\varphi = (\varphi_\alpha)_{\alpha \in \Phi}$ of maps $\varphi_\alpha : U_\alpha \rightarrow \mathbb{R} \cup \{\infty\}$ satisfying a list of conditions, see [KP23, Definition 6.1.2]. For example, one of the conditions is that for $\alpha \in \Phi$, each $r \in \mathbb{R}$ gives rise to a compact open subgroup $U_{\alpha,r} := \varphi_\alpha^{-1}([r, \infty]) \subset U_\alpha$. Moreover, $U_{\alpha,\infty} = \varphi_\alpha^{-1}(\{\infty\}) = \{1\}$ and the groups $U_{\alpha,r}$ for $r \in \mathbb{R}$ give a neighbourhood basis around 1 in U_α .

Valuations on the root datum of $(\mathcal{G}, \mathcal{S})$ exist and they are organized in the Bruhat-Tits building $\mathcal{B}(\mathcal{G}_{\text{der}})$ of the derived subgroup $\mathcal{G}_{\text{der}}$ of $\mathcal{G}$. Namely, the *apartment* $\mathcal{A}(\mathcal{S})$ in $\mathcal{B}(\mathcal{G}_{\text{der}})$ is an affine space whose points give rise to valuations of the root datum of $(\mathcal{G}, \mathcal{S})$ in a canonical way, see [KP23, §9.6] for details. The building $\mathcal{B}(\mathcal{G}_{\text{der}})$ is built out of the apartments $\mathcal{A}(\mathcal{S}')$, where $\mathcal{S}'$ runs over all maximal split tori in $\mathcal{G}$. The *chambers* of $\mathcal{B}(\mathcal{G}_{\text{der}})$ correspond to the Iwahori subgroups of G . There is an action of G on $\mathcal{B}(\mathcal{G}_{\text{der}})$, satisfying all sorts of nice properties, see [KP23, §4.1] for details. The structure of the Bruhat-Tits building together with the action of G can be used to describe compact open subgroups of G . For example, every point in $\mathcal{B}(\mathcal{G}_{\text{der}})$ gives rise to a compact open subgroup of G , and every vertex in $\mathcal{B}(\mathcal{G}_{\text{der}})$ gives rise to a maximal compact open subgroup of G .

Let us elaborate on how points in the building give rise to compact open subgroups. Let $X^*(G)$ denote the F -characters of G and consider the subgroup

$$G^1 := \{g \in G = \mathcal{G}(F) : v_F(\chi(g)) = 0 \ \forall \chi \in X^*(G)\} = \bigcap_{\chi \in X^*(G)} \ker |\chi|_F$$

of G . We warn the reader that in the literature, this group knows different notations, which might lead to confusion. It is denoted by G^1 in [KP23], by ${}^0G$ in [Ren10] and by G^0 in [FP22; Roc09]. Here, we will stick to the notation G^1 . A point $x \in \mathcal{B}(\mathcal{G}_{\text{der}})$ gives rise to the compact open subgroup $G^1 \cap G_x = G_x^1$ of G , where G_x denotes the stabilizer of x in G . If $x \in \mathcal{B}(\mathcal{G}_{\text{der}})$ is a vertex, then G_x^1 is a maximal compact open subgroup of G . Conversely, every maximal compact subgroup of G is of the form G_x^1 for a point (not necessarily a vertex) $x \in \mathcal{B}(\mathcal{G}_{\text{der}})$ [KP23, Theorem 4.2.15]. One may also work with the *extended building* of $\mathcal{G}$, namely

$$\mathcal{B}(\mathcal{G}) := \mathcal{B}(\mathcal{G}_{\text{der}}) \times (X_*(C) \otimes_{\mathbb{Z}} \mathbb{R}),$$

where $X_*(C)$ denotes the F -cocharacters of $\mathcal{C} = Z(\mathcal{G})^0$. The advantage of the extended building is that for a point $\tilde{x} \in \mathcal{B}(\mathcal{G})$, its stabilizer $G_{\tilde{x}}$ in G is itself a compact open subgroup. It equals $G_{\tilde{x}} = G^1 \cap G_x = G_x^1$, where x denotes the image of $\tilde{x}$ in $\mathcal{B}(\mathcal{G}_{\text{der}})$, see [KP23, §4.3] for details. A *special* vertex $x \in \mathcal{A}(\mathcal{S})$ gives rise to a *good* maximal compact subgroup $K = G_x^1$ of G , which means that K contains representatives of every element of the Weyl group $W(G, S) = N_G(S)/Z_G(S)$. For such K , the Iwasawa decomposition holds.

Theorem 2.1.8 (Iwasawa decomposition [KP23, Theorem 5.3.4]). *Let P be a parabolic subgroup of G and let K be any good maximal compact subgroup of G associated to a special vertex x in the Bruhat-Tits building $\mathcal{B}(\mathcal{G}_{\text{der}})$ of G . Then*

$$G = PK = KP.$$

In particular, let $P = MU_P$ be a Levi decomposition of a parabolic subgroup of G , then the Iwasawa decomposition implies that any $g \in G$ can be

written as

$$g = m_P(g)u_P(g)k_P(g), \text{ where } m_P(g) \in M, u_P(g) \in U_P \text{ and } k_P(g) \in K.$$

This will be important later, in Chapter 4.

We conclude this section by discussing the unramified characters of G . The group of *unramified characters* of G is

$$X_{\text{nr}}(G) := \text{Hom}(G/G^1, \mathbb{C}^\times),$$

which has the structure of a complex algebraic torus since the quotient G/G^1 is a free abelian group of finite rank, see [Roc09]. The group of unramified characters plays an important role in the representation theory of G . We will discuss it a bit more in the next section, since there are natural generalizations of the definitions of G^1 and $X_{\text{nr}}(G)$ to covering groups. Using Bruhat-Tits theory, it can be shown that G^1 is generated by all compact subgroups of G . This result will be mentioned in the next section (see Lemma 2.1.13), since it also holds for covering groups.

2.1.3 Covering groups

Let F be a non-archimedean local field and let $G' = \mathcal{G}'(F)$ be the group of F -points of a connected reductive linear algebraic group $\mathcal{G}'$ over F .

Definition 2.1.9. A (*finite central*) *covering group* G of G' is an extension of topological groups

$$1 \rightarrow A \rightarrow G \rightarrow G' \rightarrow 1,$$

where A is a finite abelian group, $A \hookrightarrow Z(G)$ and the projection $p_G : G \rightarrow G'$ is a topological covering map.

An example of such a covering group is the following.

Example 2.1.10. Consider $G' = \text{SL}_2(F)$, which is generated by the subgroups

- $T' = \left\{ t(x) := \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} : x \in F^\times \right\} \cong F^\times,$
- $U = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in F \right\} \cong F^+,$
- $\bar{U} = \left\{ \begin{pmatrix} 1 & 0 \\ \bar{u} & 1 \end{pmatrix} : \bar{u} \in F \right\} \cong F^+.$

There is a covering group

$$1 \rightarrow \{\pm 1\} \rightarrow \widetilde{\text{SL}_2(F)} \rightarrow \text{SL}_2(F) \rightarrow 1.$$

As a set, $\widetilde{\text{SL}_2(F)} = \{\pm 1\} \times \text{SL}_2(F)$. As a group, it is generated by subgroups $U, \bar{U}$ and T . The multiplication on $T = \{\pm 1\} \times T'$ is given by

$$(\varepsilon, t(x))(\varepsilon', t(x')) = (\varepsilon\varepsilon'(x, x')_2, t(xx')),$$

where the sign is determined by the quadratic Hilbert symbol

$$(x, x')_2 = \begin{cases} 1 & \text{if } 1 = \alpha^2 x + \beta^2 x' \text{ for some } \alpha, \beta \in F \\ -1 & \text{otherwise.} \end{cases}$$

$\widetilde{\mathrm{SL}}_2(F)$ is an example of the *Metaplectic group*, which is a double cover of the symplectic group $\mathrm{Sp}_{2N}(F)$. It is also an example of a Brylinski-Deligne covering group, which we will discuss more elaborately in Section 2.1.4.

Like G' , a covering group G of a reductive group G' is a locally compact totally disconnected group. However, in general G can not be seen as the group of F -rational points of a linear algebraic group over F . Still, many methods and results in the (representation) theory of reductive p -adic groups generalize to covering groups. We list here some properties and results about the structure of covering groups that will be important to us. Representation theory will be discussed in Section 2.2.

Definition 2.1.11. Let G be a covering group of a reductive p -adic group G' . A *parabolic* (resp. *Levi*) subgroup of G is the inverse image of a parabolic (resp. Levi) subgroup of G' .

Covering groups split canonically over unipotent subgroups. Let $G'_{\mathrm{unip}} := G'_{\mathrm{unip}}(F)$ where $\mathcal{G}'_{\mathrm{unip}}$ is the subvariety of $\mathcal{G}'$ of unipotent elements.

Proposition 2.1.12 ([Li14, Proposition 2.2.1], [MW94, Appendice I]). *Let $p_G : G \rightarrow G'$ be a covering group. There is a unique continuous section $s : G'_{\mathrm{unip}} \rightarrow G$ of p_G such that for every unipotent subgroup U of G' , $s|_U$ is a group homomorphism and such that s is invariant under conjugation.*

The above result implies that the unipotent subgroups of G' may canonically be identified with the unipotent subgroups of G . This justifies the notation U instead of U' for unipotent subgroups in G' . In particular, let $P = p_G^{-1}(P')$ be a parabolic subgroup of G . Then a Levi decomposition $P' = M'U$ in G' lifts to a Levi decomposition $P = MU$ in G , where $M = p_G^{-1}(M')$. Another decomposition in G' that lifts to a decomposition in G is the Iwasawa decomposition. Namely, the decomposition $G' = P'K' = K'P'$ from Theorem 2.1.8 lifts to an Iwasawa composition $G = PK = KP$ in G , where $K := p_G^{-1}(K')$ is a good maximal compact subgroup of G .

We will now discuss the unramified characters of a covering group. Recall the groups

$$G'^1 = \cap_{\chi \in X^*(G')} \ker |\chi|_F \text{ and } X_{\mathrm{nr}}(G') = \mathrm{Hom}(G'/G'^1, \mathbb{C}^\times)$$

of the previous section, that were defined for a reductive p -adic group G' . For a covering group G of G' , we have natural generalizations of these groups, namely

$$G^1 := p_G^{-1}(G'^1) \text{ and } X_{\mathrm{nr}}(G) := \mathrm{Hom}(G/G^1, \mathbb{C}^\times) = X_{\mathrm{nr}}(G').$$

The following result follows from Bruhat-Tits theory and implies that the unramified characters of G are the complex characters of G that are trivial on all compact subgroups of G .

Lemma 2.1.13. *Let G be a covering group of G' . Then the following two statements hold:*

- (i) *The group G^1 is generated by all compact subgroups of G .*
- (ii) *If G' is semisimple, then $G^1 = G$ is generated by its compact subgroups.*

Proof. The first statement is [FP22, Lemma 4.8]. The second statement then follows from the definition of G^1 and the fact that a semisimple group has no nontrivial rational characters. $\square$

This concludes our discussion on general covering groups. In the next section, we will discuss a particular class of covering groups in more detail, namely Brylinski-Deligne covering groups. Many covering groups fall into this class and they have a nice combinatorial description which allows for explicit computation in such groups.

2.1.4 Brylinski-Deligne covering groups

In [BD01], Brylinski and Deligne study central extensions of $\mathcal{G}'$ by $\mathbf{K}_2$ in the category of sheaves of groups on the big Zariski site of $\mathrm{Spec}(F)$, or of a more general base scheme S instead of $\mathrm{Spec}(F)$. We will attempt to shortly explain these notions without going into too much detail. The important part for us is the classification of such extensions. Brylinski and Deligne showed that the category of central extensions of $\mathcal{G}'$ by $\mathbf{K}_2$ has a nice combinatorial description, in terms of Weyl group-invariant quadratic forms on the cocharacter lattice of a maximal torus of $\mathcal{G}'$ and some additional data. This classification is similar to the classification of linear algebraic groups using root data. By taking F -points and pushing forward these extensions along the Hilbert symbol, we obtain a class of covering groups of G' with an explicit presentation that allows us to do actual computations for such groups. Examples of p -adic covering groups that are obtained this way, are metaplectic covering groups and all covering groups (cf. Definition 2.1.9) of G' when $\mathcal{G}'$ is split and simply connected, as illustrated in [BD01, §0]. In this section, we give a brief summary of some of the classification results of Brylinski and Deligne. Besides the original paper [BD01], a nice reference for the interested reader is [GG18, §2-3], where the emphasis is on covering groups of reductive p -adic groups obtained by pushing forward Brylinski-Deligne extensions along the Hilbert symbol. This section is largely based on these two sources.

Although the theory of Brylinski and Deligne works for not necessarily split groups over a general base scheme S , we will restrict our attention to the case where $\mathcal{G}'$ is a split reductive linear algebraic group over a non-archimedean local field F . The *big Zariski site* $\mathrm{Spec}(F)_{\mathrm{Zar}}$ of the scheme $\mathrm{Spec}(F)$ is taken to be the category of schemes of finite type over $\mathrm{Spec}(F)$ [BD01, §1.4]. The group $\mathcal{G}'$ determines a sheaf of groups on $\mathrm{Spec}(F)_{\mathrm{Zar}}$ by $S \mapsto \mathrm{Hom}_{\mathrm{Spec}(F)}(S, \mathcal{G}')$.

For a scheme S , let $K_2(S)$ be Quillen's K_2 -group of the exact category of vector bundles on S , see [Qui73] for Quillen's original construction. If $S = \mathrm{Spec}(R)$ is the spectrum of a ring R , then $K_2(S) = K_2(R)$, where $K_2(R)$

is the second degree algebraic K -theory group of R . For $S = \operatorname{Spec}(F)$, the group $K_2(\operatorname{Spec}(F)) = K_2(F)$ is generated by the symbols $\{x, y\}$, where

$$\{ , \} : F^\times \times F^\times \rightarrow K_2(F)$$

is the universal symbol. The K_2 -groups define a presheaf of groups on S , given by $U \mapsto K_2(U)$ for U open in S . Let $\mathbf{K}_2$ be the sheafification of this presheaf on S . This gives a system of sheaves on the schemes S in $\operatorname{Spec}(F)_{\text{zar}}$, satisfying the required compatibility conditions (see [BD01, §1.4]) in order for it to correspond to a sheaf on $\operatorname{Spec}(F)_{\text{zar}}$, which we will also denote by $\mathbf{K}_2$.

Brylinski and Deligne classified the extensions

$$1 \rightarrow \mathbf{K}_2 \rightarrow \mathcal{G} \rightarrow \mathcal{G}' \rightarrow 1$$

in the category of sheaves of groups on $\operatorname{Spec}(F)_{\text{zar}}$. Such extensions are also called – or rather, can be viewed as – *multiplicative $\mathbf{K}_2$ -torsors* on $\mathcal{G}'$. We will use these two names (central extensions of $\mathcal{G}'$ by $\mathbf{K}_2$ / multiplicative $\mathbf{K}_2$ -torsors) interchangeably. We will first summarize the classification results. Later, we will explain how these extensions give rise to p -adic covering groups.

The first classification result concerns the category of multiplicative $\mathbf{K}_2$ -torsors on a torus. Fix a split maximal torus $\mathcal{T}'$ of $\mathcal{G}'$ and let Y be the cocharacter lattice of $\mathcal{T}'$. For a quadratic form $Q : Y \rightarrow \mathbb{Z}$, we denote by B_Q the symmetric bilinear form on Y associated to Q , which is given by

$$B_Q(y, z) := Q(y + z) - Q(y) - Q(z).$$

Proposition 2.1.14 ([BD01, Theorem 3.16] with $S = \operatorname{Spec}(F)$). *The category of central extensions of $\mathcal{T}'$ by $\mathbf{K}_2$ is equivalent to the category whose objects are pairs $(Q, \mathcal{E})$ where*

- $Q : Y \rightarrow \mathbb{Z}$ is a quadratic form on Y ;
- $\mathcal{E}$ is a central extension of groups

$$1 \rightarrow F^\times \rightarrow \mathcal{E} \rightarrow Y \rightarrow 1$$

with commutator $[-, -] : Y \times Y \rightarrow F^\times$, $[y, z] = (-1)^{B_Q(y, z)}$.

The set of morphisms between $(Q, \mathcal{E})$ and $(Q', \mathcal{E}')$ is empty if $Q \neq Q'$. If $Q = Q'$, the set of morphism from $(Q, \mathcal{E})$ to $(Q, \mathcal{E}')$ is the set of isomorphisms of central extensions

$$\begin{array}{ccccc} F^\times & \longrightarrow & \mathcal{E} & \longrightarrow & Y \\ \parallel & & \downarrow \cong & & \parallel \\ F^\times & \longrightarrow & \mathcal{E}' & \longrightarrow & Y. \end{array}$$

Let $W = N_{\mathcal{G}'}(\mathcal{T}')/\mathcal{T}'$ be the Weyl group of $(\mathcal{G}', \mathcal{T}')$. For simply connected groups, we have the following classification result.

Proposition 2.1.15 ([BD01, Theorem 4.7] with $S = \operatorname{Spec}(F)$). *Suppose that $\mathcal{G}'$ is simply connected. The category of central extensions of $\mathcal{G}'$ by $\mathbf{K}_2$ is equivalent to the category whose objects are W -invariant quadratic forms $Q : Y \rightarrow \mathbb{Z}$. The only morphisms are the identity morphisms.*

Now, let $\mathcal{G}'$ be a connected split reductive group and denote its root datum by $(X, \Phi, Y, \Phi^\vee)$, with simple roots $\Delta \subset \Phi$ and simple coroots $\Delta^\vee \subset \Phi^\vee$. Let $\mathcal{G}'_{\text{sc}}$ be the simply connected cover of the derived group of $\mathcal{G}'$. Then we have a map $q : \mathcal{G}'_{\text{sc}} \rightarrow \mathcal{G}'$ and an inclusion of cocharacter lattices $Y_{\text{sc}} \hookrightarrow Y$. Whenever we are given a W -invariant quadratic form Q on Y , we may restrict it to Y_{sc} and by Proposition 2.1.15, we obtain a central extension of $\mathcal{G}'_{\text{sc}}$ by $\mathbf{K}_2$. This extension may in turn be restricted to $\mathcal{T}'_{\text{sc}}$, and by Proposition 2.1.14 we obtain a central extension

$$1 \rightarrow F^\times \rightarrow \mathcal{E}_{\text{sc}} \rightarrow Y_{\text{sc}} \rightarrow 1. \quad (2.1.1)$$

As explained in [GG18, §2.4], the simple roots $\alpha \in \Delta$ give rise to a section $s_{\text{sc}} : \Delta^\vee \rightarrow \mathcal{E}_{\text{sc}}$. Then $\mathcal{E}_{\text{sc}}$ is generated by the elements $s_{\text{sc}}(\alpha^\vee)$ for $\alpha^\vee \in \Delta^\vee$ and $a \in F^\times$, subject to the relations:

- all elements $a \in F^\times$ are central in $\mathcal{E}_{\text{sc}}$;
- $[s_{\text{sc}}(\alpha^\vee), s_{\text{sc}}(\beta^\vee)] = (-1)^{B_Q(\alpha^\vee, \beta^\vee)}$ for all $\alpha^\vee, \beta^\vee \in \Delta^\vee$.

The category of central extensions of $\mathcal{G}'$ by $\mathbf{K}_2$ has a description which is a combination of Propositions 2.1.14 and 2.1.15.

Proposition 2.1.16 ([BD01, Theorem 6.2] with $S = \text{Spec}(F)$). *The category of central extensions of $\mathcal{G}'$ by $\mathbf{K}_2$ is equivalent to the category whose objects are triples $(Q, \mathcal{E}, f)$, where*

- $Q : Y \rightarrow \mathbb{Z}$ is a W -invariant quadratic form on Y . By (2.1.1), it defines an extension $\mathcal{E}_{\text{sc}}$ of Y_{sc} by $F^\times$;
- $\mathcal{E}$ is a central extension of Y by $F^\times$ with commutator $[y, z] = (-1)^{B_Q(y, z)}$;
- $f : \mathcal{E}_{\text{sc}} \rightarrow \mathcal{E}$ makes the diagram

$$\begin{array}{ccccc} F^\times & \longrightarrow & \mathcal{E}_{\text{sc}} & \longrightarrow & Y_{\text{sc}} \\ \parallel & & \downarrow f & & \downarrow \\ F^\times & \longrightarrow & \mathcal{E} & \longrightarrow & Y \end{array}$$

commute.

The set of morphisms between $(Q, \mathcal{E}, f)$ and $(Q', \mathcal{E}', f')$ is empty if $Q \neq Q'$. If $Q = Q'$, the set of morphisms from $(Q, \mathcal{E}, f)$ to $(Q, \mathcal{E}', f')$ is the set of isomorphisms of central extensions

$$\begin{array}{ccccc} F^\times & \longrightarrow & \mathcal{E} & \longrightarrow & Y \\ \parallel & & \downarrow \cong & & \parallel \\ F^\times & \longrightarrow & \mathcal{E}' & \longrightarrow & Y \end{array}$$

such that the diagram

$$\begin{array}{ccc} \mathcal{E}_{\text{sc}} & \xlongequal{\quad} & \mathcal{E}_{\text{sc}} \\ \downarrow f & & \downarrow f' \\ \mathcal{E} & \xrightarrow{\cong} & \mathcal{E}' \end{array}$$

commutes.

By Proposition 2.1.16, a BD extension $\mathcal{G}$ of $\mathcal{G}'$ comes from a BD extension $\mathcal{G}_{\text{sc}}$ of $\mathcal{G}'_{\text{sc}}$ and they fit in a commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \mathbf{K}_2 & \longrightarrow & \mathcal{G}_{\text{sc}} & \longrightarrow & \mathcal{G}'_{\text{sc}} \longrightarrow 1 \\
 & & \downarrow & & \downarrow q & & \downarrow q \\
 1 & \longrightarrow & \mathbf{K}_2 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G}' \longrightarrow 1.
 \end{array} \tag{2.1.2}$$

In particular, the projection map $q : \mathcal{G}'_{\text{sc}} \rightarrow \mathcal{G}'$ lifts to a projection map $q : \mathcal{G}_{\text{sc}} \rightarrow \mathcal{G}$.

As described in [GG18, §2.6] we may replace the category of triples $(Q, \mathcal{E}, f)$ by a different category.

Definition 2.1.17. Let $Q : Y \rightarrow \mathbb{Z}$ be a quadratic form with associated symmetric bilinear form B_Q . A *bisector* D of Q is a bilinear form $D : Y \times Y \rightarrow \mathbb{Z}$, not necessarily symmetric, such that

$$D(y, y) = Q(y) \text{ and } B_Q(y, z) = D(y, z) + D(z, y)$$

for all $y, z \in Y$.

For each W -invariant quadratic form Q on Y , define $\text{Bis}_{\mathcal{G}', Q}$ to be the category whose objects are pairs (D, η) where D is a bisector of Q and $\eta : Y_{\text{sc}} \rightarrow F^\times$ is a group homomorphism. The set of morphisms between (D, η) and (D', η') is the set of maps $\xi : Y \rightarrow F^\times$ such that

- $\xi(y+z)\xi(y)^{-1}\xi(z)^{-1} = (-1)^{D(y,z)-D'(y,z)}$ for all $y, z \in Y$;
- $\xi(\alpha^\vee) = \eta'(\alpha^\vee)/\eta(\alpha^\vee)$ for all $\alpha^\vee \in \Delta^\vee$.

Now consider the category

$$\text{Bis}_{\mathcal{G}'} := \bigcup_Q \text{Bis}_{\mathcal{G}', Q}$$

which contains each $\text{Bis}_{\mathcal{G}', Q}$ as a full subcategory. Denote the category of triples $(Q, \mathcal{E}, f)$ as in Proposition 2.1.16 by $\text{BD}_{\mathcal{G}'}$. Then there exists an equivalence of categories

$$\text{Bis}_{\mathcal{G}'} \rightarrow \text{BD}_{\mathcal{G}'}$$

which is given by an explicit functor called the *incarnation functor*. It sends a pair (D, η) to the triple $(Q, \mathcal{E}, f)$ given by:

- $Q(y) = D(y, y)$ for all $y \in Y$;
- $\mathcal{E} = F^\times \times Y$ with group law

$$(a, y)(b, z) = ((-1)^{D(y,z)}ab, y+z);$$

- $f : \mathcal{E}_{\text{sc}} \rightarrow \mathcal{E}$ is the map

$$f(s_{\text{sc}}(\alpha^\vee)) = (\eta(\alpha^\vee), \alpha^\vee) \text{ for all } \alpha^\vee \in \Delta^\vee,$$

where $s_{\text{sc}} : \Delta^\vee \rightarrow \mathcal{E}_{\text{sc}}$ is the section that was mentioned just before Proposition 2.1.16.

Now, let us explain how we obtain covering groups of reductive p -adic groups from BD extensions, see also [BD01, Construction 10.3] and [GG18, §3.1]. Assume that $(Q, \mathcal{E}, f)$ is a triple in $\text{BD}_{G'}$ incarnated by (D, η) . It corresponds to a multiplicative $\mathbf{K}_2$ -torsor on $\mathcal{G}'$,

$$1 \rightarrow \mathbf{K}_2 \rightarrow \mathcal{G} \rightarrow \mathcal{G}' \rightarrow 1.$$

By [BD01, §1.4], taking F -points and since $H^1(F, \mathbf{K}_2) = 0$, we obtain a central extension of groups

$$1 \rightarrow K_2(F) \rightarrow \mathcal{G}(F) \rightarrow \mathcal{G}'(F) \rightarrow 1.$$

Let n be an integer such that F contains all n -th roots of unity, i.e. the set $\mu_n(F) = \{x \in F : x^n = 1\}$ has n elements. Recall from Section 2.1.1 that the n -th Hilbert symbol gives a map

$$(-, -)_n : K_2(F) \rightarrow \mu_n(F).$$

We may push forward the extension above along the Hilbert symbol to obtain

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(F) & \longrightarrow & \mathcal{G}(F) & \longrightarrow & \mathcal{G}'(F) & \longrightarrow & 1 \\ & & \downarrow (-, -)_n & & \downarrow & & \parallel & & \\ 1 & \longrightarrow & \mu_n(F) & \longrightarrow & G & \longrightarrow & G' & \longrightarrow & 1. \end{array}$$

The lower exact sequence is a finite central extension of G' by $\mu_n(F)$ and by [BD01, §10], there exists a topology on G such that $G \rightarrow G'$ becomes a topological covering map, which is essentially because the Hilbert symbol is locally constant. We will call such a covering group a BD covering group of G' associated to the data $(Q, \mathcal{E}, f, n)$ or (D, η, n) . In later chapters of this thesis, we will always work with BD covering groups incarnated by $(D, \eta = 1, n)$.

Example 2.1.18. We give some more context to the construction above and explain that if G' is simply connected, BD extensions give rise to all covering groups of G' , see also [BD01, §0]. The classification of covers of split simply connected groups was already known due to the work of Steinberg, Moore and Matsumoto [Ste62; Moo68; Mat69]. The theory of Brylinski and Deligne was developed as a more general framework.

If G' is split and simply connected, there is a universal central extension [Moo68, §1]

$$1 \rightarrow \pi_1(\mathcal{G}'(F)) \rightarrow \mathcal{G}(F) \rightarrow \mathcal{G}'(F) \rightarrow 1.$$

Assume G' is moreover simple, then $\pi_1(\mathcal{G}'(F)) = K_2(F)$ and the universal extension is also called *Matsumoto's central extension*. Any central extension of $\mathcal{G}'(F)$ by some abelian group A is obtained by pushing forward the universal extension along a homomorphism $K_2(F) \rightarrow A$. In the Brylinski-Deligne framework, Matsumoto's extension corresponds to the BD extension associated to the Weyl group-invariant quadratic form Q on Y given by $Q(\alpha^\vee) = 1$

for all simple coroots $\alpha^\vee$. By Proposition 2.1.15, this extension is universal, since all other Weyl group-invariant quadratic forms on Y are multiples of Q . We will denote this universal Weyl group-invariant quadratic form on Y as $Q = 1$.

To obtain a topological cover, we must push forward Matsumoto's extension along a symbol which is locally constant. By [Moo68, §3], there is a universal locally constant symbol on F , namely the Hilbert symbol

$$(\ , \)_{\# \mu(F)} : K_2(F) \rightarrow \mu(F)$$

where $\mu(F)$ denotes the (finite, cyclic) group of roots of unity of F . If F contains all n -th roots of unity, i.e. $n \mid \# \mu(F)$, then the n -th Hilbert symbol on F equals $(\ , \)_n = (\ , \)_{\# \mu(F)/n}$.

Thus for G' split, simple and simply connected, Matsumoto's universal central extension and the universal Hilbert symbol give rise to a universal covering group of G' , namely

$$\begin{array}{ccccccc} 1 & \longrightarrow & K_2(F) & \longrightarrow & \mathcal{G}(F) & \longrightarrow & \mathcal{G}'(F) \longrightarrow 1 \\ & & \downarrow (\cdot, \cdot)_{\# \mu(F)} & & \downarrow & & \parallel \\ 1 & \longrightarrow & \mu(F) & \longrightarrow & G_{\text{univ}} & \longrightarrow & G' \longrightarrow 1. \end{array}$$

All other covering groups of G' arise from this universal cover G_{univ} , by pushing forward along a homomorphism $\mu(F) \rightarrow A$, for some finite abelian group A . For example, let Q be a Weyl-group invariant quadratic form on Y corresponding to some integer $\ell \in \mathbb{Z}$ and let G be the BD cover of G' associated to the data $(Q = \ell, n)$. Then G is obtained by pushing forward G_{univ} along the map

$$\mu(F) \rightarrow \mu_n(F), \quad a \mapsto a^{\ell \cdot \# \mu(F)/n}.$$

Example 2.1.19. Let $\text{Sp}_{2N}(F)$ be the symplectic group over F , see [Spr09, 2.1.4. Examples. (4)(h)]. It is a split, simply connected reductive group over F . The *metaplectic group* $\text{Mp}_{2N}(F)$ is a double cover of $\text{Sp}_{2N}(F)$, fitting in an exact sequence

$$1 \rightarrow \{\pm 1\} \rightarrow \text{Mp}_{2N}(F) \rightarrow \text{Sp}_{2N}(F) \rightarrow 1.$$

It is the BD cover of $\text{Sp}_{2N}(F)$ associated to the data $(Q = 1, n = 2)$. By the discussion in Example 2.1.18, this is the unique genuine double cover of $\text{Sp}_{2N}(F)$.

More generally, in the literature the terminology *metaplectic covers* is also used for the covers that we call “BD covers” here, i.e. general n -fold BD covers of general reductive p -adic groups.

Brylinski-Deligne covering groups have an explicit presentation which we will describe now, starting with the covering torus T . Let us first consider the extension

$$1 \rightarrow K_2(F) \rightarrow \mathcal{T}(F) \rightarrow \mathcal{T}'(F) \rightarrow 1$$

obtained by taking F -points (so not yet pushed forward along the Hilbert symbol). As a set, it can be realized as $\mathcal{T}(F) = K_2(F) \times \mathcal{T}'(F)$. Recall that X is the character lattice of $\mathcal{T}'$ and Y is the cocharacter lattice of $\mathcal{T}'$. The bisector D of Q , which is a $\mathbb{Z}$ -valued bilinear form on Y , may be written as

$$D = \sum_i x_1^i \otimes x_2^i \in X \otimes_{\mathbb{Z}} X.$$

Then the multiplication on $\mathcal{T}(F) = K_2(F) \times \mathcal{T}'(F)$ is given by

$$(1, t_1)(1, t_2) = (\sigma(t_1, t_2), t_1 t_2),$$

where σ is the 2-cocycle

$$\sigma(t_1, t_2) = \prod_i \{x_1^i(t_1), x_2^i(t_2)\},$$

see [Wei11, §3.2]. Here, $\{ , \}$ is the universal symbol. Pushing this forward along the Hilbert symbol, we obtain

$$1 \rightarrow \mu_n(F) \rightarrow T \rightarrow T' \rightarrow 1.$$

As a set, $T = \mu_n(F) \times T'$ and the multiplication satisfies

$$(1, y(a))(1, z(b)) = ((a, b)_n^{D(y, z)}, y(a)z(b))$$

for all $y, z \in Y$ and $a, b \in F^\times$. With this, we can describe the covering torus T in terms of generators and relations. Namely, T is generated by $\mu_n(F) \subset Z(T)$ and the elements $\mathbf{s}(y(a)) := (1, y(a))$ for $y \in Y$ and $a \in F^\times$, subject to the following relations:

- $[\mathbf{s}(y(a)), \mathbf{s}(z(b))] = (a, b)_n^{B_Q(y, z)};$
- $\mathbf{s}(y(a))\mathbf{s}(z(a)) = (a, a)_n^{D(y, z)}\mathbf{s}((y + z)(a));$
- $\mathbf{s}(y(a))\mathbf{s}(y(b)) = (a, b)_n^{Q(y)}\mathbf{s}(y(ab))$

for all $y, z \in Y$ and $a, b \in F^\times$. The covering torus T is in general not commutative, since the Hilbert symbol is not symmetric for $n > 2$, and D is not necessarily symmetric either. The center $Z(T)$ of T is generated by $\mu_n(F)$ and all elements $\mathbf{s}(y(a))$ where $a \in F^\times$ and y lies in the sublattice

$$Y_{Q, n} := \{y \in Y : B_Q(y, z) \in n\mathbb{Z} \text{ for all } z \in Y\} \subset Y.$$

The group G is generated by T and the unipotent subgroups U_α for $\alpha \in \Phi$. For each $\alpha \in \Phi$, there exists a splitting $e_\alpha : F \rightarrow U_\alpha \subset G$. Namely, if $\{e'_\alpha : \alpha \in \Phi\}$ denotes an épinglage or pinning for (G', T') , then we take e_α to be the canonical lift of the isomorphism $e'_\alpha : F \rightarrow U_\alpha \subset G'$. For each root $\alpha \in \Delta$, define the elements

$$n_\alpha(x) = e_\alpha(x)e_{-\alpha}(-x^{-1})e_\alpha(x) \in N(T)$$

and elements $h_\alpha(x) \in T$ given by

$$h_\alpha(x) := n_\alpha(x)n_\alpha(-1).$$

By [GG18, Proposition 2.5], since we assumed that $\eta = 1$, the $h_\alpha(x)$ satisfy

$$h_\alpha(x) = (1, \alpha^\vee(x)) = \mathbf{s}(\alpha^\vee(x)).$$

It follows that for $a, b \in F^\times$, we have that

$$\sigma_\alpha(a, b) := h_\alpha(a)h_\alpha(b)h_\alpha(ab)^{-1} = (a, b)_n^{Q(\alpha^\vee)}.$$

Then we have the following multiplication relations in G , for $\alpha, \beta \in \Delta$, $z, v \in F^\times$ and $u \in F$:

$$\begin{aligned} n_\alpha(z)e_\alpha(u)n_\alpha(z)^{-1} &= e_{-\alpha}(-z^2u); \\ n_\alpha(z)e_\beta(u)n_\alpha(z)^{-1} &= e_\gamma(\epsilon z^{-\langle \beta, \alpha^\vee \rangle}u); \\ n_\alpha(z)h_\beta(v)n_\alpha(z)^{-1} &= h_\gamma(v)\sigma_\gamma(v, \epsilon z^{-\langle \beta, \alpha^\vee \rangle})^{-1} \\ h_\alpha(z)e_\beta(u)h_\alpha(z)^{-1} &= e_\beta(z^{\langle \beta, \alpha^\vee \rangle}u), \end{aligned}$$

where $\gamma := \beta - \langle \beta, \alpha^\vee \rangle \alpha$ and $\epsilon = \epsilon(\alpha, \beta) = \pm 1$. These relations follow from the relations in [Wei11, §2.1], which hold for split simply connected groups, so they hold for G_{sc} and we can compose them with the projection map $q : G_{\text{sc}} \rightarrow G$ from (2.1.2) to obtain the relations above.

We conclude this section by describing a maximal abelian subgroup of T . Recall from Lemma 2.1.1 and Corollary 2.1.2 that $F^{\times n} \mathcal{O}_F^\times$ is a maximal subgroup of $F^\times$ such that $(a, b)_n = 1$ for all a, b in this subgroup. We may use this result to show that $Z(T)T(\mathcal{O}_F)$ is a maximal abelian subgroup of T .

Lemma 2.1.20. *Let T be a BD extension of a split torus T' . Then the subgroup $A := Z(T)T(\mathcal{O}_F)$ of T is maximal abelian.*

Proof. By Lemma 2.1.1, A is abelian and $T(F^{\times n} \mathcal{O}_F^\times) \subset A$. Let $\{y_1, \dots, y_r\}$ be a basis of the cocharacter lattice Y of T' . Then T is generated by $\mu_n \subset Z(T)$ and all $\mathbf{s}(y_i(a))$ where $1 \leq i \leq r$, $a \in F^\times$. Suppose that $t \in T$ with $t \notin A$, then we will show that there exists an element $t' \in T(\mathcal{O}_F^\times)$ with $[t, t'] \neq 1$. So let $t \notin A$. We may write

$$t = \varepsilon \cdot \mathbf{s}(y_1(a_1))\mathbf{s}(y_2(a_2)) \dots \mathbf{s}(y_r(a_r)),$$

where $\varepsilon \in \mu_n(F)$ and each $a_i \in F^\times$. We want to find an element $t' \in T(\mathcal{O}_F^\times)$ that does not commute with t , so we may ignore the factor ε , since it lies in $Z(T)$. Moreover, since $T(\mathcal{O}_F^\times)$ is abelian, we may assume that each $a_i = \varpi_F^{\ell_i}$ for some integer ℓ_i , where ϖ_F is a uniformizer of F . So

$$t = t_1 t_2 \dots t_r, \text{ where each } t_i := \mathbf{s}(y_i(\varpi_F^{\ell_i})).$$

Since $t \notin Z(T)$, there exists an element $t'' \in T$ that does not commute with t . We may assume that t'' is of the form $t'' = \mathbf{s}(y_j(b))$ for some $b \in F^\times$, $1 \leq j \leq r$. Then, because $[s_1, s_2] \in Z(T)$ for all $s_1, s_2 \in T$,

$$1 \neq [t, t''] = [t_1, t''] [t_2, t''] \dots [t_r, t''] = (\varpi_F, b)_n^{\ell_1 B_Q(y_1, y_j) + \dots + \ell_r B_Q(y_r, y_j)}. \quad (2.1.3)$$

In particular, $\ell_1 B_Q(y_1, y_j) + \dots + \ell_r B_Q(y_r, y_j)$ is not a multiple of n . Let $x_F \in \mathcal{O}_F^\times$ be as in Lemma 2.1.1(ii). Then $(\varpi_F, x_F)_n$ is a primitive n -th root of unity, so also for $t' := \mathbf{s}(y_j(x_F)) \in T(\mathcal{O}_F^\times)$ we have

$$[t, t'] = [t_1, t'] \dots [t_r, t'] = (\varpi_F, x_F)_n^{\ell_1 B_Q(y_1, y_j) + \dots + \ell_r B_Q(y_r, y_j)} \neq 1.$$

This shows that $t \notin A$ does not commute with $t' \in A$, which completes the proof. $\square$

The group A from Lemma 2.1.20 can be characterized as the unique maximal abelian subgroup of T that contains the unique maximal compact subgroup $T(\mathcal{O}_F)$ of T . This result will be used in Chapter 5.

2.2 Representation theory

In this section, we discuss the “representation” part of the preliminaries, namely the basics of smooth complex representations of tdlc groups, the Bernstein decomposition, Harish-Chandra’s intertwining operators and μ -function.

2.2.1 Smooth representations

Let G be a covering group of a reductive p -adic group G' . In this thesis, we are interested in smooth complex representations of G . We discuss the basic concepts, see also [Ren10], [FP22].

Definition 2.2.1. A *complex representation* (π, V) of G is a $\mathbb{C}$ -vector space V together with a group homomorphism $\pi : G \rightarrow \mathrm{Aut}(V)$.

Definition 2.2.2. A complex representation (π, V) of G is *smooth* if for every $v \in V$, the stabilizer

$$\mathrm{Stab}(v) := \{g \in G : \pi(g)v = v\}$$

is open in G , or equivalently if $V = \cup_K V^K$, where K runs over the compact open subgroups of G .

Definition 2.2.3. Let (π, V) be a smooth complex representation of G . The dual space V^* of V is equipped with the representation

$$\pi^*(g)f(v) = f(\pi(g^{-1})v) \text{ for } g \in G, v \in V, f \in V^*.$$

This representation is not necessarily smooth. Therefore, we restrict to the subspace $V^\vee := \cup_K (V^*)^K$ where K runs over the compact open subgroups of G . The space $V^\vee$ is called the *smooth dual* of V and is equipped with the smooth representation $\pi^\vee := \pi^*|_{V^\vee}$. The representation $(\pi^\vee, V^\vee)$ is called the *contragredient representation* of (π, V) .

From now on, whenever we say “representation” we mean a smooth complex representation. We denote the category of smooth complex representations of G by $\mathrm{Rep}(G)$.

Definition 2.2.4. A smooth complex representation (π, V) of G is called *admissible* if for every compact open subgroup K of G , the space V^K is finite dimensional.

Definition 2.2.5. A representation (π, V) of G is called *Hermitian* if there exists a non-degenerate Hermitian form $\langle \cdot, \cdot \rangle$ on V which is G -invariant, i.e. for every $v, w \in V$ and $g \in G$,

$$\langle v, w \rangle = \langle \pi(g) \cdot v, \pi(g) \cdot w \rangle.$$

Given a representation (π, V) of G , its *complex conjugate* representation $(\bar{\pi}, \bar{V})$ is defined as follows. The complex conjugate vector space is

$$\bar{V} := \{\bar{v} \mid v \in V\} \text{ with scalar multiplication } z\bar{v} = \overline{zv}.$$

Then G acts on $\bar{V}$ via $\bar{\pi}(g)\bar{v} := \overline{\pi(g)v}$. The *Hermitian dual* of (π, V) is $(\pi^\vee, \bar{V}^\vee) \cong (\bar{\pi}^\vee, \bar{V}^\vee)$ and the representation (π, V) is Hermitian if and only if $(\pi, V) \cong (\pi^\vee, \bar{V}^\vee)$ [Ren10, §IV.2.1]. A stronger definition is that of a unitary representation.

Definition 2.2.6. A representation (π, V) of G is called *unitary* if there exists a positive definite Hermitian form on V which is G -invariant (see Definition 2.2.5).

For H a closed subgroup of G , we have restriction and induction functors going between $\text{Rep}(G)$ and $\text{Rep}(H)$. The restriction functor restricts a representation (π, V) of G to H , i.e.

$$\text{Res}_H^G(\pi, V) = (\pi|_H, V) \in \text{Rep}(H).$$

For a representation (π, V) of H , the induced representation is given by

$$\text{Ind}_H^G(V) := \{f : G \rightarrow V : f(hg) = h \cdot f(g) \ \forall g \in G, h \in H\}^s \in \text{Rep}(G)$$

on which G acts via right multiplication. The superscript s means taking the smooth vectors with respect to this action (like in Definition 2.2.3), which means only taking the functions that are right-invariant under some compact open subgroup of G . The functor Ind_H^G has a subfunctor ind_H^G called *compact induction*. It is defined by

$$\text{ind}_H^G(V) := \{f \in \text{Ind}_H^G(V) \mid f \text{ has compact support modulo } H\}.$$

For $P = MU_P$ a parabolic subgroup of G , we also have induction and restriction functors, namely the functors of (*normalized*) *parabolic induction* and (*normalized*) *parabolic restriction*. These functors go between $\text{Rep}(G)$ and $\text{Rep}(M)$. Let π be a representation of M . Then we can inflate π to get a representation of P , i.e. $\pi(mu) = \pi(m)$ for all $m \in M, u \in U_P$. Because of the Iwasawa decomposition $G = PK$, we have that G/P is compact, so $\text{ind}_P^G = \text{Ind}_P^G$. The normalized parabolic induction functor is given by

$$I_P^G(\pi) := \text{Ind}_P^G(\delta_P^{1/2} \otimes \pi) \in \text{Rep}(G).$$

Here, δ_P is the modular function on P , where we use the opposite convention for the modular function compared to the one in [Ren10, p. II.3.7]. For each $m \in M$, $u \in U_P$, it is given by [Ren10, §V.5.4] (to the power of -1)

$$\delta_P(mu) = \delta_P(m) = \prod_{\alpha \in \Sigma(P)} |\alpha(m)|_F^{\dim U_\alpha}, \quad (2.2.1)$$

where each U_α is the unipotent subgroup of G corresponding to the root α . Concretely, the parabolically induced representation is given on the vector space

$$I_P^G(V) = \{f : G \rightarrow V \mid f(mug) = \delta_P^{1/2}(m)\pi(m)f(g) \ \forall m \in M, u \in U_P, g \in G\}^s,$$

with G acting via right-multiplication.

Now let (π, V) be a representation of G . The normalized parabolic restriction functor is given by

$$R_P^G(\pi, V) := (\delta_P^{-1/2} \otimes \pi|_M, V_{U_P}) \in \text{Rep}(M)$$

The representation is given on the vector space $V_{U_P} := V/V(U_P)$, where $V(U_P)$ is generated by the elements $\pi(u) \cdot v - v$ for $v \in V$, $u \in U_P$. The action of M on the space V_{U_P} is well-defined since M normalizes U_P .

Definition 2.2.7. A smooth representation (π, V) of G is called *supercuspidal* if for every proper parabolic subgroup P of G , $R_P^G(V) = 0$.

Theorem 2.2.8 ([FP22, Theorem 6.2], [Ren10, §VI.2.1]). *Let (π, V) be an irreducible smooth representation of G . Then it is supercuspidal if and only if all its matrix coefficients have compact support modulo $Z(G)$.*

A more general concept is that of representations which are (*essentially*) *square integrable modulo center*. Let $d\bar{g}$ denote a bi-invariant Haar measure on $G/Z(G)$ and let χ be a unitary character of $Z(G)$. Let $L^2(G, \chi, d\bar{g})$ denote the space of locally constant functions $f : G \rightarrow \mathbb{C}$ such that

- $f(gz) = \chi(z)f(g)$ for all $z \in Z(G)$ and $g \in G$, and
- $\int_{G/Z(G)} |f(g)|^2 d\bar{g} < \infty$.

Definition 2.2.9. Let (π, V) be a smooth representation of G . Then:

- (i) (π, V) is called *square integrable modulo center* if the central character $\text{cc}(\pi)$ of π is unitary, and additionally every matrix coefficient of π lies in $L^2(G, \text{cc}(\pi), d\bar{g})$.
- (ii) (π, V) is called *essentially square integrable* if there exists a character $\chi \in X_{\text{nr}}(G)$ such that $\pi \otimes \chi$ is square integrable modulo center.

Recall that in principle, we are interested in representations of covering groups

$$1 \rightarrow A \rightarrow G \rightarrow G' \rightarrow 1, \quad (2.2.2)$$

with A a finite abelian group, $A \hookrightarrow Z(G)$ and G' a reductive p -adic group. Since A is finite and $A \subset Z(G)$, the representations of G decompose according to the characters $\chi \in \text{Hom}(A, \mathbb{C}^\times)$. All finite subgroups of $\mathbb{C}^\times$ lie on the unit circle S^1 , hence $\text{Hom}(A, \mathbb{C}^\times) = \text{Hom}(A, S^1)$. Let $\chi : A \rightarrow \mu_n$ be a character of A , with μ_n the group of n -th roots of unity in $\mathbb{C}^\times$. Moreover, choose n such that χ surjects onto μ_n . Then by pushing forward the extension (2.2.2) along χ , we obtain another covering group $\tilde{G}$ of G' , namely

$$\begin{array}{ccccccc} 1 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & G' \longrightarrow 1 \\ & & \downarrow \chi & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mu_n & \longrightarrow & \tilde{G} & \longrightarrow & G' \longrightarrow 1. \end{array}$$

Studying the representations of G whose restriction to A equals χ is equivalent to studying the representations of $\tilde{G}$ whose restriction to μ_n is the identity character $\text{id}_{\mu_n} : \mu_n \xrightarrow{\sim} \mu_n$. Representations of the latter kind are also called *genuine* representations.

2.2.2 Bernstein decomposition and types

In this section, we discuss the Bernstein decomposition of the category of smooth complex representations of a p -adic covering group G . We also discuss how Bernstein blocks can be described using type theory. The main references for the Bernstein decomposition are [Ren10], [Roc09], [FP22].

Definition 2.2.10. A *cuspidal datum* $(M, (\sigma, V))$ for G consists of a Levi subgroup M of G together with an irreducible supercuspidal representation (σ, V) of M .

An equivalence relation on the class of all cuspidal data is given as follows. Two cuspidal data $(M_1, (\sigma_1, V_1))$ and $(M_2, (\sigma_2, V_2))$ are *inertially equivalent* if

$$gM_1g^{-1} = M_2 \text{ and } g \cdot \sigma_1 \cong \sigma_2 \otimes \chi$$

for some $g \in G$ and $\chi \in X_{\text{nr}}(M_2)$, where $g \cdot \sigma_1$ is the representation of gM_1g^{-1} given by $x \mapsto \sigma_1(g^{-1}xg)$. For a cuspidal datum $(M, (\sigma, V))$ we denote its inertial equivalence class for G by $[M, (\sigma, V)]_G$. The category $\text{Rep}(G)$ has full subcategories

$$\begin{aligned} \text{Rep}_{[M, (\sigma, V)]_G}(G) &= \{\pi \in \text{Rep}(G) : \text{all irreducible subquotients of } \pi \\ &\quad \text{belong to the inertial equivalence class } [M, (\sigma, V)]_G\}. \end{aligned}$$

Theorem 2.2.11 (Bernstein decomposition [FP22, Theorem 7.14]). *The category $\text{Rep}(G)$ of smooth complex representations of G decomposes as*

$$\text{Rep}(G) = \prod_{\mathfrak{s}} \text{Rep}_{\mathfrak{s}}(G),$$

where the product is indexed by all cuspidal data $\mathfrak{s} = [M, (\sigma, V)]_G$ up to inertial equivalence.

Each Bernstein block $\text{Rep}_{\mathfrak{s}}(G)$ is equivalent to a module category.

Proposition 2.2.12 ([FP22, §9]). *Let $(M, (\sigma, V))$ be a cuspidal datum. Write $\mathfrak{s}_M$ for its inertial equivalence class in M and $\mathfrak{s}_G$ for its inertial equivalence class in G . Let P be a parabolic subgroup of G with M as a Levi subgroup. Then the following two statements hold.*

- (i) *The representation $\text{ind}_{M^1}^M(\sigma|_{M^1})$ of M is a progenerator for the category $\text{Rep}_{\mathfrak{s}_M}(M)$. In other words, there is an equivalence of categories*

$$\text{Rep}_{\mathfrak{s}_M}(M) \simeq \text{End}_M(\text{ind}_{M^1}^M(\sigma|_{M^1})) - \text{Mod}.$$

- (ii) *The representation $I_P^G(\text{ind}_{M^1}^M(\sigma|_{M^1}))$ of G is a progenerator for the category $\text{Rep}_{\mathfrak{s}_G}(G)$. In other words, there is an equivalence of categories*

$$\text{Rep}_{\mathfrak{s}_G}(G) \simeq \text{End}_G(I_P^G(\text{ind}_{M^1}^M(\sigma|_{M^1}))) - \text{Mod}.$$

For σ a cuspidal representation of M with inertial equivalence class $\mathfrak{s}_M$, let

$$X_{\text{nr}}(M, \sigma) := \{\chi \in X_{\text{nr}}(M) : \sigma \otimes \chi \cong \sigma\}.$$

Then the quotient $\mathcal{O} := X_{\text{nr}}(M)/X_{\text{nr}}(M, \sigma)$ has the structure of a complex algebraic variety and it is in bijection with the inertial equivalence class $\mathfrak{s}_M$. Let $\mathbb{C}[\mathcal{O}]$ denote the ring of regular functions on $\mathcal{O}$. Let $W(M, \mathcal{O}) := N_G(\mathfrak{s}_M)/M$, where $N_G(\mathfrak{s}_M)$ is the subgroup of G consisting of the $g \in G$ such that $[gMg^{-1}, g \cdot \sigma]_M = \mathfrak{s}_M$. The following result about the structure of $\text{Rep}_{\mathfrak{s}_G}(G)$ is due to Bernstein and also works for covering groups.

Proposition 2.2.13 ([FP22, Theorem 9.12]). *Let $\mathfrak{s}_G = [M, \sigma]_G$ be an inertial equivalence class for G . Then the following statements hold:*

- (i) *The algebra $\text{End}_G(I_P^G(\text{ind}_{M^1}^M(\sigma|_{M^1})))$ contains $\mathbb{C}[\mathcal{O}]$ as a subalgebra and is finitely generated over it as a module.*
- (ii) *The center of $\text{Rep}_{\mathfrak{s}_G}(G)$ equals*

$$Z(\text{Rep}_{\mathfrak{s}_G}(G)) = Z(\text{End}_G(I_P^G(\text{ind}_{M^1}^M(\sigma|_{M^1})))) = \mathbb{C}[\mathcal{O}]^{W(M, \mathcal{O})},$$

i.e. it is the ring of $W(M, \mathcal{O})$ -invariant regular functions on the complex algebraic variety $\mathcal{O}$.

One way to further describe the structure of Bernstein blocks, is to further investigate the structure of the endomorphism algebras of Bernstein's progenerators. This was done for reductive groups in [Sol22] and we will elaborate on this in Chapter 5. Another way to study Bernstein blocks is with the use of type theory, which we will discuss now. It was developed for reductive groups in [BK98] but it also works for covering groups, see for example [Wan24, §3], [Zou25, §3].

Let K be a compact open subgroup of G and let (ρ, W) be a smooth irreducible representation of K . To (K, ρ) we associate the following objects:

- $\mathcal{H}(G, \rho)$ denotes the convolution algebra of compactly supported functions $f : G \rightarrow \text{End}_{\mathbb{C}}(W)$ such that

$$f(k_1 g k_2) = \rho(k_1) f(g) \rho(k_2) \text{ for all } k_1, k_2 \in K, g \in G.$$

- $\text{Rep}_{\rho}(G)$ is the full subcategory of $\text{Rep}(G)$ consisting of all representations (π, V) that are generated by the ρ -isotypic subspace V^{ρ} of V .
- $\mathcal{M}_{\rho} : \text{Rep}_{\rho}(G) \rightarrow \mathcal{H}(G, \rho) - \text{Mod}$ is the functor $\pi \mapsto \text{Hom}_K(\rho, \pi)$.

Remark 2.2.14. The algebra denoted by $\mathcal{H}(G, \rho)$ in [BK98] is the algebra denoted by $\mathcal{H}(G, \rho^{\vee})$ using our convention above. We use this opposite convention for $\mathcal{H}(G, \rho)$, because ultimately we want to compare the endomorphism algebras arising from types with the endomorphism algebras arising from Bernstein's progenerators. We did not mention it explicitly before, but the modules in Proposition 2.2.12 are in fact right modules. So for the comparison, it is natural to work with right $\mathcal{H}(G, \rho)$ -modules instead of left modules. For $\pi \in \text{Rep}_{\rho}(G)$, Bushnell and Kutzko work with $\text{Hom}_K(\rho, \pi)$ as a left $\mathcal{H}(G, \rho^{\vee})$ -module (our notation). Since $\mathcal{H}(G, \rho^{\vee})$ and $\mathcal{H}(G, \rho)$ are anti-isomorphic, we may consider $\text{Hom}_K(\rho, \pi)$ as a right $\mathcal{H}(G, \rho)$ -module.

Definition 2.2.15. Let $\mathfrak{s} = [L, \sigma]_G$ be an inertial equivalence class for G , let K be a compact open subgroup of G and let ρ be an irreducible K -representation. Then (K, ρ) is called an $\mathfrak{s}$ -type if the following equivalent conditions hold:

- $\text{Rep}_{\rho}(G) = \text{Rep}_{\mathfrak{s}}(G)$,
- the functor $\mathcal{M}_{\rho} : \text{Rep}_{\rho}(G) \rightarrow \mathcal{H}(G, \rho) - \text{Mod}$ is an equivalence of categories.

There are more equivalent conditions to the ones above, but we will not mention them here. If (K, ρ) is an $\mathfrak{s}$ -type, then by [BK98, (2.6)]

$$\mathcal{H}(G, \rho) \cong \text{End}_G(\text{ind}_K^G(\rho)), \quad (2.2.3)$$

with $\text{End}_G(\text{ind}_K^G(\rho))$ acting on the left by Remark 2.2.14. So $\text{ind}_K^G(\rho)$ is a progenerator for $\text{Rep}_{\mathfrak{s}}(G) \simeq \mathcal{H}(G, \rho) - \text{Mod}$ and the algebras $\mathcal{H}(G, \rho)$ and $\text{End}_G(\text{ind}_K^G(\rho))$ are not only Morita equivalent, but they are in fact isomorphic.

Now let L be a proper Levi subgroup of G and suppose there exists a type (K_L, ρ_L) for $\mathfrak{s}_L = [L, \sigma]_L$. Let $\mathcal{P}(L)$ be the set of parabolic subgroups of G with Levi subgroup L . Suppose that the type (K, ρ) for $\mathfrak{s} = [L, \sigma]_G$ is a *cover* of (K_L, ρ_L) (see [BK98, Definition 8.1] for the definition). Then for all $P \in \mathcal{P}(L)$ there exists a ring homomorphism

$$t_P : \mathcal{H}(L, \rho_L) \rightarrow \mathcal{H}(G, \rho),$$

making the diagrams

$$\begin{array}{ccc} \text{Rep}_{\mathfrak{s}}(G) & \xrightarrow{\mathcal{M}_{\rho}} & \mathcal{H}(G, \rho) - \text{Mod} \\ \downarrow R_P^G & & \downarrow t_P^* \\ \text{Rep}_{\mathfrak{s}_L}(L) & \xrightarrow{\mathcal{M}_{\rho_L}} & \mathcal{H}(L, \rho_L) - \text{Mod} \end{array} \quad \begin{array}{ccc} \text{Rep}_{\mathfrak{s}}(G) & \xrightarrow{\mathcal{M}_{\rho}} & \mathcal{H}(G, \rho) - \text{Mod} \\ I_P^G \uparrow & & (t_P)_* \uparrow \\ \text{Rep}_{\mathfrak{s}_L}(L) & \xrightarrow{\mathcal{M}_{\rho_L}} & \mathcal{H}(L, \rho_L) - \text{Mod}, \end{array}$$

commute. Here, t_P^* denotes restriction and $(t_P)_*$ is its right adjoint functor

$$V \mapsto \mathrm{Hom}_{\mathcal{H}(L, \rho_L)}(\mathcal{H}(G, \rho), V).$$

The diagram on the right follows from the diagram on the left, since I_P^G is the right adjoint functor of R_P^G . In [BK98], this compatibility for covers of types is shown to hold with respect to the un-normalized parabolic induction and restriction functors (so without the factor $\delta_P^{\pm 1/2}$). By [Dat99, Théorème 1.3.1, Remarque] there is also a compatibility with respect to the normalized functors of parabolic induction and restriction. By Bernstein's second adjointness theorem, I_P^G is the left adjoint of $R_{\overline{P}}^G$, where $\overline{P}$ denotes the parabolic subgroup of G with Levi L , opposite to P . So there is a third commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}_{\mathfrak{s}}(G) & \xrightarrow{\mathcal{M}_\rho} & \mathcal{H}(G, \rho) - \mathrm{Mod} \\ I_P^G \uparrow & & \uparrow *(t_{\overline{P}}) \\ \mathrm{Rep}_{\mathfrak{s}_L}(L) & \xrightarrow{\mathcal{M}_{\rho_L}} & \mathcal{H}(L, \rho_L) - \mathrm{Mod}, \end{array}$$

where $*(t_{\overline{P}})$ is the left adjoint of $t_{\overline{P}}^*$, i.e.

$$V \mapsto \mathcal{H}(G, \rho) \otimes_{\mathcal{H}(L, \rho_L)} V.$$

Finally, if (K, ρ) is a cover of (K_L, ρ_L) and $P \in \mathcal{P}(L)$, then by [Dat99],

$$\mathrm{ind}_K^G(\rho) \cong I_P^G(\mathrm{ind}_{K_L}^L(\rho_L)). \quad (2.2.4)$$

It is not straightforward to construct types and they may not even always exist. In [Roc98], types were constructed for principal series representations of split reductive groups, with some restrictions on the residue characteristic of F . Moreover, an explicit description is given of the Hecke algebra $\mathcal{H}(G, \rho)$ arising from a type. These methods were generalized to principal series representations of Brylinski-Deligne covers of split groups in [Wan24]. In Chapter 5, we will describe the structure of principal series blocks for BD covers of split groups without restrictions on the residue characteristic, essentially by combining the type theory and the results on endomorphism algebras from [Sol22]. The structure of the algebras in question is closely related to the structure of affine Hecke algebras. We refer to [Sol21] for a survey on this subject and will not elaborate on it here.

2.2.3 Intertwining operators and the Harish-Chandra μ -function

Chapter 4 of this thesis is about intertwining operators and a formula for the Harish-Chandra μ -function for covering groups. In this section, we give an introduction to these subjects, that assumes less prerequisites than the introduction given in Chapter 4.

An *intertwining operator* is a G -linear map between two representations (π_1, V_1) and (π_2, V_2) of G . We will consider specific intertwining operators,

namely the operators $J_{Q|P}$ associated to two parabolic subgroups P, Q of G with the same Levi subgroup M . For a representation (π, V) of M , the operator $J_{Q|P}(\pi)$ is an intertwining operator between the parabolically induced representations $(I_P^G(\pi), I_P^G(V))$ and $(I_Q^G(\pi), I_Q^G(V))$. These intertwining operators were extensively studied in [Wal03] and generalized to covering groups in [Li12]. The Harish-Chandra μ -function is defined using the intertwining operators $J_{\overline{P}|P}$ and $J_{P|\overline{P}}$, where $\overline{P}$ is the parabolic subgroup of G opposite to P .

Let $P = MU_P$, $Q = MU_Q$ be parabolic subgroups of G with the same Levi subgroup M . Let I_P^G and I_Q^G be the functors of normalized parabolic induction. Let (π, V) be a smooth admissible M -representation, then

$$\begin{aligned} I_P^G(V) &= \{f : G \rightarrow V \mid f(mug) = \delta_P^{1/2}(m)\pi(m)f(g) \ \forall m \in M, u \in U_P, g \in G\}^s, \\ I_Q^G(V) &= \{f : G \rightarrow V \mid f(mug) = \delta_Q^{1/2}(m)\pi(m)f(g) \ \forall m \in M, u \in U_Q, g \in G\}^s. \end{aligned}$$

Recall that the superscript ^{s} in $I_P^G(V), I_Q^G(V)$ means only taking smooth vectors with respect to the action of G . The action of G on these vector spaces is given by multiplication on the right, defining the representations $I_P^G(\pi)$ and $I_Q^G(\pi)$, respectively.

Definition 2.2.16 ([Wal03, §IV.1]). In the context above, the intertwining operator

$$J_{Q|P}(\pi) : I_P^G(V) \rightarrow I_Q^G(V)$$

is defined by

$$f \mapsto \left[g \mapsto \int_{(U_P \cap U_Q) \backslash U_Q} f(ug) du \right]$$

(provided that the integral converges).

This operator does not always converge, but assuming that π is such that $J_{Q|P}(\pi)$ does converge, let us confirm that this defines an intertwining operator between the G -representations $(I_P^G(\pi), I_P^G(V))$ and $(I_Q^G(\pi), I_Q^G(V))$. The step that is not very obvious, is to show that for $f \in I_P^G(V)$, indeed $J_{Q|P}(\pi)f$ lies in $I_Q^G(V)$. For any $u \in U_Q$ and $g \in G$ we have that

$$\begin{aligned} J_{Q|P}(\pi)f(ug) &= \int_{(U_P \cap U_Q) \backslash U_Q} f(u_1ug) du_1 \\ &= \int_{(U_P \cap U_Q) \backslash U_Q} f(u_1g) du_1 \\ &= J_{Q|P}(\pi)f(g), \end{aligned}$$

since unipotent groups are unimodular. Moreover, for $m \in M$ and $g \in G$, we

have

$$\begin{aligned}
J_{Q|P}(\pi)f(mg) &= \int_{(U_P \cap U_Q) \setminus U_Q} f(u_1 mg) du_1 \\
&= \int_{(U_P \cap U_Q) \setminus U_Q} f(mm^{-1}u_1 mg) du_1 \\
&= \delta_P^{1/2}(m)\pi(m) \int_{(U_P \cap U_Q) \setminus U_Q} f(m^{-1}u_1 mg) du_1 \\
&= \delta_P^{1/2}(m)\pi(m)\delta_{Q \cap \bar{P}}(m) \int_{U_Q \cap U_{\bar{P}}} f(u_1 g) du_1 \\
&= \delta_Q^{1/2}(m)\pi(m)J_{Q|P}(\pi)f(g),
\end{aligned}$$

where the second to last equality follows from [Car79, p. 145] and the fact that $(U_P \cap U_Q) \setminus U_Q \cong U_Q \cap U_{\bar{P}}$. Their Haar measures differ by a scalar and we may normalize the Haar measures to have this scalar be equal to 1. The last equality then holds because of Equation (2.2.1), which implies

$$\begin{aligned}
\delta_P(m)^{1/2}\delta_{Q \cap \bar{P}}(m) &= \prod_{\alpha \in \Sigma(P)} |\alpha(m)|_F^{\dim U_\alpha/2} \prod_{\alpha \in \Sigma(\bar{P}) \cap \Sigma(Q)} |\alpha(m)|_F^{\dim U_\alpha} \\
&= \prod_{\alpha \in \Sigma(P)} |\alpha(m)|_F^{\dim U_\alpha/2} \prod_{\alpha \in \Sigma(P) \cap \Sigma(\bar{Q})} |\alpha(m)|_F^{-\dim U_\alpha} \\
&= \prod_{\alpha \in \Sigma(P) \cap \Sigma(\bar{Q})} |\alpha(m)|_F^{-\dim U_\alpha/2} \prod_{\alpha \in \Sigma(P) \cap \Sigma(Q)} |\alpha(m)|_F^{\dim U_\alpha/2} \\
&= \prod_{\alpha \in \Sigma(\bar{P}) \cap \Sigma(Q)} |\alpha(m)|_F^{\dim U_\alpha/2} \prod_{\alpha \in \Sigma(P) \cap \Sigma(Q)} |\alpha(m)|_F^{\dim U_\alpha/2} \\
&= \prod_{\alpha \in \Sigma(Q)} |\alpha(m)|_F^{\dim U_\alpha/2} \\
&= \delta_Q(m)^{1/2}.
\end{aligned}$$

This confirms that whenever $J_{P|Q}(\pi)$ converges, it defines an intertwining operator between $(I_P^G(\pi), I_P^G(V))$ and $(I_Q^G(\pi), I_Q^G(V))$.

Let us now discuss the convergence of $J_{P|Q}(\pi)$. For (π, V) an admissible representation of M , we let

$$\mathcal{O} := \pi \otimes X_{\text{nr}}(M) = \{\pi \otimes \chi \mid \chi \in X_{\text{nr}}(M)\},$$

where the $\pi \otimes \chi$ are considered up to isomorphism. Let

$$X_{\text{nr}}(M, \pi) := \{\chi \in X_{\text{nr}}(M) : \pi \otimes \chi \cong \pi\}.$$

Then we can view $\mathcal{O}$ as a complex algebraic variety via the bijection

$$X_{\text{nr}}(M)/X_{\text{nr}}(M, \pi) \rightarrow \mathcal{O}.$$

The following is as discussed in [Wal03, §I.1], generalized to covering groups. Recall that M is the inverse image of a Levi subgroup M' in G' . Let $A_{M'}$ be

a maximal split torus in $Z(M')$ and let $\Sigma(M)$ be the weights of the adjoint representation of $A_{M'}$ on the Lie algebra of G' . A parabolic subgroup P of G with Levi M determines a system of positive roots $\Sigma(P)$ in $\Sigma(M)$. Recall that $X_{\text{nr}}(M) = \text{Hom}(M/M^1, \mathbb{C}^\times)$. There is a surjection

$$X^*(A_{M'}) \otimes_{\mathbb{Z}} \mathbb{C} \twoheadrightarrow X_{\text{nr}}(M),$$

$$\chi \otimes z \mapsto (m \mapsto |\chi(m)|_F^z),$$

where $X^*(A_{M'})$ denotes the group of F -characters of $A_{M'}$. For $\chi \in X_{\text{nr}}(M)$, let $\tilde{\chi} \in X^*(A_{M'}) \otimes_{\mathbb{Z}} \mathbb{C}$ be so that it maps to χ under the surjection above. Then the real part $\text{Re}(\tilde{\chi}) \in X^*(A_{M'}) \otimes_{\mathbb{Z}} \mathbb{R}$ does not depend on the choice of $\tilde{\chi}$, and we define $\text{Re}(\chi) := \text{Re}(\tilde{\chi})$. We have a pairing

$$\langle \cdot, \cdot \rangle : (X^*(A_{M'}) \otimes_{\mathbb{Z}} \mathbb{R}) \times (X_*(A_{M'}) \otimes_{\mathbb{Z}} \mathbb{R}) \rightarrow \mathbb{R},$$

so for any root $\alpha \in \Sigma(M)$, we can define the element $\alpha^\vee \in X_*(A_{M'} \cap M'_{\text{der}}) \otimes_{\mathbb{Z}} \mathbb{R}$ to be determined by

$$\langle \alpha|_{A_{M'} \cap M'_{\text{der}}}, \alpha^\vee \rangle = 2.$$

Then for $\chi \in X_{\text{nr}}(M)$, the condition for $J_{Q|P}(\pi \otimes \chi)$ to converge is that π has finite length and that $\langle \text{Re}(\chi), \alpha^\vee \rangle$ is large enough for all $\alpha \in \Sigma(P) \cap \Sigma(\overline{Q})$.

Alternatively, let $\Sigma_{\text{red}}(M)$ denote the reduced (i.e. indivisible) roots in $\Sigma(M)$. Then for $\alpha \in \Sigma_{\text{red}}(M)$, we can define the variable $r_\alpha(\chi) \in \mathbb{R}$ determined by

$$|\chi|_{M \cap M_{\alpha, \text{der}}} = \delta_{P_\alpha}^{r_\alpha(\chi)}|_{M \cap M_{\alpha, \text{der}}}.$$

Here, $M_\alpha = \langle M, U_\alpha, U_{-\alpha} \rangle$ is the unique Levi subgroup of G containing the root subgroup U_α and containing M as a maximal Levi subgroup. Moreover, P_α is the parabolic subgroup $P_\alpha = P \cap M_\alpha = MU_\alpha$ of M_α . The condition for $J_{Q|P}(\pi \otimes \chi)$ to converge is that $\langle \text{Re}(\chi), \alpha^\vee \rangle$ is large enough for all $\alpha \in \Sigma(P) \cap \Sigma(\overline{Q})$, which is equivalent to requiring that $r_\alpha(\chi)$ is large enough for all reduced roots $\alpha \in \Sigma_{\text{red}}(P) \cap \Sigma_{\text{red}}(\overline{Q})$.

Theorem 2.2.17 ([Wal03, Théorème IV.1.1]). *Suppose that π is a finite length representation of M . Then there exists $R \in \mathbb{R}$ such that $J_{Q|P}(\pi \otimes \chi)$ converges for every χ with $r_\alpha(\chi) > R$ for all $\alpha \in \Sigma_{\text{red}}(P) \cap \Sigma_{\text{red}}(\overline{Q})$. Moreover, $J_{Q|P}(\pi \otimes \chi)$ defines a rational operator on the open cone*

$$\mathcal{U}_R := \{\pi \otimes \chi \in \mathcal{O} \mid r_\alpha(\chi) > R \ \forall \alpha \in \Sigma_{\text{red}}(P) \cap \Sigma_{\text{red}}(\overline{Q})\} \subset \mathcal{O}.$$

Let us explain what it actually means for $J_{Q|P}(\pi \otimes \chi)$ to be a rational operator. Let K be a maximal compact subgroup of G . For every $\chi \in X_{\text{nr}}(M)$, we have

$$I_P^G(\pi \otimes \chi, V) \cong I_{P \cap K}^K((\pi \otimes \chi)|_K, V) = I_{P \cap K}^K(\pi|_K, V).$$

So every representation $I_P^G(\pi \otimes \chi)$ can be realized on the same vector space $I_{P \cap K}^K(V)$. The operator $J_{Q|P}(\pi \otimes \chi)$ fits in the commutative diagram

$$\begin{array}{ccc} I_P^G(\pi \otimes \chi, V) & \xrightarrow{J_{Q|P}(\pi \otimes \chi)} & I_Q^G(\pi \otimes \chi, V) \\ \downarrow \cong & & \downarrow \cong \\ I_{P \cap K}^K(\pi|_K, V) & \xrightarrow{J_{Q|P}^K(\pi \otimes \chi)} & I_{Q \cap K}^K(\pi|_K, V), \end{array}$$

where the operator $J_{Q|P}^K(\pi \otimes \chi)$ is defined as the operator that makes the diagram commute (warning: it does not equal the operator $J_{Q \cap K|P \cap K}(\pi|_K)$). In the top row of the diagram, both the operator $J_{Q|P}(\pi \otimes \chi)$ and the vector spaces $I_P^G(V)$, $I_Q^G(V)$ depend on the character $\chi \in X_{\text{nr}}(M)$. In the bottom row, the operator $J_{Q|P}^K(\pi \otimes \chi)$ depends on χ , but the vector spaces $I_{P \cap K}^K(V)$ and $I_{Q \cap K}^K(V)$ do not depend on χ . The statement “ $J_{Q|P}(\pi \otimes \chi)$ is rational in χ ” now means that the operator $J_{Q|P}^K(\pi \otimes \chi)$ is an operator between the vector spaces $I_{P \cap K}^K(V)$ and $I_{Q \cap K}^K(V)$ that depends rationally on χ . More precisely, let $B := \mathbb{C}[X_{\text{nr}}(M)]$ be the ring of regular functions on the complex algebraic torus $X_{\text{nr}}(M)$. Then there exists $b \in B$ such that for every χ with $\pi \otimes \chi \in \mathcal{U}_R$:

- $b(\chi) \neq 0$ and

- for every $f \in I_{P \cap K}^K(V)$, there exist $f_1, \dots, f_n \in I_{Q \cap K}^K(V)$ and $b_1, \dots, b_n \in B$ such that

$$b(\chi)J_{Q|P}^K(\pi \otimes \chi)f = \sum_{i=1}^n b_i(\chi)f_i. \quad (2.2.5)$$

So $J_{Q|P}(\pi \otimes \chi)$ defines a rational operator on the open cone $\mathcal{U}_R \subset \mathcal{O}$ and, with the description in (2.2.5), it has a meromorphic continuation on the whole space $\mathcal{O}$.

Now, let $Q = \overline{P}$ and let π be irreducible. Then for generic $\pi' \in \mathcal{O}$, the representation $I_P^G(\pi', V)$ is irreducible [Sau97, Théorème 3.2], see also Theorem 4.3.2. For those π' , by Schur’s lemma, the composition

$$j_P(\pi') := J_{P|\overline{P}}(\pi') \circ J_{\overline{P}|P}(\pi') : I_P^G(\pi', V) \rightarrow I_P^G(\pi', V) \quad (2.2.6)$$

equals multiplication by some complex scalar. So j_P defines a rational function on an open $\mathcal{U} \subset \mathcal{O}$, which means that there exist $b'_1, b'_2 \in B$ such that for all χ with $\pi \otimes \chi \in \mathcal{U}$, we have $b'_1(\chi) \neq 0$ and $b'_1(\chi)j_P(\pi \otimes \chi) = b'_2(\chi)$. In fact, the function j_P only depends on the cuspidal datum $(M, (\pi, V))$ and not on the choice of parabolic subgroup P [Wal03, §IV.3], so from now on we write $j := j_P$.

Recall that $A_{M'}$ is a maximal split torus in $Z(M')$ and that $\Sigma_{\text{red}}(M) \subset X^*(A_{M'})$ is the set of reduced roots coming from the adjoint representation of $A_{M'}$ on the Lie algebra of G' . Then we have [Wal03, p. 287]

$$j = \prod_{\alpha} j_{\alpha}, \text{ where } \alpha \text{ runs over all positive roots in } \Sigma_{\text{red}}(M). \quad (2.2.7)$$

Each j_{α} is computed in the Levi subgroup $M_{\alpha} := \langle M, U_{\alpha}, U_{-\alpha} \rangle$ of G that contains M as a maximal Levi subgroup. The μ -function is defined as follows.

Definition 2.2.18 ([Wal03, §V.2]). The Harish-Chandra μ -function is

$$\mu(\pi \otimes \chi) := \gamma \cdot j(\pi \otimes \chi)^{-1},$$

where γ is the positive real scalar $\gamma := \prod_{\alpha} \gamma(M_{\alpha}|M)^2$, where α runs over the positive roots in $\Sigma_{\text{red}}(M)$ and $\gamma(M_{\alpha}|M)$ is as defined in [Wal03, §I.1].

Since we are only interested in the poles and zeros of μ , we will not elaborate on the factor γ here. By the definition of μ and by (2.2.7), the μ -function also satisfies

$$\mu = \prod_{\alpha} \mu_{\alpha}, \text{ where } \alpha \text{ runs over all positive roots in } \Sigma_{\text{red}}(M).$$

Each μ_{α} is computed in M_{α} . In Chapter 4, we give an explicit formula for μ_{α} as a rational function on $\mathbb{P}^1(\mathbb{C})$. The zeros and poles of the μ -function are used in [Sol22] to describe the structure of endomorphism algebras of Bernstein blocks in $\text{Rep}(G)$, for reductive groups G . Namely, let (M, σ) be a cuspidal datum for G and denote by $\mathfrak{s}$ its inertial equivalence class. Recall from the Bernstein decomposition that

$$\text{Rep}_{\mathfrak{s}}(G) \simeq \text{End}_G(I_P^G(\text{ind}_{M^1}^M(\sigma|_{M^1}))) - \text{Mod}.$$

The roots α such that μ_{α} has a zero on $\mathcal{O} = \sigma \otimes X_{\text{nr}}(M)$ determine a root system. An affine Hecke algebra is then associated to this root system, whose q -parameters are determined by the poles of μ_{α} . This Hecke algebra is shown to be closely related to the structure of the endomorphism algebra $\text{End}_G(I_P^G(\text{ind}_{M^1}^M(\sigma|_{M^1})))$. The strategy generalizes to covering groups using the results in Chapter 4. We will discuss this more elaborately in Chapter 5, where we will use these methods to describe endomorphism algebras for principal series blocks for Brylinski-Deligne covering groups.

Chapter 3

Examples of intertwining operators for (a BD cover of) $\mathrm{SL}_2(\mathbb{Q}_p)$

In this chapter, we calculate $j(\chi) = J_{P|\bar{P}}(\chi) \circ J_{\bar{P}|P}(\chi)$ from (2.2.6) explicitly in some particular cases. First, we consider the case where $G = \mathrm{SL}_2(\mathbb{Q}_p)$ and χ is an unramified character of the maximal torus T of G . Then we consider again $G = \mathrm{SL}_2(\mathbb{Q}_p)$, but with χ a ramified character of T . Finally, we consider the case where G is a metaplectic cover of $\mathrm{SL}_2(\mathbb{Q}_p)$ and χ is a ramified character of T . These examples are meant to illustrate that even in the simplest cases, it turns out to be quite complicated to evaluate the integrals $J_{P|\bar{P}}(\chi)$ and $J_{\bar{P}|P}(\chi)$ explicitly, especially for covering groups.

3.1 Unramified principal series

Example 3.1.1. In this example, $F = \mathbb{Q}_p$. We have that $\varpi_F = p$ and we write $v := v_F$. Consider $G = \mathrm{SL}_2(F)$, with maximal torus

$$T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in F^\times \right\} \cong F^\times,$$

unipotent subgroups

$$U = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \in F \right\} \cong F^+ \text{ and } \bar{U} = \left\{ \begin{pmatrix} 1 & 0 \\ \bar{u} & 1 \end{pmatrix} \mid \bar{u} \in F \right\} \cong F^+,$$

and Borel subgroups $B = TU$, $\bar{B} = T\bar{U}$ consisting of the upper, resp. lower triangular matrices in $\mathrm{SL}_2(F)$. Let χ be an unramified character of T , so of the form $t \mapsto z^{v(t)}$ for some $z \in \mathbb{C}^\times$. We want to know the zeros/poles of

$$J_{B/\bar{B}} J_{\bar{B}/B} : I_B^G(\chi) \rightarrow I_{\bar{B}}^G(\chi)$$
$$f \mapsto \left(g \mapsto \int_U \int_{\bar{U}} f(\bar{u}ug) d\bar{u} du \right).$$

The composition $J_{B/\bar{B}}J_{\bar{B}/B}$ is only described by the double integral above in the range where it converges. We calculate the integral above for a specific f and $g = 1$. We consider $f \in I_B^G(\chi)$ given by

$$f(utk) = \chi(t)|t| \text{ for } u \in U, t \in T \text{ and } k \in \begin{pmatrix} 1 & 0 \\ \mathfrak{p}_F & 1 \end{pmatrix}$$

and f is zero everywhere else. We have

$$\begin{pmatrix} 1 & 0 \\ \bar{u} & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & tu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t\bar{u} & 1 \end{pmatrix},$$

where $t = 1/(1 + \bar{u}u)$. Then the double integral becomes

$$\int_U \int_{\bar{U}} f(\bar{u}u) d\bar{u}du = \int_U \int_{\bar{U}} \chi \left(\frac{1}{1 + \bar{u}u} \right) \left| \frac{1}{1 + \bar{u}u} \right| 1_{\mathfrak{p}_F} \left(\frac{\bar{u}}{1 + \bar{u}u} \right) d\bar{u}du.$$

First we consider the inner integral over $\bar{U}$. After a substitution $t = 1/(1 + \bar{u}u)$, it suffices to calculate the integral

$$\int_F \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt = \sum_{n=-\infty}^{\infty} \int_{v(t)=n} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt.$$

Let us consider the integrals over $v(t) = n$ for different n .

For $v(t) = n > 0$, we have $v(1-t) = 0$, so we only get something nonzero for $v(u) < 0$. Therefore

$$\begin{aligned} \int_{v(t)=n} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt &= p^{v(u)+n} z^n (p^{-n} - p^{-(n+1)}) 1_{F \setminus \mathcal{O}_F}(u) \\ &= p^{v(u)} z^n (1 - p^{-1}) 1_{F \setminus \mathcal{O}_F}(u), \end{aligned}$$

and the sum over all $n > 0$ becomes

$$\begin{aligned} \int_{v(t)>0} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt &= p^{v(u)} (1 - p^{-1}) 1_{F \setminus \mathcal{O}_F}(u) \sum_{n>0} z^n \\ &= p^{v(u)} (1 - p^{-1}) \frac{z}{1-z} 1_{F \setminus \mathcal{O}_F}(u). \end{aligned}$$

For $v(t) = n < 0$ we have $v(1-t) = v(t) = n$, so we only get something nonzero for $v(u) < n$. In this case,

$$\begin{aligned} \int_{v(t)<0} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt &= p^{v(u)} (1 - p^{-1}) \sum_{v(u)<n<0} z^n \\ &= p^{v(u)} (1 - p^{-1}) \frac{1 - z^{v(u)+1}}{z - 1} 1_{F \setminus \mathcal{O}_F}(u). \end{aligned}$$

For $v(t) = 0$, we further break up the integral, so that

$$\begin{aligned} \int_{v(t)=0} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt \\ = \sum_{n=0}^{\infty} \int_{v(t)=0, v(1-t)=n} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt. \end{aligned}$$

The set where $v(t) = 0, v(1-t) = 0$ has measure $(p-2)/p$, and the set where $v(t) = 0, v(1-t) = n > 0$ has measure $p^{-n} - p^{-n-1}$, so the integral above becomes

$$\begin{aligned} p^{v(u)} \frac{p-2}{p} 1_{F \setminus \mathcal{O}_F}(u) + p^{v(u)} \sum_{n>0, n>v(u)} (p^{-n} - p^{-n-1}) \\ = p^{v(u)} \frac{p-2}{p} 1_{F \setminus \mathcal{O}_F}(u) + p^{v(u)-N_u}, \end{aligned}$$

where $N_u = \max(1, v(u) + 1)$. Adding all terms together, we get

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \int_{v(t)=n} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt \\ = p^{v(u)} (1 - p^{-1}) 1_{F \setminus \mathcal{O}_F}(u) \left[\frac{z}{1-z} + \frac{z^{v(u)+1} - 1}{1-z} \right] \\ + p^{v(u)} \frac{p-2}{p} 1_{F \setminus \mathcal{O}_F}(u) + p^{v(u)-N_u}. \end{aligned}$$

Rearranging the terms, the expression becomes

$$p^{v(u)} (1 - p^{-1}) \frac{z^{v(u)+1}}{1-z} 1_{F \setminus \mathcal{O}_F}(u) - p^{v(u)-1} 1_{F \setminus \mathcal{O}_F}(u) + p^{v(u)-N_u}.$$

Now it remains to calculate the integral of the function above over all $u \in F$. For $v(u) \geq 0$, the only nonzero term is $p^{v(u)-N_u} = p^{-1}$, and the set where $v(u) \geq 0$ has measure 1.

So the integral over all u of the function above becomes

$$\begin{aligned} p^{-1} + \frac{(1-p^{-1})}{1-z} \sum_{m \leq -1} p^m (p^{-m} - p^{-m-1}) z^{m+1} = p^{-1} + \frac{(1-p^{-1})^2}{1-z} \sum_{m \geq 0} z^{-m} \\ = p^{-1} + \frac{(1-p^{-1})^2}{(1-z)(1-z^{-1})}, \end{aligned}$$

which is in line with the result of the calculation in [Sol25, Section 4.1].

3.2 Ramified principal series

Example 3.2.1. Again we consider $G = \mathrm{SL}_2(\mathbb{Q}_p)$. Now we do a similar calculation for a ramified character χ of T . We can write

$$T = \mathbb{Z} \times (1 + p\mathbb{Z}_p) \times \mathbb{Z}/(p-1)\mathbb{Z}.$$

Then consider the character χ given by

$$(n, a, m \bmod p-1) \mapsto z^n \zeta_{p-1}^m,$$

where ζ_{p-1} is a primitive $(p-1)$ -th root of unity. Similar to the previous example, we calculate the integral

$$\int_U \int_{\bar{U}} \chi \left(\frac{1}{1 + \bar{u}u} \right) \left| \frac{1}{1 + \bar{u}u} \right| 1_{\mathfrak{p}_F} \left(\frac{\bar{u}}{1 + \bar{u}u} \right) d\bar{u}du.$$

First considering the inner integral over $\bar{U}$ and doing a substitution, we again obtain the integral

$$\int_F \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt = \sum_{n=-\infty}^{\infty} \int_{v(t)=n} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt.$$

We calculate the smaller integrals over $v(t) = n$ separately. For $v(t) = n > 0$, we have $v(1-t) = 0$, so

$$\begin{aligned} \int_{v(t)=n} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt \\ = 1_{F \setminus \mathcal{O}_F}(u) p^{v(u)+n} z^n (p^{-n} - p^{-n-1}) \int_{\mathbb{Z}/(p-1)\mathbb{Z}} \zeta_{p-1}^m dm = 0, \end{aligned}$$

since this comes down to the integral of a nontrivial character of a finite group. By a similar argument, all integrals over $v(t) = n < 0$ vanish. So it remains to calculate the integral over $v(t) = 0$, and for this we again break up the integral into smaller parts:

$$\begin{aligned} \int_{v(t)=0} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt \\ = \sum_{n=0}^{\infty} \int_{v(t)=0, v(1-t)=n} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt. \end{aligned}$$

Note that if $v(t) = 0, v(1-t) = n > 0$, then t must be of the form $t = 1 + p^n b$ for some $b \in \mathbb{Z}_p^\times$. So χ is constant on this set with value 1. Moreover, the measure of this set is $p^{-n} - p^{-n-1}$. Hence,

$$\begin{aligned} \int_{v(t)=0, v(1-t)>0} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt &= \sum_{n=N_u}^{\infty} p^{v(u)} (p^{-n} - p^{-n-1}) \\ &= p^{v(u)-N_u}, \end{aligned}$$

where $N_u = \max(1, 1 + v(u))$. If $v(t) = 0, v(1-t) = 0$, then t must be of the form $t = a + pb$ where $b \in \mathbb{Z}_p$ and $1 \neq a \in \mathbb{F}_p^\times$. So we get

$$\begin{aligned} \int_{v(t)=0, v(1-t)=0} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt \\ = 1_{F \setminus \mathcal{O}_F}(u) (1 - p^{-1}) p^{v(u)} \int_{\mathbb{Z}/(p-1)\mathbb{Z} - \{0\}} \zeta_{p-1}^m dm \\ = -1_{F \setminus \mathcal{O}_F}(u) (1 - p^{-1}) p^{v(u)} \frac{1}{p-1} = -1_{F \setminus \mathcal{O}_F}(u) p^{v(u)-1}, \end{aligned}$$

where the equality in the second line follows from the fact that

$$\int_{\mathbb{Z}/(p-1)\mathbb{Z}} \zeta_{p-1}^m dm = 0,$$

hence

$$\int_{\mathbb{Z}/(p-1)\mathbb{Z}-\{0\}} \zeta_{p-1}^m = -\frac{1}{p-1} \zeta_{p-1}^0 = -\frac{1}{p-1}.$$

It remains to calculate

$$\int_U \left(p^{v(u)-N_u} - 1_{F \setminus \mathcal{O}_F}(u) p^{v(u)-1} \right) du.$$

For $v(u) \geq 0$, the only nonzero term is $p^{v(u)-N_u} = p^{-1}$, and the measure of this set is 1. For $v(u) < 0$ we have $N_u = 1$, and so the two terms in the integral cancel. So the integral over the whole of U equals p^{-1} (which is independent of z , as expected).

3.3 Ramified principal series for BD cover

Example 3.3.1. Now we consider the double covering group $\widetilde{\mathrm{SL}_2(F)}$ where $F = \mathbb{Q}_p$ (recall Example 2.1.10). This group has maximal torus $T = \{1, -1\} \times T'$, and unipotent subgroups $U, \bar{U}$. We have Borel subgroups $B = TU$ and $\bar{B} = T\bar{U}$. The multiplication on T is given by

$$(\varepsilon, t(a))(\varepsilon', t(a')) = (\varepsilon\varepsilon'(a, a')_2, t(aa')),$$

where the sign is determined by the quadratic Hilbert symbol

$$(a, a')_2 = \begin{cases} 1 & \text{if } 1 = \alpha^2 a + \beta^2 a' \text{ for some } \alpha, \beta \in F \\ -1 & \text{otherwise.} \end{cases}$$

The Hilbert symbol is symmetric, it is equal to 1 if one of the entries is a square, and it is bimultiplicative. We know that as a set,

$$T = \{1, -1\} \times T' = \{1, -1\} \times \mathbb{Z} \times (1 + p\mathcal{O}_F) \times \mathbb{F}_p^\times.$$

Concretely, the identification is given by

$$\{1, -1\} \times \mathbb{Z} \times (1 + p\mathcal{O}_F) \times \mathbb{F}_p^\times \ni (\varepsilon, n, y, x^m) \mapsto (\varepsilon, 1)(1, p)^n(1, y)(1, x)^m \in T,$$

where x is a generator of $\mathbb{F}_p^\times$. The element on the right hand side corresponds to either $(1, p^n y x^m) \in T$ or $(-1, p^n y x^m) \in T$ and we want to specify which one of the two it is. In other words, we are looking for a formula $\varepsilon(n, m)$ such that

$$(\varepsilon, 1)(1, p)^n(1, y)(1, x)^m = (\varepsilon \cdot \varepsilon(n, m), p^n y x^m).$$

By induction, one can show that for all $t \in T'$, $n \in \mathbb{Z}$,

$$(1, t)^n = ((t, t)_2^{(n-1)n/2}, t^n).$$

Lemmas 2.1.4 and 2.1.6 tell us that

$$(x, x)_2 = 1, (p, x)_2 = -1 \text{ and } (p, p)_2 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4} \\ -1 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

It follows that

$$\begin{aligned} (1, p)^n (1, x)^m &= ((p, p)_2^{(n-1)n/2}, p^n) (1, x^m) \\ &= ((p, p)_2^{(n-1)n/2} (p^n, x^m)_2, p^n x^m) \\ &= ((p, p)_2^{(n-1)n/2} (p, x)_2^{nm}, p^n x^m) \\ &= ((p, p)_2^{(n-1)n/2} (-1)^{nm}, p^n x^m) \end{aligned}$$

Since $y \in 1 + p\mathcal{O}_F$ is always a square (or by Lemma 2.1.3), multiplying by $(1, y)$ in T does not change the sign. Similarly, multiplying by $(\varepsilon, 1)$ just means multiplying the sign by ε , since 1 is a square. So one can conclude that we have the following multiplication rule:

$$(\varepsilon, n, y, x^m) \mapsto (\varepsilon, 1)(1, p)^n (1, y)(1, x)^m = (\varepsilon \cdot \varepsilon(n, m), p^n y x^m),$$

where

$$\varepsilon(n, m) = (p, p)_2^{(n-1)n/2} (-1)^{nm} = \begin{cases} (-1)^{nm} & \text{if } p \equiv 1 \pmod{4} \\ (-1)^{(n-1)n/2} (-1)^{nm} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

We consider the character χ on T given by

$$\chi : (\varepsilon, 1)(1, p)^n (1, y)(1, x)^m \mapsto \varepsilon z^n \zeta_{p-1}^m,$$

where $\zeta_{p-1} \in \mathbb{C}^\times$ is a primitive $(p-1)$ -th root of unity. We want to know the zeros/poles of

$$\begin{aligned} J_{B/\bar{B}} J_{\bar{B}/B} : I_B^G(\chi) &\rightarrow I_{\bar{B}}^G(\chi) \\ f &\mapsto \left(g \mapsto \int_U \int_{\bar{U}} f(\bar{u}ug) d\bar{u} du \right). \end{aligned}$$

The composition $J_{B/\bar{B}} J_{\bar{B}/B}$ is only described by the double integral above in the range where it converges. We calculate the integral above for a specific f and $g = 1$. We consider $f \in I_B^G(\chi)$ given by

$$f(u\tilde{t}k) = \chi(\tilde{t})|\tilde{t}| \text{ for } u \in U, \tilde{t} \in T \text{ and } k \in \begin{pmatrix} 1 & 0 \\ \mathfrak{p}_F & 1 \end{pmatrix}$$

and f is zero everywhere else. Here, for $\tilde{t} \in T$, $|\tilde{t}|$ is the absolute value of $\tilde{t}$ after projecting to $T' \subset \text{SL}_2(F)$. We have

$$\begin{pmatrix} 1 & 0 \\ \bar{u} & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & tu \\ 0 & 1 \end{pmatrix} \left(\varepsilon(t, u), \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ t\bar{u} & 1 \end{pmatrix} \in UT\bar{U},$$

where $t = 1/(1 + \bar{u}u)$ and $\varepsilon(t, u) = (t, -u)_2$, see Lemma 3.3.2 below. Then the double integral becomes (with abuse of notation)

$$\int_U \int_{\bar{U}} f(\bar{u}u) d\bar{u}du = \int_U \int_{\bar{U}} \chi\left(\varepsilon(t, u), \frac{1}{1 + \bar{u}u}\right) \left| \frac{1}{1 + \bar{u}u} \right| 1_{\mathfrak{p}_F}\left(\frac{\bar{u}}{1 + \bar{u}u}\right) d\bar{u}du.$$

First we consider the inner integral over $\bar{U}$. After a substitution $t = 1/(1 + \bar{u}u)$, it suffices to calculate the integral

$$\begin{aligned} \int_F \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(\varepsilon(t, u), t) 1_{\mathfrak{p}_F}\left(\frac{1-t}{u}\right) dt \\ = \sum_{n=-\infty}^{\infty} \int_{v(t)=n} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(\varepsilon(t, u), t) 1_{\mathfrak{p}_F}\left(\frac{1-t}{u}\right) dt. \end{aligned}$$

For $v(t) > 0$, we have $v(1-t) = 0$. Then we only get something nonzero for $v(u) < 0$. So

$$\begin{aligned} \int_{v(t)=n} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(\varepsilon(t, u), t) 1_{\mathfrak{p}_F}\left(\frac{1-t}{u}\right) dt \\ = 1_{F \setminus \mathcal{O}_F}(u) p^{v(u)+n} \int_{v(t)=n} \chi(\varepsilon(t, u), t) dt. \quad (3.3.1) \end{aligned}$$

We have to take a closer look at the integral $\int_{v(t)=n} \chi(\varepsilon(t, u), t) dt$. Recall that we have an identification

$$\{1, -1\} \times \mathbb{Z} \times (1 + p\mathcal{O}_F) \times \mathbb{F}_p^\times \xrightarrow{\sim} \{1, -1\} \times T' = T.$$

We take the integral over all elements $(\varepsilon(t, u), t) \in T$ with $v(t) = n$, i.e. all elements of the form $(\varepsilon, 1)(1, p)^n(1, y)(1, x)^m = (\varepsilon(p^n y x^m, u), p^n y x^m)$ (note that this is only “half” of the group T). For each n , the problem boils down to calculating the integral of a nontrivial character of the finite group $\{1, -1\} \times \mathbb{F}_p^\times$, over only half of the group. Namely, we only consider the $(\varepsilon, 1)(1, x)^m = (\varepsilon, x^m)$ such that $(\varepsilon, 1)(1, p)^n(1, x)^m = (\varepsilon(p^n x^m, u), p^n x^m)$. More precisely, we have to compare the two signs

$$\varepsilon \cdot \varepsilon(n, m) = \varepsilon \cdot (p, p)_2^{(n-1)n/2} (-1)^{nm} \text{ and } \varepsilon(p^n x^m, u) = (p, -u)_2^n (x, -u)_2^m.$$

We only integrate over the $(\varepsilon, 1)(1, x)^m$ so that both signs in the equation above are equal. Note that what this subset looks like exactly, depends on the value of $p \bmod 4$, on the value of n , and also on u . But in each particular case of calculating the integral (3.3.1), these three values are fixed, and our only variables are ε and m . The task remains to see for which ε and m the equality of the signs above holds. After observing the formulas a bit more carefully, one can conclude that the set of solutions to the equation above can only take one of the following forms (as a subset of $\{1, -1\} \times \mathbb{F}_p^\times$):

$$\begin{aligned} & \bullet \{1\} \times \mathbb{F}_p^\times \\ & \bullet \{-1\} \times \mathbb{F}_p^\times \\ & \bullet \{1\} \times \{x^m \mid m \text{ even}\} \cup \{-1\} \times \{x^m \mid m \text{ odd}\} \\ & \bullet \{1\} \times \{x^m \mid m \text{ odd}\} \cup \{-1\} \times \{x^m \mid m \text{ even}\}. \end{aligned} \quad (3.3.2)$$

Now, we take the integral of the character $\chi|_{\{1, -1\} \times \mathbb{F}_p^\times} : (\varepsilon, x^m) \mapsto \varepsilon \cdot \zeta_{p-1}^m$, over one of the subsets above. This will still essentially be an integral of a nontrivial character over a finite group (or a linear combination of such integrals). Hence, they all vanish. By a similar argument, the integral over $v(t) < 0$ vanishes.

It remains to consider the case where $v(t) = 0$. We do something similar as in Example 3.2.1 and again break up the integral into smaller parts:

$$\begin{aligned} \int_{v(t)=0} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(\varepsilon(t, u), t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt \\ = \sum_{n=0}^{\infty} \int_{v(t)=0, v(1-t)=n} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(\varepsilon(t, u), t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt. \end{aligned}$$

If $v(t) = 0, v(1-t) = n > 0$, then t must be of the form $t = 1 + p^n b$ for some $b \in \mathbb{Z}_p^\times$. Then t is a square, hence $\varepsilon(t, u) = 1$. It follows that χ is constant on this set of $(1, t)$ -s, with value 1. Moreover, the measure of this set is $p^{-n} - p^{-n-1}$. Hence,

$$\begin{aligned} \int_{v(t)=0, v(1-t)>0} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(\varepsilon(t, u), t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt &= \sum_{n=N_u}^{\infty} p^{v(u)} (p^{-n} - p^{-n-1}) \\ &= p^{v(u)-N_u}, \end{aligned}$$

where $N_u = \max(1, 1 + v(u))$. If $v(t) = 0, v(1-t) = 0$, then t must be of the form $t = a + pb$ where $b \in \mathbb{Z}_p$ and $1 \neq a \in \mathbb{F}_p^\times$. So we get

$$\begin{aligned} \int_{v(t)=0, v(1-t)=0} \left| \frac{1}{u} \right| \left| \frac{1}{t} \right| \chi(\varepsilon(t, u), t) 1_{\mathfrak{p}_F} \left(\frac{1-t}{u} \right) dt \\ = 1_{F \setminus \mathcal{O}_F}(u) (1 - p^{-1}) p^{v(u)} \int_{A - \{(1,1)\}} \chi|_{\{1, -1\} \times \mathbb{F}_p^\times}(t) dt \\ = -1_{F \setminus \mathcal{O}_F}(u) (1 - p^{-1}) p^{v(u)} \frac{1}{p-1} \\ = -1_{F \setminus \mathcal{O}_F}(u) p^{v(u)-1}, \end{aligned}$$

where A is a subset of one of the forms in (3.3.2). Note that A contains $(1, 1)$ because $(1, 1)$ corresponds to a square $t \in 1 + p\mathcal{O}_F$. The equality in the second line follows from the fact that

$$\int_A \chi|_{\{1, -1\} \times \mathbb{F}_p^\times}(t) dt = 0$$

(like we argued before), hence

$$\int_{A - \{(1,1)\}} \chi|_{\{1, -1\} \times \mathbb{F}_p^\times}(t) dt = -\frac{1}{p-1} \chi|_{\{1, -1\} \times \mathbb{F}_p^\times}(1, 1) = -\frac{1}{p-1}.$$

It remains to calculate

$$\int_U \left(p^{v(u)-N_u} - 1_{F \setminus \mathcal{O}_F}(u) p^{v(u)-1} \right) du.$$

For $v(u) \geq 0$, the only nonzero term is $p^{v(u)-N_u} = p^{-1}$, and the measure of this set is 1. For $v(u) < 0$ we have $N_u = 1$, and so the two terms in the integral cancel. So the integral over the whole of U equals p^{-1} (which is independent of z , as expected). Also note that the result is equal to the result in Example 3.2.1.

Lemma 3.3.2. *In the group $\widetilde{\mathrm{SL}_2(\mathbb{Q}_p)}$ of Example 3.3.1, we have*

$$\begin{pmatrix} 1 & 0 \\ \bar{u} & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & tu \\ 0 & 1 \end{pmatrix} \left((t, -u)_2, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ t\bar{u} & 1 \end{pmatrix} \in UT\bar{U},$$

where $t = 1/(1 + u\bar{u})$.

Proof. We use notation and results from the end of Section 2.1.4. The map $h_\alpha : F^\times \rightarrow T$ is given by $t \mapsto (1, t)$. Then

$$\sigma_\alpha(t_1, t_2) = (1, t_1)(1, t_2)(1, t_1 t_2)^{-1} = ((t_1, t_2)_2, 1),$$

i.e. the cocycle σ_α is given by the Hilbert symbol. Write $t = 1/(1 + u\bar{u})$. Then we want to show that

$$e_\alpha(-tu)e_{-\alpha}(\bar{u})e_\alpha(u)e_{-\alpha}(-t\bar{u}) = \sigma_\alpha(t, -u)h_\alpha(t).$$

One can check the following equalities:

$$\begin{aligned} & e_\alpha(-tu)e_{-\alpha}(\bar{u})e_\alpha(u)e_{-\alpha}(-t\bar{u}) \\ &= e_\alpha(-tu)e_{-\alpha}(t^{-1}u^{-1})e_{-\alpha}(-u^{-1})e_\alpha(u)e_{-\alpha}(-u^{-1})e_{-\alpha}(tu^{-1}) \\ &= e_\alpha(-tu)e_{-\alpha}(t^{-1}u^{-1})n_{-\alpha}(-u^{-1})e_{-\alpha}(tu^{-1}) \\ &= e_\alpha(-tu)e_{-\alpha}(t^{-1}u^{-1})e_\alpha(-tu)n_{-\alpha}(-u^{-1}) \\ &= n_\alpha(-ut)n_{-\alpha}(-u^{-1}) \\ &= n_\alpha(-ut)n_\alpha(u) \\ &= h_\alpha(-ut)h_\alpha(-u)^{-1} \\ &= \sigma_\alpha(t, -u)^{-1}h_\alpha(t) \\ &= \sigma_\alpha(t, -u)h_\alpha(t), \end{aligned}$$

which is what we wanted to show. □

Chapter 4

Intertwining operators for covers

This chapter is, up to some minor modifications, the preprint [FS25], which is joint with Maarten Solleveld. The preprint has been accepted for publication in the Journal of Lie Theory. Its content might overlap a little with the content of previous chapters. Sections 4.3 and 4.4.2 are written by the co-author Solleveld; the other sections are written by the author of this thesis. The results were obtained in collaboration.

In Chapter 3, it turned out to be quite tricky to determine the Harish-Chandra μ -function by explicit computation of the integrals, even in simple examples and especially for covering groups. In this chapter, we determine a general formula for μ using other methods. We obtain a formula for μ as a rational function, but still without knowing the explicit values of the poles. Those will be determined in Chapter 5 for principal series representations of covers of split groups.

4.1 Introduction

In this paper, we prove a formula for the Harish-Chandra μ -function for covering groups of reductive p -adic groups. By a covering group G of a reductive p -adic group G' , we mean a central extension of G' by a finite abelian group, which is moreover a topological covering. An example of such a group is the metaplectic group, which is a double cover of the symplectic group. Many methods and results from the representation theory of reductive p -adic groups generalize to covering groups. For example, we have notions of parabolic induction and restriction functors, and there is a Bernstein decomposition, analogous to the reductive case. See [FP22] for an overview.

The Harish-Chandra μ -function is defined using intertwining operators $J_{\bar{P}|P}$ and $J_{P|\bar{P}}$ between parabolically induced representations, for opposite parabolic subgroups P and $\bar{P}$ of G . These intertwining operators are studied in [Wal03]; in fact, they can be considered more generally for parabolic

subgroups P, Q which have the same Levi subgroup, but are not necessarily opposite. Such intertwining operators also exist for covering groups, see [Li12]. An important ingredient in our work is an adjointness relation of these intertwining operators, which was stated but not proved in [Wal03]. Section 4.3 of this paper gives a concrete proof of this adjointness property, using Bruhat-Tits theory. In Section 4.4, we prove a formula for the Harish-Chandra μ function, as a complex rational function. The formula was already given for reductive groups in [Sil80]. Here, we provide an alternative proof of the result, which also works for covering groups.

To be able to more precisely state our results, let us give some more context. Let M be a Levi subgroup of a covering group G and let $P = MU_P, Q = MU_Q$ be parabolic subgroups of G with Levi M . Let π be any irreducible smooth M -representation, and let I_P^G be the functor of normalized parabolic induction. Consider the intertwining operators

$$J_{Q|P}(\pi) : I_P^G(\pi) \rightarrow I_Q^G(\pi)$$

$$f \mapsto \left[g \mapsto \int_{(U_P \cap U_Q) \backslash U_Q} f(ug) du \right],$$

and $J_{P|Q}$ which is defined in a similar way. Let $\bar{\pi}^\vee$ denote the Hermitian dual of π . The M -invariant pairing $\langle \cdot, \cdot \rangle$ between π and $\bar{\pi}^\vee$ gives rise to a G -invariant pairing $\langle \cdot, \cdot \rangle$ between $I_P^G(\pi)$ and $I_P^G(\bar{\pi}^\vee)$. Similarly, we get a pairing $\langle \cdot, \cdot \rangle$ between $I_Q^G(\pi)$ and $I_Q^G(\bar{\pi}^\vee)$. With respect to these pairings, we will prove the following result.

Theorem 4.1.1. *The intertwining operators $J_{Q|P}(\pi)$ and $J_{P|Q}(\bar{\pi}^\vee)$ are adjoint, in the sense that for $f_1 \in I_P^G(\pi), f_2 \in I_Q^G(\bar{\pi}^\vee)$ we have*

$$\langle J_{Q|P}(\pi)f_1, f_2 \rangle = \langle f_1, J_{P|Q}(\bar{\pi}^\vee)f_2 \rangle.$$

There is a similar result for the contragredient representation $\pi^\vee$ instead of $\bar{\pi}^\vee$. In the special case where π is Hermitian, we obtain the adjointness relation that was already stated in [Wal03] and which we will need to prove the result in Section 4.4.

Now, let us discuss the Harish-Chandra μ -function. Consider an irreducible supercuspidal representation σ of M . Let $P = MU$ be a parabolic subgroup of G with Levi M , and let $\bar{P} = M\bar{U}$ be the parabolic subgroup of G opposite to P . We have intertwining operators $J_{\bar{P}|P}(\sigma), J_{P|P}(\sigma)$ as above; they are integrals over $\bar{U}$ and U , respectively.

The composition

$$j(\sigma) := J_{P|\bar{P}}(\sigma) \circ J_{\bar{P}|P}(\sigma) : I_P^G(\sigma) \rightarrow I_P^G(\sigma)$$

is a scalar, and does not depend on the choice of parabolic subgroup P [Wal03]. The Harish-Chandra μ -function is, up to a positive real scalar, defined to be j^{-1} . This function μ plays a crucial role in the Plancherel formula for reductive p -adic groups [Wal03], generalized to covering groups in [Li12].

One may view $\mu(\sigma \otimes \chi)$ as a complex rational function in the variable $\chi \in X_{\text{nr}}(M)$, where $X_{\text{nr}}(M)$ is the complex algebraic torus of unramified

characters of M . The function μ then decomposes as a product $\mu = \prod_{\alpha} \mu_{\alpha}$, where α runs over the positive roots in the reduced root system corresponding to M in G . Each μ_{α} can be computed in a Levi subgroup M_{α} of G , which has M as a maximal Levi subgroup. Moreover, μ_{α} can be seen as a complex function in a single variable $z = \chi(h_{\alpha}^{\vee}) \in \mathbb{C}^{\times}$, where $h_{\alpha}^{\vee}$ is suitably chosen element of M .

Theorem 4.1.2. *In the above context, there exists a unitary $\sigma_0 \in \sigma \cdot X_{\text{nr}}(M)$ such that μ_{α} has the form*

$$\mu_{\alpha}(\sigma_0 \otimes \chi) = \mu_{\alpha}(\sigma_0, z) = c \cdot \frac{(1-z)(1-z^{-1})}{(1-qz)(1-qz^{-1})} \frac{(1+z)(1+z^{-1})}{(1+q'z)(1+q'z^{-1})}, \quad (4.1.1)$$

where $c \in \mathbb{R}_{>0}$ and $q, q' \in \mathbb{R}_{\geq 1}$.

Both the zeros and the poles of μ_{α} have clear representation-theoretic significance. If $\sigma \otimes \chi$ is a pole of μ_{α} , then $I_{P \cap M_{\alpha}}^{M_{\alpha}}(\sigma \otimes \chi)$ is reducible (but not decomposable). On the other hand, the set of roots α with $\mu_{\alpha}(\sigma \otimes \chi) = 0$ forms a root system. It plays a role in the Knapp-Stein theory of intertwining operators [Li12; Sil78], which determines the decomposition of $I_P^G(\sigma \otimes \chi)$ in indecomposable representations.

Let us give an outline of the proof of the theorem. By setting $\mu_{\alpha}(\sigma, 0) := \lim_{z \rightarrow 0} \mu_{\alpha}(\sigma, z)$ and $\mu_{\alpha}(\sigma, \infty) := \lim_{z \rightarrow \infty} \mu_{\alpha}(\sigma, z)$, we can view μ_{α} as a rational function on the projective curve $\mathbb{P}^1(\mathbb{C})$. Suppose $\mu_{\alpha}(\sigma, z)$ is not constant, then there exists a unitary $\sigma_0 \in \sigma \cdot X_{\text{nr}}(M)$ with $\mu_{\alpha}(\sigma_0) = 0$. The only other possibility for a zero of $\mu_{\alpha}(\sigma_0, z)$ in $\mathbb{C}^{\times}$ is at $z = -1$. Every zero must be of order two, which corresponds to $J_{\bar{P}|P}$ and $J_{P|\bar{P}}$ each having a pole of order one at that point. So $\mu(\sigma_0, z)$ has at most two zeroes in $\mathbb{C}^{\times}$, located at $z = \pm 1$, each of order two. Moreover, we will show that $\mu(\sigma_0, z)$ is not zero at $z = 0$ and $z = \infty$. We will also find two poles for every zero, using techniques concerning Hermitian and unitary representations.

An application of our result is to give a more explicit description of Bernstein blocks in the category of smooth representations of G . It is known that Bernstein blocks can be realized as module categories of endomorphism algebras, see [FP22]. The work in [Sol23] describes the structure of these endomorphism algebras for reductive p -adic groups. With the formula for the μ -function for covering groups, and other results that are already known for covering groups, the results in [Sol23] should generalize.

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4.2 Notation

F : non-archimedean local field

$G' = \mathcal{G}'(F)$: reductive group over F

$S' \subset G'$: maximal F -split torus

$M' \subset G'$: Levi subgroup containing S'

$P' = M'U_{P'}$: parabolic subgroup of G'

$\bar{P}' = M'U_{\bar{P}'}$: opposite parabolic subgroup

$p_G : G \rightarrow G'$: finite central covering

$M = p_G^{-1}(M')$: Levi subgroup of G

$P = p_G^{-1}(P')$, $\bar{P} = p_G^{-1}(\bar{P}')$: parabolic subgroups of G

Since p_G splits over unipotent subgroups, $U_{P'}$ can be identified with a subgroup $U_P \subset P$, and similarly $U_{\bar{P}'} \cong U_{\bar{P}} \subset \bar{P}$. Then $P = MU_P$ and $\bar{P} = MU_{\bar{P}}$.

$K' = G'_x$: good maximal compact subgroup of G' , associated to a special vertex x in the apartment for S' in the Bruhat–Tits building $\mathcal{B}(G')$

$K = p_G^{-1}(K') = G_x$: good maximal compact subgroup of G

π : irreducible smooth M -representation on $\mathbb{C}$ -vector space V_π

$I_P^G(\pi)$: normalized parabolic induction of π , on the vector space $I_P^G(V_\pi)$

$(\bar{\pi}^\vee, \bar{V}_\pi^\vee)$: Hermitian dual of (π, V_π)

$(\pi^\vee, V_\pi^\vee)$: contragredient of (π, V_π)

$(\sigma, V_\sigma) \in \text{Irr}(M)_{\text{cusp}}$: irreducible supercuspidal representation of M

$X_{\text{nr}}(M)$: the torus of unramified characters of M

$\text{Irr}(M)_{[M, \sigma]} = \sigma \cdot X_{\text{nr}}(M)$: the inertial equivalence class of σ for M

4.3 Adjointness relations of intertwining operators

Let G be a covering group of a reductive p -adic group and let M be a Levi subgroup of G . In this section, we prove adjointness relations for the intertwining operators $J_{P|Q}$ and $J_{Q|P}$, where P and Q are parabolic subgroups of G with Levi subgroup M . The adjointness relations will first be proved for opposite parabolic subgroups P and $\bar{P}$, after which they will be generalized to arbitrary parabolic subgroups P and Q with the same Levi M .

Let (π, V_π) be an irreducible smooth representation of M , and denote by $(\bar{\pi}^\vee, \bar{V}_\pi^\vee)$ the Hermitian dual of π . The M -invariant sesquilinear pairing between π and $\bar{\pi}^\vee$ gives rise to a G -invariant sesquilinear pairing between $I_P^G(\pi)$ and $I_{\bar{P}}^G(\bar{\pi}^\vee)$, namely

$$\langle f, f' \rangle = \int_K \langle f(k), f'(k) \rangle dk \quad f \in I_P^G(V_\pi), f' \in I_{\bar{P}}^G(\bar{V}_\pi^\vee). \quad (4.3.1)$$

By the Iwasawa decomposition $G = PK$, $\int_K$ in (4.3.1) is up to a positive scalar factor the same as $\int_{P \backslash G}$. The pairing (4.3.1) is nondegenerate, so provides an isomorphism $I_P^G(\bar{\pi}^\vee) \cong \overline{I_{\bar{P}}^G(\pi)}^\vee$.

There are intertwining operators

$$J_{\bar{P}|P}(\pi) : I_P^G(\pi) \rightarrow I_{\bar{P}}^G(\pi), \quad J_{P|\bar{P}}(\bar{\pi}^\vee) : I_{\bar{P}}^G(\bar{\pi}^\vee) \rightarrow I_P^G(\bar{\pi}^\vee), \quad (4.3.2)$$

defined by

$$(J_{\bar{P}|P}(\pi)f)(g) = \int_{U_{\bar{P}}} f(\bar{u}g) d\bar{u}, \quad (J_{P|\bar{P}}(\bar{\pi}^\vee)f')(g) = \int_{U_P} f'(ug) du. \quad (4.3.3)$$

These arise via meromorphic continuation, as in [Wal03, §IV.1].

We can build two G -invariant sesquilinear pairings $I_P^G(V_\pi) \times I_P^G(\overline{V_\pi}^\vee) \rightarrow \mathbb{C}$:

$$(f_1, f_2) \mapsto \langle J_{\bar{P}|P}(\pi)f_1, f_2 \rangle \quad (4.3.4)$$

$$(f_1, f_2) \mapsto \langle f_1, J_{P|\bar{P}}(\bar{\pi}^\vee)f_2 \rangle, \quad (4.3.5)$$

which are well-defined provided Condition 4.3.1(i) below.

Condition 4.3.1. (i) $J_{\bar{P}|P}(\pi)$ and $J_{P|\bar{P}}(\bar{\pi}^\vee)$ are regular, i.e. the meromorphic continuation of (4.3.3) does not have a pole at π or $\bar{\pi}^\vee$.

(ii) $I_P^G(\pi)$ and $I_{\bar{P}}^G(\pi)$ are irreducible. This implies that $I_P^G(\bar{\pi}^\vee) \cong \overline{I_P^G(\pi)}^\vee$ and $I_{\bar{P}}^G(\bar{\pi}^\vee) \cong \overline{I_{\bar{P}}^G(\pi)}^\vee$ are also irreducible.

The second condition holds for irreducible supercuspidal M -representations in general position by [Ren10, Théorème VI.8.5], and (more generally) for irreducible M -representations in general position by [Sau97, Théorème 3.2]. Although these results are written for reductive p -adic groups, the arguments apply also to finite covers like G . We elaborate on this below.

Theorem 4.3.2. *Let π be an irreducible representation of M . Let $\mathcal{V}$ be an open neighbourhood of 1 in the variety $X_{\text{nr}}(M)$ with the analytic topology. Then there exists a dense open subset $\mathcal{V}'$ of $\mathcal{V}$ such that $I_P^G(\pi \otimes \chi)$ is irreducible for all $\chi \in \mathcal{V}'$.*

Proof. For reductive groups, this was proved in [Sau97, Théorème 3.2]. We explain how the proof generalizes to covering groups. Most results of [Sau97] hold for locally compact unimodular groups, so in particular for finite central covering groups of reductive p -adic groups. The result in [Sau97, Théorème 3.2] is specific for reductive p -adic groups and uses the following additional properties of representations of such groups: supercuspidal representations, parabolic induction, Jacquet restriction, Frobenius reciprocity, Bernstein's geometrical lemma [Ren10, Théorème IV.5.1], unramified characters and supercuspidal support. All of these concepts are worked out in [Ren10] and generalize to covering groups, as worked out in [FP22]. Therefore, [Sau97, Théorème 3.2] generalizes to covering groups. $\square$

Lemma 4.3.3. *Assuming Condition 4.3.1, the pairings (4.3.4) and (4.3.5) differ by a nonzero scalar factor.*

Proof. Any nonzero G -invariant sesquilinear pairing $V \times W \rightarrow \mathbb{C}$ defines a nonzero homomorphism of G -representations $\phi : V \rightarrow \overline{W}^\vee$, and conversely, by the formula

$$(v, w) = \phi(v)(w).$$

If V and W are irreducible G -representations, then ϕ is unique up to scalars from $\mathbb{C}^\times$, by Schur's lemma. The pairings $I_P^G(V_\pi) \times I_{\bar{P}}^G(\overline{V_\pi}^\vee) \rightarrow \mathbb{C}$ in (4.3.4) and (4.3.5) are nonzero because (4.3.1) is nondegenerate and the operators (4.3.2) are nonzero [Wal03, p. 283]. Hence they differ only by a factor from $\mathbb{C}^\times$. $\square$

We want to show that the pairings (4.3.4) and (4.3.5) are equal. This is claimed, but not proven, for reductive p -adic groups in [Wal03, p. 287]. It requires some preparation.

Let $\Phi(G', S')$ be the root system of G' and write it as

$$\Phi(G', S') = \Phi(M', S') \cup \Phi(U_P, S') \cup \Phi(U_{\bar{P}}, S').$$

Here, $\Phi(U_P, S') \subset \Phi(G', S')$ is the set of roots such that U_P is generated by all root subgroups U_α , where $\alpha \in \Phi(U_P, S')$. The subsets $\Phi(M', S')$ and $\Phi(U_{\bar{P}}, S')$ are given similarly.

For each $r \in \mathbb{R}$, the valuated root datum of G provides compact open subgroups $U_{\alpha, r} \subset U_\alpha$. We normalize the valuations on the root subgroups U_α (of G and of G') so that

$$U_{\alpha, 0} = U_\alpha \cap K = U_{\alpha, x}.$$

By [KP23, Lemma 7.3.11.(2)] there are well-defined subgroups

$$U_{P, r} = \prod_{\alpha \in \Phi(U_P, S')} U_{\alpha, r}, \quad U_{\bar{P}, r} = \prod_{\alpha \in \Phi(U_{\bar{P}}, S')} U_{\alpha, r}.$$

For $r \geq 0$ we also have the subgroups $Z_G(S')_r = p_G^{-1}(Z_{G'}(S')_r)$ of $Z_G(S')$ and the Moy–Prasad groups [KP23, §13.2]

$$G'_{x, r} \subset G' \text{ and } G_{x, r} = p_G^{-1}(G'_{x, r}).$$

Recall that $p_G : G \rightarrow G'$ is a local homeomorphism and that the groups $G'_{x, r}$ with $r \in \mathbb{R}_{\geq 0}$ form a neighborhood basis of e in G' [KP23, Proposition 13.2.5]. Restricting p_G to a neighborhood of e in G on which it is a homeomorphism, we find $r_x > 0$ and a homomorphic splitting $s_G : G'_{x, r_x} \rightarrow G$ of p_G . Then

$$G_{x, r} = s_G(G'_{x, r}) \times \ker(p_G) \quad \forall r \geq r_x.$$

The advantage of this construction is that the subgroups $s_G(G'_{x, r})$ with $r \geq r_x$ form a neighborhood basis of e in G , in contrast with the $G_{x, r}$. As x lies in an apartment of $\mathcal{B}(M')$, we have $M' \cap K' = M'_x$ and $M \cap K = M_x$, and the groups $M'_{x, r}$ and $M_{x, r} = p_G^{-1}(M'_{x, r})$ are defined.

Let $v_1 \in V_\pi, v_2 \in \overline{V}_\pi^\vee$, and pick $r \in \mathbb{R}_{> 0}$ so that $s_G(M'_{x, r})$ fixes v_1 and v_2 . We define $f_1 \in I_P^G(V_\pi)$ and $f_2 \in I_{\bar{P}}^G(\overline{V}_\pi^\vee)$ by the conditions

- $f_1(e) = v_1$ and $f_2(e) = v_2$,
- $\text{supp}(f_1) = PU_{\bar{P}, r}$ and $\text{supp}(f_2) = \bar{P}U_{P, r}$,
- f_1 is right $U_{\bar{P}, r}$ -invariant and f_2 is right $U_{P, r}$ -invariant.

We note that these conditions are not overdetermined because the multiplication maps

$$U_P \times M \times U_{\bar{P}} \rightarrow G \quad \text{and} \quad U_{\bar{P}} \times M \times U_P \rightarrow G \quad (4.3.6)$$

are injective. That follows from the analogous statements for G' , which can be found for instance in [CGP15, Proposition 2.1.8.(3)].

We want to analyse

$$\langle J_{\bar{P}|P}(\pi)f_1, f_2 \rangle = \int_{U_{\bar{P}}} \int_K \langle f_1(\bar{u}k), f_2(k) \rangle dk d\bar{u}. \quad (4.3.7)$$

Lemma 4.3.4. *The only nonzero contributions to (4.3.7) come from $k \in U_{\bar{P},0}M_xU_{P,r}$ and $\bar{u} \in U_{\bar{P},0}$ such that $\bar{u}k \in U_{\bar{P},r}M_xU_{P,r}$.*

Proof. From the support of f_2 we see that we can only get nonzero contributions to (4.3.7) when $k \in K \cap \bar{P}U_{P,r}$. Write

$$k = \bar{u}_1 m_1 u_2 \text{ with } \bar{u}_1 \in U_{\bar{P}}, m_1 \in M, u_2 \in U_{P,r}.$$

As $U_{P,r} \subset K$, we have

$$\bar{u}_1 m_1 \in K \cap U_{\bar{P}}M = G_x \cap U_{\bar{P}}M = G_x \cap \bar{P}. \quad (4.3.8)$$

By [KP23, Proposition 8.3.1] there exists an integral model $\mathcal{G}'_x$ of $\mathcal{G}'$, such that $\mathcal{G}'_x(\mathfrak{o}_F) = G'_x$. Then $\bar{P}' \cap G'_x, U_{\bar{P}'} \cap G'_x$ and $M' \cap G'_x$ determine $\mathfrak{o}_F$ -subgroup schemes $\bar{P}'_x, \mathcal{U}'_{\bar{P}',x}$ and $\mathcal{M}'_x$ of $\mathcal{G}'_x$. Since $\bar{P} = \mathcal{U}_{\bar{P}} \rtimes \mathcal{M}'$ in $\mathcal{G}'$ and $\mathcal{G}'_x$ is an integral model of $\mathcal{G}'$, also

$$\bar{P}'_x = \mathcal{U}'_{\bar{P}',x} \rtimes \mathcal{M}'_x.$$

Taking $\mathfrak{o}_F$ -rational points, we obtain

$$\bar{P}' \cap G'_x = \bar{P}'_x(\mathfrak{o}_F) = \mathcal{U}'_{\bar{P}',x}(\mathfrak{o}_F) \rtimes \mathcal{M}'_x(\mathfrak{o}_F) = (U_{\bar{P}'} \cap G'_x) \rtimes (M' \cap G'_x) = U_{\bar{P}',0} \rtimes M'_x.$$

Applying p_G^{-1} , we find that

$$\bar{P} \cap G_x = U_{\bar{P},0} \rtimes M_x.$$

Now (4.3.8) says that $\bar{u}_1 \in U_{\bar{P},0}$ and $m_1 \in M_x$, so

$$\begin{aligned} k &= \bar{u}_1 m_1 u_2 \in U_{\bar{P},0} M_x U_{P,r}, \\ \bar{u}k &= \bar{u} \bar{u}_1 m_1 u_2 \in U_{\bar{P}} M_x U_{P,r}. \end{aligned}$$

The support of f_1 shows that $f_1(\bar{u}k)$ can only be nonzero when $\bar{u}k \in U_P M U_{\bar{P},r}$. We write

$$\bar{u}k = u_3 m_2 \bar{u}_5 \text{ with } u_3 \in U_P, m_2 \in M, \bar{u}_5 \in U_{\bar{P},r}.$$

Then

$$\bar{u}k u_2^{-1} = \bar{u} \bar{u}_1 m_1 = u_3 m_2 \bar{u}_5 u_2^{-1} \in U_{\bar{P}} M_x \cap U_P M U_{\bar{P},r} U_{P,r}. \quad (4.3.9)$$

The Iwahori decomposition of the Moy–Prasad group $G'_{x,r}$ [KP23, §13.2] entails that the multiplication map

$$M_{x,r} \times U_{P,r} \times U_{\bar{P},r} \rightarrow G_{x,r} \text{ is bijective.} \quad (4.3.10)$$

This also works with any other ordering of the three factors. Hence

$$\begin{aligned} U_P M U_{\bar{P},r} U_{P,r} &= U_P M M_{x,r} U_{\bar{P},r} U_{P,r} = \\ &= U_P M M_{x,r} U_{P,r} U_{\bar{P},r} = U_P M U_{P,r} U_{\bar{P},r} = U_P M U_{\bar{P},r}. \end{aligned}$$

This enables us to rewrite (4.3.9) as

$$\bar{u}\bar{u}_1 m_1 = u_6 m_3 \bar{u}_7 \text{ with } u_6 \in U_P, m_3 \in M, \bar{u}_7 \in U_{\bar{P},r}.$$

Next we observe that

$$u_6 m_3 = \bar{u}\bar{u}_1 m_1 \bar{u}_7^{-1} \in U_P M \cap U_{\bar{P}} M_x U_{\bar{P},r} = U_P M \cap M_x U_{\bar{P}}.$$

By (4.3.6), the last intersection is just M_x . This forces $u_6 = 1, m_3 = m_1 \in M_x$ and

$$\bar{u}\bar{u}_1 = m_1 \bar{u}_7 m_1^{-1} \in m_1 U_{\bar{P},r} m_1^{-1} = U_{\bar{P},r}.$$

As $\bar{u}_1 \in U_{\bar{P},0} \supset U_{\bar{P},r}$, that also forces $\bar{u} \in U_{\bar{P},0}$. $\square$

Assume that the Haar measure on $G_{x,r} = U_{\bar{P},r} M_{x,r} U_{P,r}$ is the product of the Haar measures on the three factors. Now we are ready to evaluate (4.3.7).

Lemma 4.3.5.

$$\langle J_{\bar{P}|P}(\pi) f_1, f_2 \rangle = \text{vol}(U_{\bar{P},r}) \text{vol}(U_{\bar{P},0}) \text{vol}(M_x) \text{vol}(U_{P,r}) \langle v_1, v_2 \rangle.$$

Proof. By Lemma 4.3.4 we may replace the integral over $k \in K$ in (4.3.7) by the integral over $\bar{u}_1 m_1 u_2 \in U_{\bar{P},0} \times M_x \times U_{P,r}$. Lemma 4.3.4 also says that we may replace the integral over $\bar{u} \in U_{\bar{P},0}$ by the integral over $\bar{u}_2 = \bar{u}\bar{u}_1^{-1} \in U_{\bar{P},r}$. Using the properties of f_2 we compute

$$\begin{aligned} \langle J_{\bar{P}|P}(\pi) f_1, f_2 \rangle &= \int_{U_{\bar{P},r}} \int_{U_{\bar{P},0}} \int_{M_x} \int_{U_{P,r}} \langle f_1(\bar{u}_2 m_1 u_2), f_2(\bar{u}_1 m_1 u_2) \rangle du_2 dm_1 d\bar{u}_1 d\bar{u}_2 \\ &= \text{vol}(U_{\bar{P},0}) \int_{U_{\bar{P},r}} \int_{M_x} \int_{U_{P,r}} \langle f_1(\bar{u}_2 m_1 u_2), f_2(m_1) \rangle du_2 dm_1 d\bar{u}_2. \end{aligned}$$

Since M_x is compact and normalizes $U_{\bar{P},r}$, we may exchange the order of $\bar{u}_2$ and m_1 . That gives

$$\begin{aligned} \text{vol}(U_{\bar{P},0}) \int_{U_{\bar{P},r}} \int_{M_x} \int_{U_{P,r}} \langle f_1(m_1 \bar{u}_2 u_2), f_2(m_1) \rangle du_2 dm_1 d\bar{u}_2 &= \\ \text{vol}(U_{\bar{P},0}) \int_{U_{\bar{P},r}} \int_{M_x} \int_{U_{P,r}} \langle \pi(m_1) f_1(\bar{u}_2 u_2), \bar{\pi}^\vee(m_1) f_2(e) \rangle du_2 dm_1 d\bar{u}_2 &= \\ \text{vol}(U_{\bar{P},0}) \text{vol}(M_x) \int_{U_{\bar{P},r}} \int_{U_{P,r}} \langle f_1(\bar{u}_2 u_2), f_2(e) \rangle du_2 d\bar{u}_2. \end{aligned}$$

Here $\bar{u}_2 u_2 \in s_G(G'_{x,r})$, so by (4.3.10) it can be written as

$$\bar{u}_2 u_2 = u_3 m_3 \bar{u}_3 \text{ with } u_3 \in U_{P,r}, m_3 \in s_G(M'_{x,r}), \bar{u}_3 \in U_{\bar{P},r}.$$

The construction of f_1 entails that

$$f_1(\bar{u}_2 u_2) = f_1(u_3 m_3 \bar{u}_3) = f_1(m_3) = \pi(m_3) f_1(e) = \pi(m_3) v_1 = v_1.$$

Thus our integral simplifies to

$$\begin{aligned} \text{vol}(U_{\bar{P},0}) \text{vol}(M_x) \int_{U_{\bar{P},r}} \int_{U_{P,r}} \langle v_1, f_2(e) \rangle du_2 d\bar{u}_2 = \\ \text{vol}(U_{\bar{P},0}) \text{vol}(M_x) \text{vol}(U_{\bar{P},r}) \text{vol}(U_{P,r}) \langle v_1, v_2 \rangle. \quad \square \end{aligned}$$

Completely analogous to Lemma 4.3.5, one can show that

$$\langle f_1, J_{P|\bar{P}}(\bar{\pi}^\vee) f_2 \rangle = \text{vol}(U_{P,0}) \text{vol}(M_x) \text{vol}(U_{\bar{P},r}) \text{vol}(U_{P,r}) \langle v_1, v_2 \rangle. \quad (4.3.11)$$

For a better comparison, we need to normalize the Haar measures on G , M , U_P , $U_{\bar{P}}$. That can be done by fixing the volume of one compact open subgroup in each of these groups:

$$\begin{aligned} \text{vol}(M_x) = 1, \text{vol}(U_{P,0}) = 1, \text{vol}(U_{\bar{P},0}) = 1, \\ \text{vol}(G_{x,r}) = \text{vol}(U_{\bar{P},r}) \text{vol}(M_{x,r}) \text{vol}(U_{P,r}). \end{aligned}$$

The last definition says that (locally) the Haar measure on G is the product of the Haar measures on $U_{\bar{P}}$, M and U_P .

We will also formulate a version of the upcoming theorem for the pairing between a representation π and its contragredient $\pi^\vee$. By the same formula as in (4.3.1), the M -invariant bilinear pairing between π and $\pi^\vee$ yields a G -invariant bilinear pairing between $I_P^G(\pi)$ and $I_{\bar{P}}^G(\pi^\vee)$.

Theorem 4.3.6. *Let π be any irreducible smooth M -representation.*

(a) *For any $f_1 \in I_P^G(V_\pi)$, $f_2 \in I_{\bar{P}}^G(\bar{V}_\pi^\vee)$:*

$$\langle J_{\bar{P}|P}(\pi) f_1, f_2 \rangle = \langle f_1, J_{P|\bar{P}}(\bar{\pi}^\vee) f_2 \rangle.$$

(b) *For any $f_1 \in I_P^G(V_\pi)$, $f \in I_{\bar{P}}^G(V_\pi^\vee)$:*

$$\langle J_{\bar{P}|P}(\pi) f_1, f \rangle = \langle f_1, J_{P|\bar{P}}(\pi^\vee) f \rangle.$$

Remark 4.3.7. If $J_{\bar{P}|P}(\pi)$ is singular, then the two sides of the equalities in (a) and (b) of Theorem 4.3.6 may be infinite.

Proof of Theorem 4.3.6. (a) First we assume that Condition 4.3.1 holds. By Lemma 4.3.3, it suffices to find one pair (f_1, f_2) such that $\langle J_{\bar{P}|P}(\pi) f_1, f_2 \rangle$ equals $\langle f_1, J_{P|\bar{P}}(\bar{\pi}^\vee) f_2 \rangle$ and is nonzero. This is achieved by Lemma 4.3.5 and (4.3.11) with v_1, v_2 such that $\langle v_1, v_2 \rangle \neq 0$. That proves the theorem assuming Condition 4.3.1.

Consider a family of irreducible smooth M -representations $\pi_0 \otimes \chi$, where χ runs through the group $X_{\text{nr}}(M)$ of unramified characters of M . By the rationality of $J_{\bar{P}|P}(\pi_0 \otimes \chi)$ and $J_{P|\bar{P}}(\bar{\pi} \otimes \bar{\chi}^\vee)$ as functions of $\chi \in X_{\text{nr}}(M)$

[Wal03, Théorème IV.1.1] and by Theorem 4.3.2, Condition 4.3.1 holds for $\pi_0 \otimes \chi$ with χ in some nonempty open subset of $X_{\text{nr}}(M)$ with the analytic topology. In other words, we have already proven the theorem for all π in a nonempty open subset of the variety

$$\{\pi_0 \otimes \chi : \chi \in X_{\text{nr}}(M)\}$$

with the analytic topology. Consider a χ_0 for which Condition 4.3.1 fails, and let $f_3 \in I_P^G(\pi \otimes \chi_0)$ and $f_4 \in I_P^G(\overline{\pi_0 \otimes \chi_0}^\vee)$. Via the linear bijections

$$I_P^G(V_{\pi_0 \otimes \chi_0}) \cong I_{P \cap K}^K(V_{\pi_0 \otimes \chi_0}) \cong I_{P \cap K}^K(V_{\pi_0 \otimes \chi}) \cong I_P^G(V_{\pi_0 \otimes \chi})$$

we can regard f_3 also as an element of $I_P^G(V_{\pi_0 \otimes \chi})$ for any $\chi \in X_{\text{nr}}(M)$. Similarly $f_4 \in I_P^G(\overline{\pi_0 \otimes \chi}^\vee)$. Now the expressions

$$\langle J_{\bar{P}|P}(\pi_0 \otimes \chi) f_3, f_4 \rangle \quad \text{and} \quad \langle f_3, J_{P|\bar{P}}(\overline{\pi_0 \otimes \chi}^\vee) f_4 \rangle$$

are rational functions of $\chi \in X_{\text{nr}}(M)$. They coincide on an analytic open, hence Zariski-dense subset of $X_{\text{nr}}(M)$, so they are equal.

(b) This can be shown in the same way as part (a), using $\pi^\vee$ instead of $\bar{\pi}^\vee$ in all the previous arguments. $\square$

Theorem 4.3.6.a says that $J_{\bar{P}|P}(\pi) : I_P^G(\pi) \rightarrow I_P^G(\pi)$ is the adjoint operator of

$$J_{P|\bar{P}}(\bar{\pi}^\vee) : \overline{I_P^G(\pi)}^\vee \cong I_P^G(\bar{\pi}^\vee) \rightarrow I_P^G(\bar{\pi}^\vee) \cong \overline{I_P^G(\pi)}^\vee.$$

In the special case that π is Hermitian (e.g. unitary), we may replace $\bar{\pi}^\vee$ by the isomorphic representation π . Then Theorem 4.3.6.a says that

$$J_{\bar{P}|P}(\pi) : I_P^G(\pi) \rightarrow I_P^G(\pi) \text{ is the adjoint of } J_{P|\bar{P}}(\pi) : I_P^G(\pi) \rightarrow I_P^G(\pi), \quad (4.3.12)$$

with respect to the G -invariant Hermitian forms (4.3.1) on $I_P^G(V_\pi)$ and on $I_{\bar{P}}^G(V_\pi)$.

Theorem 4.3.6.b is mentioned (for reductive p -adic groups) in [Wal03, p. 283], but unfortunately without a proof. We note that in the versions of Theorem 4.3.6 in [Wal03], the two parabolic subgroups share the same Levi factor, but they need not be opposite. We want to extend Theorem 4.3.6 to that generality, for our covering group G .

Let $A_{M'}$ be the maximal F -split torus in the centre of M' . We say that a root of $A_{M'}$ is a character of $A_{M'}$ appearing in the adjoint action on $\text{Lie}(G) = \text{Lie}(G')$, and that a root α is reduced if $q\alpha$ is not a root for $q \in (0, 1)$. Let $Q = MU_Q$ be another parabolic subgroup of G with Levi factor M . As in [Wal03, p. 279], we define the distance between P and Q as

$$d(P, Q) = \text{number of reduced roots of } A_{M'} \text{ appearing in } \text{Lie}(U_{\bar{P}} \cap U_Q).$$

In this more general context the definition of $J_{Q|P}(\pi)$ becomes

$$(J_{Q|P}(\pi)f)(g) = \int_{(U_P \cap U_Q) \backslash U_Q} f(ug) \, du.$$

Proposition 4.3.8. *Theorem 4.3.6 holds also with Q instead of $\bar{P}$, that is, it holds for any two parabolic subgroups with the same Levi factor.*

Proof. We can find a sequence of parabolic subgroups

$$P = P_0, P_1, \dots, P_d = Q$$

such that $d(P_i, P_j) = i - j$ for all $i \geq j$. According to [Wal03, p. 283],

$$J_{Q|P}(\pi) = J_{P_d|P_{d-1}}(\pi) \circ \dots \circ J_{P_2|P_1}(\pi) \circ J_{P_1|P_0}(\pi).$$

Hence it suffices to prove the proposition for two parabolic subgroups of distance 1. In other words, we only need to consider the cases with $d(P, Q) = 1$.

Now $A_{M'}$ acts on $\text{Lie}(U_{\bar{P}} \cap U_Q)$ by multiples of a unique reduced root α . We put $T'_\alpha = \ker(\alpha)^\circ$, a codimension one subtorus of $A_{M'}$. Consider the Levi subgroup $M'_\alpha := Z_{G'}(T'_\alpha)$ of G' . It is generated by

$$M' \cup (U_{\bar{P}} \cap U_Q) \cup (U_P \cap U_{\bar{Q}})$$

and contains M' as a maximal Levi subgroup. Then $M_\alpha := p_G^{-1}(M'_\alpha)$ is a Levi subgroup of G with M as maximal Levi subgroup. Since

$$(U_P \cap U_Q) \backslash U_Q \cong U_{\bar{P}} \cap U_Q$$

and $U_{\bar{P}} \cap U_Q \subset M_\alpha$, both $J_{Q|P}(\pi)$ and $J_{Q \cap M_\alpha | P \cap M_\alpha}(\pi)$ are given by integration over $U_{\bar{P}} \cap U_Q$. It follows that $J_{Q|P}(\pi)$ can be obtained from $J_{Q \cap M_\alpha | P \cap M_\alpha}(\pi)$ by applying the functor $I_{P M_\alpha}^G = I_{Q M_\alpha}^G$.

In M_α there are only two parabolic subgroups with Levi factor M , namely $P \cap M_\alpha$ and $Q \cap M_\alpha$, and they are opposite. Thus Theorem 4.3.6 applies to $J_{Q \cap M_\alpha | P \cap M_\alpha}(\pi)$, for any irreducible smooth M -representation π . We note that the group $K \cap M_\alpha = M_{\alpha, x}$ is the good maximal compact subgroup of M_α associated to the special vertex x . For $f_1 \in I_P^G(V_\pi)$ and $f_2 \in I_Q^G(\bar{V}_\pi^\vee)$ we compute

$$\begin{aligned} & \langle J_{Q|P}(\pi) f_1, f_2 \rangle \\ &= \int_K \langle (J_{Q|P}(\pi) f_1)(k), f_2(k) \rangle dk \\ &= \int_{K \cap M_\alpha \backslash K} \int_{K \cap M_\alpha} \langle (J_{Q|P}(\pi) f_1)(k_\alpha k), f_2(k_\alpha k) \rangle dk_\alpha dk \quad (4.3.13) \\ &= \int_{K \cap M_\alpha \backslash K} \int_{K \cap M_\alpha} \langle (J_{Q|P}(\pi) \pi(k) f_1)(k_\alpha), (\bar{\pi}^\vee(k) f_2)(k_\alpha) \rangle dk_\alpha dk. \end{aligned}$$

The restriction of $\pi(k) f_1 : G \rightarrow V_\pi$ to M_α can be regarded as an element of $I_{P \cap M_\alpha}^{M_\alpha}(V_\pi)$, and similarly for $\bar{\pi}^\vee(k) f_2$. Then Theorem 4.3.6.a says that

(4.3.13) equals

$$\begin{aligned}
& \int_{K \cap M_\alpha \backslash K} \langle J_{Q \cap M_\alpha | P \cap M_\alpha}(\pi)(\pi(k)f_1)|_{M_\alpha}, (\bar{\pi}^\vee(k)f_2)|_{M_\alpha} \rangle dk = \\
& \int_{K \cap M_\alpha \backslash K} \langle (\pi(k)f_1)|_{M_\alpha}, J_{P \cap M_\alpha | Q \cap M_\alpha}(\bar{\pi}^\vee)(\bar{\pi}^\vee(k)f_2)|_{M_\alpha} \rangle dk = \\
& \int_{K \cap M_\alpha \backslash K} \int_{K \cap M_\alpha} \langle f_1(k_\alpha k), (J_{P \cap M_\alpha | Q \cap M_\alpha}(\bar{\pi}^\vee)f_2)(k_\alpha k) \rangle dk_\alpha dk = \\
& \int_K \langle f_1(k), (J_{P|Q}(\bar{\pi}^\vee)f_2)(k) \rangle dk = \langle f_1, J_{P|Q}(\bar{\pi}^\vee)f_2 \rangle.
\end{aligned} \tag{4.3.14}$$

The combination of (4.3.13) and (4.3.14) shows that Theorem 4.3.6.a holds for $J_{Q|P}(\pi)$ and $J_{P|Q}(\bar{\pi}^\vee)$. The same argument applies with $\pi^\vee$ instead of $\bar{\pi}^\vee$, and that generalizes Theorem 4.3.6.b. $\square$

4.4 Poles and zeros of the Harish-Chandra μ -function

In this section, we discuss the Harish-Chandra μ -function. By [Wal03, Lemme V.2.1], it decomposes as a product $\mu = \prod_\alpha \mu_\alpha$, where α runs over the positive roots in the reduced root system corresponding to M in G . Each μ_α can be computed in a Levi subgroup M_α of G , which has M as a maximal Levi subgroup.

Thus from now on, we assume that M is a maximal Levi subgroup of G , which leads to $[N_G(M) : M] \leq 2$. Let $P = MU_P$ be a parabolic subgroup with Levi M and let $\bar{P} = MU_{\bar{P}}$ be its opposite parabolic subgroup. We consider the intertwining operators

$$J_{\bar{P}|P}(\sigma \otimes \chi) : I_P^G(\sigma \otimes \chi) \rightarrow I_{\bar{P}}^G(\sigma \otimes \chi), \quad J_{P|\bar{P}}(\sigma \otimes \chi) : I_{\bar{P}}^G(\sigma \otimes \chi) \rightarrow I_P^G(\sigma \otimes \chi),$$

for $\sigma \in \text{Irr}_{\text{cusp}}(M)$ and $\chi \in X_{\text{nr}}(M)$. They are defined by

$$(J_{\bar{P}|P}(\sigma \otimes \chi)f)(g) = \int_{U_{\bar{P}}} f(\bar{u}g) d\bar{u}, \quad (J_{P|\bar{P}}(\sigma \otimes \chi)f')(g) = \int_{U_P} f'(ug) du.$$

Their composition

$$j(\sigma \otimes \chi) := J_{P|\bar{P}}(\sigma \otimes \chi) \circ J_{\bar{P}|P}(\sigma \otimes \chi)$$

is a rational function in the variable $\chi \in X_{\text{nr}}(M)$, see [Wal03, §IV.3]. Up to a positive real scalar, the Harish-Chandra μ -function is $j(\sigma \otimes \chi)^{-1}$ [Wal03, §V.2], and we want to analyze its poles and zeros.

4.4.1 μ as a rational function

Let us explain how to view $\mu(\sigma \otimes \chi)$ as a rational function on $\mathbb{C}^\times$. We refer to [Sol22] (and the correction in [Sol23]). Let

$$X_{\text{nr}}(M, \sigma) := \{\chi \in X_{\text{nr}}(M) : \sigma \otimes \chi \cong \sigma\}.$$

Note that $X_{\text{nr}}(M, \tau) = X_{\text{nr}}(M, \sigma)$ for any $\tau \in \text{Irr}(M)_{[M, \sigma]}$. There are subgroups of M ,

$$M^1 = \cap_{\chi \in X_{\text{nr}}(M)} \ker(\chi), \quad M_\sigma^2 = \cap_{\chi \in X_{\text{nr}}(M, \sigma)} \ker(\chi).$$

The subgroup $M \subset G$ is the inverse image of a maximal Levi subgroup M' of G' . Let $A_{M'}$ be the maximal split torus in the center of M' . Then there is only one positive reduced root α coming from the adjoint representation of $A_{M'}$ on the Lie algebra of G' . Thus we have $(M_\sigma^2 \cap G^1)/M^1 \cong \mathbb{Z}$. By the following result, we can view μ as a rational function on $\mathbb{C}^\times$.

Lemma 4.4.1. *In the context above, let m be a generator of $(M_\sigma^2 \cap G^1)/M^1 \cong \mathbb{Z}$. Then $\mu(\sigma \otimes \chi)$ only depends on $\chi(m)$.*

Proof. The proof is inspired by considerations in [Sol25, §2] and we reduce to the derived subgroup. Let $\mathcal{G}'_{\text{der}}$ be the derived subgroup of $\mathcal{G}'$, $G'_{\text{der}} := \mathcal{G}'_{\text{der}}(F)$ its group of F -points, and let G_{der} be the inverse image of G'_{der} in G . The inclusion map $\mathcal{G}'_{\text{der}} \rightarrow \mathcal{G}'$ is a central isogeny. It induces an inclusion $G'_{\text{der}} \rightarrow G'$ between the groups of F -points, which lifts to an inclusion $G_{\text{der}} \rightarrow G$ between covering groups. The Levi subgroups of G are in one-to-one correspondence with the Levi subgroups of G_{der} , via the bijection $M \mapsto \tilde{M} := M \cap G_{\text{der}}$.

We have a restriction functor $-|_{\tilde{M}} : \text{Rep}(M) \rightarrow \text{Rep}(\tilde{M})$, restricting to an algebraic group homomorphism $X_{\text{nr}}(M) \rightarrow X_{\text{nr}}(\tilde{M})$. For σ an irreducible supercuspidal representation of M , the restriction $\sigma|_{\tilde{M}}$ is a finite direct sum of irreducible supercuspidal $\tilde{M}$ -representations. This was proved for reductive groups in [Sil79], but the arguments generalize to our situation of covering groups.

Since M is a maximal Levi subgroup of G , $\tilde{M}$ is a maximal Levi subgroup of G_{der} . Let $\sigma \in \text{Irr}_{\text{cusp}}(M)$ and let $\tilde{\sigma} \in \text{Irr}_{\text{cusp}}(\tilde{M})$ be an irreducible constituent of $\sigma|_{\tilde{M}}$. We claim that there exists $\tilde{c} \in \mathbb{C}^\times$ such that

$$\mu^M(\sigma \otimes \chi) = \tilde{c} \cdot \mu^{\tilde{M}}(\tilde{\sigma} \otimes \chi|_{\tilde{M}}). \quad (4.4.1)$$

For reductive groups, this was proved in [Sol25, Proposition 2.2], however the proof refers to the explicit formula for the μ -function which is known for reductive groups, but at this point we do not yet have it available for covering groups. Therefore, we have to be a bit careful. Since μ^M and $\mu^{\tilde{M}}$ are both rational functions, it suffices to show that they have the same poles and the same zeros. The poles have been taken care of in the proof of [Sol25, Proposition 2.2]; the proof generalizes to covering groups, since it only uses the definition $\mu(\sigma \otimes \chi) := j(\sigma \otimes \chi)^{-1}$ (up to a scalar), rather than the explicit formula for μ . Showing that the μ -functions have the same zeros can be done in a similar way. So the claim in Equation (4.4.1) holds.

Thus $\mu(\sigma \otimes \chi)$ only depends on $\chi|_{\tilde{M}}$ (recall that $\tilde{M} = M \cap G_{\text{der}}$). Since $\mu(\sigma \otimes \chi)$ is invariant under $X_{\text{nr}}(M, \sigma)$, it follows that $\mu(\sigma \otimes \chi)$ only depends on $\chi|_{M_\sigma^2 \cap G_{\text{der}}}$. There is a canonical homomorphism between the quotients

$$\mathbb{Z} \cong (M_\sigma^2 \cap G_{\text{der}})/(M^1 \cap G_{\text{der}}) \rightarrow (M_\sigma^2 \cap G^1)/M^1 \cong \mathbb{Z}.$$

By the considerations above, $\mu(\sigma \otimes \chi)$ only depends on $\chi(m')$, for m' a generator of $(M_\sigma^2 \cap G_{\text{der}})/(M^1 \cap G_{\text{der}})$. The image of m' in $(M_\sigma^2 \cap G_1)/M^1$ is an integer power of m , where m is a generator of $(M_\sigma^2 \cap G_1)/M^1$. So $\chi(m')$ is an integer power of $\chi(m)$. Since $\mu(\sigma \otimes \chi)$ only depends on $\chi(m')$, it only depends on $\chi(m)$. $\square$

Recall that since M is maximal, we consider only one root α . Let $h_\alpha^\vee$ be the unique generator of $(M_\sigma^2 \cap G^1)/M^1 \cong \mathbb{Z}$ such that $|\alpha(h_\alpha^\vee)|_F > 1$. Then by Lemma 4.4.1, we may view $\mu(\sigma \otimes \chi) = \mu(\sigma, z)$ as a function in the variable $z := \chi(h_\alpha^\vee) \in \mathbb{C}^\times$. Suppose $\tau = \sigma \otimes \chi_\tau \in \text{Irr}(M)_{[M, \sigma]}$ is a different choice of representative in the inertial equivalence class of σ for M . Then for $\chi \in X_{\text{nr}}(M)$, $\mu(\tau \otimes \chi) = \mu(\sigma \otimes \chi_\tau \chi)$. Hence

$$\mu(\tau, z) = \mu(\sigma, q_\tau z), \text{ where } q_\tau = \chi_\tau(h_\alpha^\vee) \in \mathbb{C}^\times. \quad (4.4.2)$$

For G' , it was shown in [Sil80, Theorem 1.6] that there exists a unitary $\sigma_0 \in \text{Irr}(M')_{[M', \sigma]}$ such that as a function in $z = \chi(h_\alpha^\vee)$, $\mu = \mu_\alpha$ has the form

$$\mu(\sigma_0 \otimes \chi) = \mu(\sigma_0, z) = c_\mu \cdot \frac{(1-z)(1-z^{-1})}{(1-qz)(1-qz^{-1})} \frac{(1+z)(1+z^{-1})}{(1+q'z)(1+q'z^{-1})}, \quad (4.4.3)$$

where $c_\mu \in \mathbb{R}_{>0}$ and $q, q' \in \mathbb{R}_{\geq 1}$. The possibility that μ is independent of χ is included in this formula; this happens when $q = q' = 1$.

Our goal is to give an alternative proof of this result, which generalizes to covering groups. The idea is as follows. Since $\mu(\sigma \otimes \chi) = \mu(\sigma, z)$ is a rational function on $\mathbb{C}^\times$, we may also view it as a rational function on the projective line $\mathbb{P}^1(\mathbb{C})$, by considering the limits $\mu(\sigma, \infty) := \lim_{z \rightarrow \infty} \mu(\sigma, z)$ and $\mu(\sigma, 0) := \lim_{z \rightarrow 0} \mu(\sigma, z)$. If $\mu(\sigma, z)$ is not constant, then there exists a unitary $\sigma_0 \in \text{Irr}(M)_{[M, \sigma]}$ with $\mu(\sigma_0) = 0$. The only other possibility for a zero of $\mu(\sigma_0, z)$ in $\mathbb{C}^\times$ will be at $z = -1$ and all zeros are of multiplicity 2. Using techniques concerning Hermitian and unitary representations (see [Ren10, §IV] for definitions), we will then locate two poles in $\mathbb{R}^\times$ for every zero of $\mu(\sigma_0, z)$. Moreover, we will show that $\mu(\sigma_0, z)$ can not be zero at $z = \infty$ and $z = 0$. Combining all these results, we obtain the formula as in (4.4.3).

The result below is similar to the reductive case and will help us to give a description of the supercuspidal representations of M .

Lemma 4.4.2. *Let $p_G : G \rightarrow G'$ be a finite central covering. Let $G^1 = p_G^{-1}(G'^1)$, where G'^1 is the subgroup of G' generated by all compact subgroups. Then $G^1 \cdot Z(G)$ has finite index in G .*

Proof. Let $g \in p_G^{-1}(Z(G'))$. Consider the commutator map

$$[g, \cdot] : G \rightarrow \ker(p_G) \subset Z(G), \quad x \mapsto gxg^{-1}x^{-1}.$$

Let $k = |\ker(p_G)|$. Then since $gxg^{-1} = [g, x]x$, we obtain

$$g^k x g^{-k} x^{-1} = g^{k-1} [g, x] x g^{1-k} x^{-1} = \dots = [g, x]^k x x^{-1} = [g, x]^k = 1.$$

Therefore, $g^k \in Z(G)$ for all $g \in p_G^{-1}(Z(G'))$. Recall that G'/G'^1 is a finitely generated free abelian group and that $Z(G')/(Z(G') \cap G'^1)$ is a subgroup of

finite index in G'/G'^1 . Hence the group of k -th powers $Z(G')^{(k)}/(Z(G')^{(k)} \cap G'^1)$ also has finite index in G'/G'^1 . Applying p_G^{-1} , it follows that $G^1 \cdot Z(G)$ has finite index in G . $\square$

Let $\text{Irr}(M)_{\text{cusp}}^{\text{unit}}$ be the set of unitary supercuspidal representations of M .

Lemma 4.4.3. *There is a bijection*

$$\text{Irr}(M)_{\text{cusp}}^{\text{unit}} \times \text{Hom}(M, \mathbb{R}_{>0}) \rightarrow \text{Irr}(M)_{\text{cusp}}.$$

Proof. First, let us show that there is a bijection

$$\text{Hom}(Z(M), \mathbb{R}_{>0}) \xrightarrow{\sim} \text{Hom}(M, \mathbb{R}_{>0}).$$

For this, we need to show that every $\chi \in \text{Hom}(Z(M), \mathbb{R}_{>0})$ extends uniquely to a character in $\text{Hom}(M, \mathbb{R}_{>0})$. Since $Z(M)$ is abelian, it has a maximal compact subgroup $Z(M)_{\text{cpt}}$. Note that χ factors through $Z(M)/Z(M)_{\text{cpt}}$, because the image of $Z(M)_{\text{cpt}}$ must be a compact subgroup of $\mathbb{R}_{>0}$, i.e. $\{1\}$. By Lemma 4.4.2, $M^1 Z(M)$ has finite index in M . Hence,

$$Z(M)/Z(M)_{\text{cpt}} \hookrightarrow M/M^1 \cong \mathbb{Z}^r,$$

is a subgroup of finite index. So $Z(M)/Z(M)_{\text{cpt}} \cong \mathbb{Z}^r$ is a free abelian subgroup of the same rank. So for $x \in M/M^1$, let n be a positive integer such that $n \cdot x \in Z(M)/Z(M)_{\text{cpt}}$. Define $\chi(x) := \chi(n \cdot x)^{1/n}$, which is well-defined because $\mathbb{R}_{>0}$ has unique n -th roots and it does not depend on the choice of n . It is a group homomorphism, since

$$\begin{aligned} \chi(n \cdot x)^{\frac{1}{n}} \chi(m \cdot y)^{\frac{1}{m}} &= \chi(nm \cdot x)^{\frac{1}{nm}} \chi(nm \cdot y)^{\frac{1}{nm}} = (\chi(nm \cdot x) \chi(nm \cdot y))^{\frac{1}{nm}} \\ &= \chi((nm \cdot x)(nm \cdot y))^{\frac{1}{nm}} = \chi(nm \cdot xy)^{\frac{1}{nm}}. \end{aligned}$$

This describes how $\chi : Z(M)/Z(M)_{\text{cpt}} \rightarrow \mathbb{R}_{>0}$ extends uniquely to a character $\chi : M/M^1 \rightarrow \mathbb{R}_{>0}$. Now we can explicitly describe the bijection. It is given by

$$\text{Irr}(M)_{\text{cusp}}^{\text{unit}} \times \text{Hom}(M, \mathbb{R}_{>0}) \rightarrow \text{Irr}(M)_{\text{cusp}}, (\pi, \chi) \mapsto \pi \otimes \chi,$$

and its inverse is given by $\sigma \mapsto (\sigma \otimes |\text{cc}(\sigma)|^{-1}, |\text{cc}(\sigma)|)$, where $\text{cc}(\sigma)$ is the central character of σ , and we extend $|\text{cc}(\sigma)|$ to a character of M as above. By [Ren10, p. 116], [FP22, Theorem 6.2], a smooth irreducible representation π of M is supercuspidal and unitary if and only if it is supercuspidal and its central character $\text{cc}(\pi)$ is unitary. That explains why the representation $\sigma \otimes |\text{cc}(\sigma)|^{-1}$ is unitary. $\square$

Remark 4.4.4. Although Lemma 4.4.3 is written for supercuspidal representations, it holds more generally for representations that are essentially square integrable, see Definition 2.2.9.

By Lemma 4.4.3, there exists a unitary $\tilde{\sigma} \in \text{Irr}(M)_{[M, \sigma]}$. Knowing $\mu(\tilde{\sigma}, z)$ is equivalent to knowing $\mu(\sigma, z)$, by (4.4.2).

Suppose that σ is unitary, then by [Ren10, §IV.2.3], the parabolically induced representation $I_P^G(\sigma)$ is unitary as well. We can use the adjointness relation from Section 4.3 to show that μ must be real and non-negative on the unit circle.

Lemma 4.4.5. *Let σ be unitary. Then the function $\mu(\sigma, z)$ has all its values in $\mathbb{R}_{\geq 0}$ for z on the unit circle $S^1 = \{e^{i\varphi} \mid \varphi \in \mathbb{R}\} \subset \mathbb{C}^\times$.*

Proof. We claim that for $z = e^{i\varphi} \in S^1$, there always exists a unitary character $\chi_z \in X_{\text{nr}}(M)$ such that $\chi_z(h_\alpha^\vee) = z$. Indeed, recall that $X_{\text{nr}}(M) = \text{Hom}(M/M^1, \mathbb{C}^\times)$. Moreover, M/M^1 is a free abelian group of finite rank, and $h_\alpha^\vee$ is a generator of $M_\sigma^2 \cap G^1/M^1$, which is a free rank 1 subgroup of M/M^1 . Therefore, we can choose a basis of M/M^1 , such that $h_\alpha^\vee$ is an integer multiple of one of these basis elements. So write $h_\alpha^\vee = n \cdot b$ with $n \in \mathbb{Z}$ and b a basis element of M/M^1 . Then we can define $\chi_z(b) = e^{i\varphi/n}$, and let χ_z be trivial on the other basis elements of M/M^1 . This gives us a unitary character with

$$\chi_z(h_\alpha^\vee) = \chi_z(n \cdot b) = \chi_z(b)^n = (z^{1/n})^n = z.$$

Since we chose σ to be unitary, $\sigma \otimes \chi_z$ is unitary as well, and the adjunction formula from Theorem 4.3.6.a applies with $\overline{\sigma \otimes \chi_z}^\vee$ replaced by $\sigma \otimes \chi_z$. Since $\sigma \otimes \chi_z$ is unitary, we may assume that the induced pairing on $I_P^G(\sigma \otimes \chi_z)$ (see (4.3.1)) is positive definite. So for $f \in I_P^G(\sigma \otimes \chi_z)$, we have

$$\begin{aligned} j(\sigma \otimes \chi_z)\langle f, f \rangle &= \langle f, J_{P|\bar{P}}(\sigma \otimes \chi_z) \circ J_{\bar{P}|P}(\sigma \otimes \chi_z)f \rangle \\ &= \langle J_{\bar{P}|P}(\sigma \otimes \chi_z)f, J_{\bar{P}|P}(\sigma \otimes \chi_z)f \rangle \in \mathbb{R}_{\geq 0} \cup \{\infty\}. \end{aligned} \quad (4.4.4)$$

Since $J_{P|\bar{P}}(\sigma \otimes \chi_z)$ is nonzero, (4.4.4) shows that $j(\sigma \otimes \chi_z) \neq 0$. Thus $z \mapsto j(\sigma \otimes \chi_z)$ is a continuous map from S^1 to $\mathbb{R}_{>0} \cup \{\infty\}$. We obtain that $\mu(\sigma, z) = \mu(\sigma \otimes \chi_z) = j(\sigma \otimes \chi_z)^{-1} \in \mathbb{R}_{\geq 0}$. $\square$

Using the result above, we can now show that for every zero (resp. pole) of μ , there must be another zero (resp. pole) of μ , of the same multiplicity.

Lemma 4.4.6. *Let σ be unitary. Then the function $\mu(\sigma, z)$ has the form*

$$\mu(\sigma, z) = c_\mu \prod_i (z - \lambda_i)^{n_i} (z^{-1} - \overline{\lambda_i})^{n_i},$$

where $c_\mu \in \mathbb{R}_{>0}$, $\lambda_i \in \mathbb{C}$ and $n_i \in \mathbb{Z}$.

Proof. By Lemma 4.4.5, we have that $\mu(\sigma, z) \in \mathbb{R}_{\geq 0}$ for all z on the unit circle. Since $\mu(\sigma, z)$ is a nonzero complex rational function in the variable z , we can write

$$\mu(\sigma, z) = c_\mu \prod_i (z - \lambda_i)^{n_i},$$

where $c_\mu \in \mathbb{C}^\times$, $\lambda_i \in \mathbb{C}$, $n_i \in \mathbb{Z}$. For every negative power $n_i \in \mathbb{Z}_{<0}$ appearing in the factorization above, we may multiply $\mu(\sigma, z)$ by $(z - \lambda_i)^{-n_i} (z^{-1} - \overline{\lambda_i})^{-n_i}$, to obtain a function of the form

$$\tilde{\mu}(\sigma, z) = c_\mu \prod_i (z - \lambda_i)^{n_i} \prod_j (z^{-1} - \gamma_j)^{\ell_j},$$

where $\lambda_i, \gamma_j \in \mathbb{C}$ and $n_i, \ell_j \in \mathbb{Z}_{>0}$. Note that $\tilde{\mu}(\sigma, z)$ is again real and non-negative on the unit circle, and it suffices to show the statement of the lemma

for $\tilde{\mu}(\sigma, z)$. For $\varphi \in \mathbb{R}$, we define $f(\varphi) := \tilde{\mu}(\sigma, e^{i\varphi})$. Then we have $f(\varphi) = \overline{f(\varphi)}$ for all $\varphi \in \mathbb{R}$, i.e.

$$c_\mu \prod_i (e^{i\varphi} - \lambda_i)^{n_i} \prod_j (e^{-i\varphi} - \gamma_j)^{\ell_j} = \overline{c_\mu} \prod_i (e^{-i\varphi} - \overline{\lambda_i})^{n_i} \prod_j (e^{i\varphi} - \overline{\gamma_j})^{\ell_j}.$$

Both sides in the equation are trigonometric polynomials, which have a unique factorization. So the set of $\overline{\lambda_i}$'s, where each $\overline{\lambda_i}$ appears n_i times, equals the set of γ_j 's, where each γ_j appears ℓ_j times. It follows that

$$\tilde{\mu}(\sigma, z) = c_\mu \prod_i (z - \lambda_i)^{n_i} (z^{-1} - \overline{\lambda_i})^{n_i}.$$

Since μ is real and non-negative on the unit circle and not identically zero, we see that $c_\mu \in \mathbb{R}_{>0}$, which completes the proof. $\square$

4.4.2 An aspect of the Iwasawa decomposition

In this subsection, we discuss an aspect of the Iwasawa decomposition. This will be necessary to study the limit of the intertwining operator $J_{\bar{P}|P}(\sigma \otimes \chi)$ and the function $\mu(\sigma, z)$ at $z = \infty$. We use the notations from Section 4.3. Recall the Iwasawa decomposition $G' = P'K' = K'P'$ from [KP23, Theorem 5.3.4]. It lifts to the Iwasawa decomposition

$$G = PK = KP. \tag{4.4.5}$$

As $P = U_P \rtimes M$, this means that for each $g \in G$ there exist $k_P(g) \in K$, $m_P(g) \in M$ and $u_P(g) \in U_P$ such that $g = u_P(g)m_P(g)k_P(g)$. The decomposition of g along (4.4.5) is unique up to $P \cap K$. Hence $k_P(g)$, $m_P(g)$ and $u_P(g)$ are unique up to, respectively, $P \cap K$, $M \cap K$ and $U_P \cap K$.

For any subgroup H of G or G we will write H_x for the stabilizer of $x \in \mathcal{B}(G')$ in H . In particular $K' = G'_x$ and $K = G_x$. We choose a positive system $\Phi(G, S)^+$ in $\Phi(G, S)$, such that $\Phi(U_P, S) \subset \Phi(G, S)^+$. Let U^+ (respectively U^-) be the subgroup of G generated by the root subgroups U_β with β in $\Phi(G, S)^+$ (respectively in $\Phi(G, S)^- = -\Phi(G, S)^+$).

We note that the normalizer $N_G(S)$ equals $p_G^{-1}(N_{G'}(S'))$.

Theorem 4.4.7. *The product map*

$$\prod_{\beta \in \Phi(G, S)^\pm} U_{\beta, x} \longrightarrow U_x^\pm = U^\pm \cap G_x$$

is a homeomorphism. Further, G_x can be obtained as

$$G_x = U_x^- U_x^+ N_G(S)_x = U_x^+ U_x^- N_G(S)_x.$$

Proof. For G' , the theorem is the combination of [BT72, Proposition 6.4.9] and the proof of [KP23, Lemma 7.7.3]. The theorem for G follows from that for G' by the covering $p_G : G \rightarrow G'$ and the canonical lifting of each root subgroup of G' to G . $\square$

We will also work with the Lie algebra $\text{Lie}(G')$, which can be defined either as the F -points of $\text{Lie}(G')$ or as the tangent space at e of G considered as manifold over the local field F . Since $p_G : G \rightarrow G'$ is a finite covering, we can identify $\text{Lie}(G')$ (defined using F -manifolds) with $\mathfrak{g} = \text{Lie}(G)$. Similarly $\text{Lie}(G'_x)$ can be viewed either as the $\mathfrak{o}_F$ -points of the Lie algebra of the $\mathfrak{o}_F$ -scheme $\mathcal{G}'_x$, or as the Lie algebra of G'_x in the sense of manifolds over $\mathfrak{o}_F$. Then $\mathfrak{g}_x = \text{Lie}(G_x)$ can be identified with $\text{Lie}(G'_x)$, and it is an $\mathfrak{o}_F$ -lattice in $\mathfrak{g}$. We write $\mathfrak{u}_\beta = \text{Lie}(U_\beta)$ and

$$\mathfrak{z} = \text{Lie}(Z_{G'}(S')) = \text{Lie}(N_{G'}(S')) = \text{Lie}(N_G(S)).$$

By $\text{Lie}(H)_x$ we mean always $\text{Lie}(H_x)$. With these notations, Theorem 4.4.7 implies that

$$\begin{aligned} \mathfrak{g}_x &= \bigoplus_{\beta \in \Phi(G,S)^-} \mathfrak{u}_{\beta,x} \oplus \bigoplus_{\beta \in \Phi(G,S)^+} \mathfrak{u}_{\beta,x} \oplus \mathfrak{z}_x \\ &= \bigoplus_{\beta \in \Phi(U_{\bar{P}},S)} \mathfrak{u}_{\beta,x} \oplus \left(\bigoplus_{\beta \in \Phi(M,S)^-} \mathfrak{u}_{\beta,x} \oplus \bigoplus_{\beta \in \Phi(M,S)^+} \mathfrak{u}_{\beta,x} \oplus \mathfrak{z}_x \right) \\ &\quad \oplus \bigoplus_{\beta \in \Phi(U_P,S)} \mathfrak{u}_{\beta,x} \\ &= \text{Lie}(U_{\bar{P},x}) \oplus \text{Lie}(M_x) \oplus \text{Lie}(U_{P,x}). \end{aligned} \tag{4.4.6}$$

Proposition 4.4.8. *Let $\bar{u} \in U_{\bar{P}}$ and write it as $\bar{u} = umk$ with $u = u_P(\bar{u}) \in U_P$, $m = m_P(\bar{u}) \in M$ and $k = k_P(\bar{u}) \in G_x$. Then $\delta_P(m_P(\bar{u})) \leq 1$.*

Proof. Since $k \in G_x$, we have

$$\bar{u}G_x\bar{u}^{-1} = G_{\bar{u}x} = G_{umkx} = G_{umx} = uG_{mx}u^{-1}.$$

By (4.4.6), the Lie algebra of this group is

$$\begin{aligned} \mathfrak{g}_{\bar{u}x} &= \text{Ad}(u)\mathfrak{g}_{mx} = \text{Ad}(u)(\text{Lie}(U_{\bar{P},mx}) \oplus \text{Lie}(M_{mx}) \oplus \text{Lie}(U_{P,mx})) \\ &= \text{Ad}(u)\text{Lie}(U_{\bar{P},mx}) \oplus \text{Ad}(u)\text{Lie}(P_{mx}). \end{aligned} \tag{4.4.7}$$

The decomposition $\mathfrak{g} = \text{Lie}(U_{\bar{P}}) \oplus \mathfrak{p}$ gives a projection $\text{pr}_{\mathfrak{g}/\mathfrak{p}} : \mathfrak{g} \rightarrow \text{Lie}(U_{\bar{P}})$ with kernel $\mathfrak{p} = \text{Lie}(P)$. Clearly $\text{pr}_{\mathfrak{g}/\mathfrak{p}}(\mathfrak{g}_{\bar{u}x})$ contains $\text{Lie}(U_{\bar{P},\bar{u}x}) = \text{Ad}(\bar{u})\text{Lie}(U_{\bar{P},x})$. From (4.4.7) we obtain, using $\text{Ad}(u)\text{Lie}(P_{mx}) \subset \mathfrak{p}$:

$$\text{Ad}(\bar{u})\text{Lie}(U_{\bar{P},x}) \subset \text{pr}_{\mathfrak{g}/\mathfrak{p}}(\mathfrak{g}_{\bar{u}x}) = \text{pr}_{\mathfrak{g}/\mathfrak{p}}(\text{Ad}(u)\text{Lie}(U_{\bar{P},mx})). \tag{4.4.8}$$

The Haar measures on the groups $G, M, U_P, P, U_{\bar{P}}, \bar{P}$ induce measures on their Lie algebras. By the unimodularity of the unipotent group $U_{\bar{P}}$, the lattice $\text{Ad}(\bar{u})\text{Lie}(U_{\bar{P},x})$ in $\text{Lie}(U_{\bar{P}})$ has the same volume as $\text{Lie}(U_{\bar{P},x})$.

Since u is unipotent, so is the F -linear operator $\text{Ad}(u) : \mathfrak{g} \rightarrow \mathfrak{g}$. Moreover $\text{Ad}(u)$ stabilizes $\mathfrak{p}$, so there exist a basis of $\mathfrak{p}$ and a basis of $\text{Lie}(U_{\bar{P}})$ such that the matrix of $\text{Ad}(u)$ with respect to the combined basis of $\mathfrak{g}$ is unipotent and upper triangular. It follows that the matrix of

$$\text{pr}_{\mathfrak{g}/\mathfrak{p}} \circ \text{Ad}(u) : \text{Lie}(U_{\bar{P}}) \rightarrow \text{Lie}(U_{\bar{P}}) \tag{4.4.9}$$

with respect to the chosen basis is also unipotent and upper triangular. Therefore (4.4.9) preserves volumes. Now we can interpret (4.4.8) in terms of volumes:

$$\mathrm{vol}(\mathrm{Lie}(U_{\bar{P},x})) \leq \mathrm{vol}(\mathrm{Lie}(U_{\bar{P},mx})).$$

This can be rewritten as

$$\begin{aligned} 1 &\leq \frac{\mathrm{vol}(\mathrm{Lie}(U_{\bar{P},mx}))}{\mathrm{vol}(\mathrm{Lie}(U_{\bar{P},x}))} = \frac{\mathrm{vol}(\mathrm{Ad}(m)\mathrm{Lie}(U_{\bar{P},x}))}{\mathrm{vol}(\mathrm{Lie}(U_{\bar{P},x}))} \\ &= |\det(\mathrm{Ad}(m) : \mathrm{Lie}(U_{\bar{P}}) \rightarrow \mathrm{Lie}(U_{\bar{P}}))|_F. \end{aligned} \quad (4.4.10)$$

The modular function $\delta_{\bar{P}}$ of $\bar{P}$ equals $\delta_{\bar{P}'} \circ p_G$, because $\ker p_G$ is finite. By [Sil80, Lemma 1.2.1.1], we have

$$\delta_{\bar{P}'}(m') = |\det(\mathrm{Ad}(m') : \mathrm{Lie}(U_{\bar{P}}) \rightarrow \mathrm{Lie}(U_{\bar{P}}))|_F \quad \text{for all } m' \in M'. \quad (4.4.11)$$

The adjoint representation of $\bar{P}$ factors through $p_G : \bar{P} \rightarrow \bar{P}'$, so the inequality (4.4.10) says that $\delta_{\bar{P}}(m) \geq 1$. Finally, we use that $\delta_{\bar{P}} = \delta_{\bar{P}'}^{-1}$ [Ren10, Lemme V.5.4]. $\square$

One may reformulate the conclusion of Proposition 4.4.8 as: conjugation by $m_P(\bar{u})$ contracts U_P and inflates $U_{\bar{P}}$. Thus the image of $\bar{U}$ in M under the Iwasawa decomposition (well-defined up to $M \cap G_x$) sits inside M in asymmetric way, like a semigroup but certainly not like a group.

Lemma 4.4.9. *In the setting of Proposition 4.4.8, suppose that P is a maximal proper parabolic subgroup of G and that $\delta_P(m_P(\bar{u})) = 1$. Then $m_P(\bar{u}) \in M^1$.*

Proof. By construction $M^1 = p_G^{-1}(M'^1)$. As the Iwasawa decomposition of G is lifted from that for G' , it suffices to prove the lemma with G' instead of G . To use that simplification without many extra primes, we assume in the remainder of the proof that $G = G'$.

The derived group $G_{\mathrm{der}} = \mathcal{G}_{\mathrm{der}}(F)$ also has an Iwasawa decomposition, namely

$$G_{\mathrm{der}} = U_P(M \cap G_{\mathrm{der}})G_{\mathrm{der},x}. \quad (4.4.12)$$

This is a restriction of the Iwasawa decomposition $G = U_P M G_x$. As $\bar{u} \in G_{\mathrm{der}}$, we can take $k_P(\bar{u})$, $m_P(\bar{u})$ and $u_P(\bar{u})$ in G_{der} as well, using (4.4.12).

By assumption $P \cap G_{\mathrm{der}} = (M \cap G_{\mathrm{der}}) \ltimes U_P$ is a maximal proper parabolic subgroup of the semisimple group G_{der} . Then the algebraic group $Z(M \cap G_{\mathrm{der}})^\circ$ is a one-dimensional F -split torus. It follows that $(M \cap G_{\mathrm{der}})/(M \cap G_{\mathrm{der}})^1$ is a free abelian group of rank one, so isomorphic to $\mathbb{Z}$.

On the right hand side of (4.4.11) we can replace $\bar{P}$ by P or by $P \cap G_{\mathrm{der}}$, both give the same expression. Hence the modular functions δ_P and $\delta_{P \cap G_{\mathrm{der}}}$ coincide on $M \cap G_{\mathrm{der}}$. As $\delta_{P \cap G_{\mathrm{der}}}$ is trivial on $(M \cap G_{\mathrm{der}})^1$, it defines a character

$$\mathbb{Z} \cong (M \cap G_{\mathrm{der}})/(M \cap G_{\mathrm{der}})^1 \rightarrow \mathbb{R}_{>0}^\times. \quad (4.4.13)$$

The restrictions of elements of $\Phi(U_P, S)$ to $Z(M \cap G_{\mathrm{der}})$ are positive powers of a single nontrivial character α . Then (4.4.11) shows that $\delta_{P \cap G_{\mathrm{der}}} |_Z(M \cap G_{\mathrm{der}})$

is a positive power of $|\alpha|$. In particular $\delta_{P \cap G_{\text{der}}}$ is nontrivial, so regarded as a character of $\mathbb{Z}$ via (4.4.13) it is injective. The assumptions of the lemma and Proposition 4.4.8 entail that

$$\delta_{P \cap G_{\text{der}}}(m_P(\bar{u})) = \delta_P(m_P(\bar{u})) = 1.$$

Therefore $m_P(\bar{u})$ lies in the kernel of (4.4.13), which means that

$$m_P(\bar{u}) \in (M \cap G_{\text{der}})^1 \subset M^1.$$

That proves the lemma for one choice of $m_P(\bar{u})$ in the Iwasawa decomposition of G . But $m_P(\bar{u})$ is unique up to the compact group $M \cap G_x$, so $m_P(\bar{u}) \in M^1$ for any other choice of $m_P(\bar{u})$ as well. $\square$

4.4.3 The limit at infinity

In this subsection, we study the limit of the intertwining operator $J_{\bar{P}|P}(\sigma \otimes \chi)$ and the function $\mu(\sigma, z)$ when $z = \chi(h_\alpha^\vee)$ goes to infinity.

By (4.4.11), the modulus character δ_P only takes values in the integer powers of q . By Proposition 4.4.8, $\delta_P(m_P(\bar{u})) \leq 1$ for all $\bar{u} \in U_{\bar{P}}$. Hence, $U_{\bar{P}}$ is the disjoint union of the subsets

$$U_{\bar{P}}(n) := \{\bar{u} \in U_{\bar{P}} : \delta_P(m_P(\bar{u})) = q^{-n}\},$$

where n runs over all non-negative integers. The set

$$U_{\bar{P}}(0) = \{\bar{u} \in U_{\bar{P}} : \delta_P(m_P(\bar{u})) = 1\}$$

is compact by [Wal03, Lemme II.3.4] and $m_P(U_{\bar{P}}(0)) \subset M^1$ by Lemma 4.4.9.

Lemma 4.4.10. *There exist $C > 0$, $R > 0$ and $d \in \mathbb{N}$ such that for every $n \geq 1$,*

$$\text{vol}(U_{\bar{P}}(n)) \leq Cq^{nR}n^d.$$

Proof. In [Wal03, Lemme II.4.1], this statement (with $R = 1/2$) is proven for $\delta_0 := \delta_{P_0}$, where P_0 is a minimal parabolic subgroup of G . So let $P_0 = M_0U_P$ with $M_0 \subset M$ a minimal Levi subgroup. Consider the Iwasawa decomposition $G = P_0K$, and for every $\bar{u} \in U_{\bar{P}}$, write

$$\bar{u} = u_0(\bar{u})m_0(\bar{u})k_0(\bar{u}),$$

where $u_0(\bar{u}) \in U_P$, $m_0(\bar{u}) \in M_0$ and $k_0(\bar{u}) \in K$. Define, for all $n \geq 1$, the subsets

$$U_{\bar{P},0}(n) := \{\bar{u} \in U_{\bar{P}} : \delta_0(m_0(\bar{u})) = q^{-n}\}.$$

Then, by [Wal03, Lemme II.4.1], there exist $C_0 > 0$ and $d_0 \in \mathbb{N}$ such that for all $n \geq 1$,

$$\text{vol}(U_{\bar{P},0}(n)) \leq C_0q^{n/2}n^{d_0}.$$

By [Wal03, Lemme II.3.4] and Proposition 4.4.8, there exist $C_1 > 0$, $r > 0$, such that for all $\bar{u} \in U_{\bar{P}}$,

$$C_1\delta_P(m_P(\bar{u}))^r \leq \delta_0(m_0(\bar{u})) \leq 1.$$

Suppose $\bar{u} \in U_{\bar{P}}(n)$, i.e. $\delta_P(m_P(\bar{u})) = q^{-n}$. Then by the inequalities above, we get

$$C_1 q^{-nr} \leq \delta_0(m_0(\bar{u})) \leq 1.$$

Write $C_1 = q^{-b}$ for some $b \in \mathbb{R}$. Let $N \in \mathbb{N}$ such that $N \geq r + \max(b, 0)$. Then for all $n \geq 1$,

$$q^{-nN} \leq q^{-nr-b} = C_1 q^{-nr} \leq \delta_0(m_0(\bar{u})) \leq 1.$$

Hence, $\bar{u} \in U_{\bar{P}}(n)$ can only be contained in one of the sets

$$U_{\bar{P},0}(nN), U_{\bar{P},0}(nN-1), \dots, U_{\bar{P},0}(0).$$

Each of the sets above has volume bounded above by $C_0 q^{nN/2} (nN)^{d_0}$. Thus $\text{vol}(U_{\bar{P}}(n)) \leq (nN+1)C_0 q^{nN/2} (nN)^{d_0} \leq n(N+1)C_0 q^{nN/2} (nN)^{d_0} = C q^{nR} n^d$, where $C = (N+1)C_0 N^{d_0}$, $R = N/2$ and $d = d_0 + 1$. $\square$

For $\chi \in X_{\text{nr}}(M)$, we define the variable $r(\chi) \in \mathbb{R}$ determined by

$$|\chi|_{M \cap G_{\text{der}}} = \delta_P^{r(\chi)}|_{M \cap G_{\text{der}}}.$$

When this variable goes to infinity, the operator $J_{\bar{P}|P}(\sigma \otimes \chi)$ converges to an operator which is independent of χ , as we will show below.

Proposition 4.4.11. *For $f \in I_{P \cap K}^K(\sigma)$, $g \in K$,*

$$\lim_{r(\chi) \rightarrow \infty} J_{\bar{P}|P}(\sigma \otimes \chi)f(g) = \int_{U_{\bar{P}}(0)} f(\bar{u}g) d\bar{u},$$

and this integral is independent of χ . In particular, $\lim_{r(\chi) \rightarrow \infty} J_{\bar{P}|P}(\sigma \otimes \chi)$ exists.

Proof. For $\chi \in X_{\text{nr}}(M)$, $f \in I_P^G(\sigma \otimes \chi) \cong I_{P \cap K}^K(\sigma)$ and $g \in K$, we have that

$$J_{\bar{P}|P}(\sigma \otimes \chi)f(g) = \int_{U_{\bar{P}}} f(\bar{u}g) d\bar{u} = \int_{U_{\bar{P}}(0)} f(\bar{u}g) d\bar{u} + \int_{\cup_{n \geq 1} U_{\bar{P}}(n)} f(\bar{u}g) d\bar{u}.$$

Recall that for $\bar{u} \in U_{\bar{P}}$, using the Iwasawa decomposition $G = PK$ we write

$$\bar{u} = u_P(\bar{u})m_P(\bar{u})k_P(\bar{u}),$$

where $u_P(\bar{u}) \in U_P$, $m_P(\bar{u}) \in M$ and $k_P(\bar{u}) \in K$. Then

$$f(\bar{u}g) = \chi \delta_P^{1/2}(m_P(\bar{u})) \sigma(m_P(\bar{u})) f(k_P(\bar{u})g).$$

For $\bar{u} \in U_{\bar{P}}(0)$, we have that $\delta_P(m_P(\bar{u})) = 1$. Then by Lemma 4.4.9, $m_P(\bar{u}) \in M^1$, so also $\chi(m_P(\bar{u})) = 1$. Hence

$$\int_{U_{\bar{P}}(0)} f(\bar{u}g) d\bar{u} = \int_{U_{\bar{P}}(0)} \sigma(m_P(\bar{u})) f(k_P(\bar{u})g) d\bar{u},$$

which does not depend on χ , since $k_P(\bar{u})g \in K$ and $f \in I_{P \cap K}^K(\sigma)$.

Now, let us analyze the integral

$$\int_{\cup_{n \geq 1} U_{\bar{P}}(n)} f(\bar{u}g) d\bar{u},$$

in particular we want to show that this integral converges to zero in the limit $r(\chi) \rightarrow \infty$. For this, we follow the proof of [Wal03, Théorème IV.1.1]. This proof shows that we can reduce the problem to showing that the integral

$$\int_{\cup_{n \geq 1} U_{\bar{P}}(n)} |\chi \delta_P^{r+1/2}(m_P(\bar{u}))| d\bar{u} = \int_{\cup_{n \geq 1} U_{\bar{P}}(n)} \delta_P^{r(\chi)+r+1/2}(m_P(\bar{u})) d\bar{u},$$

where $r \in \mathbb{R}$ is some fixed number, converges to zero as $r(\chi) \rightarrow \infty$. By Lemma 4.4.10, there exist $C > 0$, $R > 0$ and $d \in \mathbb{N}$ such that for each $n \geq 1$,

$$\text{vol}(U_{\bar{P}}(n)) \leq C q^{nR} n^d.$$

Hence, we obtain that

$$\begin{aligned} \int_{\cup_{n \geq 1} U_{\bar{P}}(n)} \delta_P^{r(\chi)+r+1/2}(m_P(\bar{u})) d\bar{u} &\leq \sum_{n \geq 1} q^{-n(r(\chi)+r+1/2)} C q^{nR} n^d \\ &= C \sum_{n \geq 1} q^{-n(r(\chi)+r')} n^d, \end{aligned}$$

where $r' \in \mathbb{R}$ is independent of n . This sum converges for $r(\chi)$ large enough, and it converges to zero when $r(\chi) \rightarrow \infty$. $\square$

Recall that for every χ , we may view $J_{\bar{P}|P}(\sigma \otimes \chi)$ as a linear operator

$$J_{\bar{P}|P}(\sigma \otimes \chi) : I_{P \cap K}^K(V_\sigma) \rightarrow I_{\bar{P} \cap K}^K(V_\sigma),$$

i.e. these are all linear operators on the same vector space, with the same range. If $z = \chi(h_\alpha^\vee) \rightarrow \infty$, then $r(\chi) \rightarrow \infty$. So by Proposition 4.4.11, $\lim_{z \rightarrow \infty} J_{\bar{P}|P}(\sigma \otimes \chi)$ exists and is given by

$$\lim_{z \rightarrow \infty} J_{\bar{P}|P}(\sigma \otimes \chi) = \int_{U_{\bar{P}}(0)} \lambda(\bar{u})^{-1} d\bar{u} : I_{P \cap K}^K(V_\sigma) \rightarrow I_{\bar{P} \cap K}^K(V_\sigma).$$

One can also directly prove that the image of this operator lies in $I_{\bar{P} \cap K}^K(V_\sigma)$, as we will do below.

Lemma 4.4.12. *Let $f \in I_{P \cap K}^K(V_\sigma)$ and let $\bar{f} := \int_{U_{\bar{P}}(0)} \lambda(\bar{u})^{-1} f d\bar{u}$. Then $\bar{f} \in I_{\bar{P} \cap K}^K(V_\sigma)$.*

Proof. Let $f \in I_{P \cap K}^K(V_\sigma)$, so $f : K \rightarrow V_\sigma$ is a function right invariant under some compact open subgroup and $f(mug) = \sigma(m)f(g)$ for all $g \in K$, $m \in M \cap K$, $u \in U_P \cap K$. We want to show that for $m \in M \cap K$, $\bar{u} \in U_{\bar{P}} \cap K$, $g \in K$, we have $\bar{f}(m\bar{u}g) = \sigma(m)\bar{f}(g)$. First, let us show that $U_{\bar{P}} \cap K \subset U_{\bar{P}}(0)$. For $\bar{u} \in U_{\bar{P}} \cap K$, write

$$\bar{u} = u_P(\bar{u})m_P(\bar{u})k_P(\bar{u})$$

using the Iwasawa decomposition. We may choose $k_P(\bar{u}) = \bar{u} \in K$. Then $m_P(\bar{u}) = 1$, hence $\delta_P(m_P(\bar{u})) = 1$ so $\bar{u} \in U_{\bar{P}}(0)$.

Now let $m \in M \cap K$, $\bar{u} \in U_{\bar{P}} \cap K$, $g \in K$. Then

$$\begin{aligned} \bar{f}(m\bar{u}g) &= \int_{U_{\bar{P}}(0)} f(\bar{u}_1 m \bar{u} g) d\bar{u}_1 = \int_{U_{\bar{P}}(0)} f(m m^{-1} \bar{u}_1 m \bar{u} g) d\bar{u}_1 \\ &= \sigma(m) \int_{U_{\bar{P}}(0)} f(m^{-1} \bar{u}_1 m \bar{u} g) d\bar{u}_1. \end{aligned}$$

We have that

$$m^{-1} \bar{u}_1 m = (m^{-1} u_P(\bar{u}_1) m) (m^{-1} m_P(\bar{u}_1) m) (m^{-1} k_P(\bar{u}_1) m),$$

therefore

$$\delta_P(m_P(m^{-1} \bar{u}_1 m)) = \delta_P(m^{-1} m_P(\bar{u}_1) m) = 1,$$

since $m \in K$ and $\bar{u}_1 \in U_{\bar{P}}(0)$. So $m^{-1} \bar{u}_1 m \in U_{\bar{P}}(0)$ and this gives

$$\text{Ad}(m^{-1}) : U_{\bar{P}}(0) \rightarrow U_{\bar{P}}(0),$$

which preserves measures since $m \in K \cap M$, so $\delta_P(m) = 1$. Thus

$$\bar{f}(m\bar{u}g) = \sigma(m) \int_{U_{\bar{P}}(0)} f(m^{-1} \bar{u}_1 m \bar{u} g) d\bar{u}_1 = \sigma(m) \int_{U_{\bar{P}}(0)} f(\bar{u}_1 \bar{u} g) d\bar{u}_1.$$

We can write

$$\bar{u}_1 \bar{u} = u_P(\bar{u}_1) m_P(\bar{u}_1) (k_P(\bar{u}_1) \bar{u}),$$

so $\bar{u}_1 \bar{u} \in U_{\bar{P}}(0)$, and we may change the variable $\bar{u}_1 \bar{u}$ to $\bar{u}_2 \in U_{\bar{P}}(0)$, by right invariance of the Haar measure on $U_{\bar{P}}$. So we obtain

$$\bar{f}(m\bar{u}g) = \sigma(m) \int_{U_{\bar{P}}(0)} f(\bar{u}_2 g) d\bar{u}_2 = \sigma(m) \bar{f}(g),$$

hence $\bar{f} \in I_{\bar{P} \cap K}^K(V_\sigma)$. □

Since

$$J_{P|\bar{P}}(\sigma \otimes \chi) \circ J_{\bar{P}|P}(\sigma \otimes \chi) = j(\sigma \otimes \chi) = \mu(\sigma \otimes \chi)^{-1}$$

is a rational function in the variable $z = \chi(h_\alpha^\vee)$, it has only finitely many poles and zeros. So for $z = \chi(h_\alpha^\vee)$ large enough, $J_{\bar{P}|P}(\sigma \otimes \chi)$ is invertible. We can use this to show that $\lim_{z \rightarrow \infty} J_{\bar{P}|P}(\sigma \otimes \chi)$ is invertible.

Lemma 4.4.13. *The operator*

$$A := \int_{U_{\bar{P}}(0)} \lambda(\bar{u})^{-1} d\bar{u} : I_{\bar{P} \cap K}^K(V_\sigma) \rightarrow I_{\bar{P} \cap K}^K(V_\sigma)$$

is invertible.

Proof. It suffices to show that for every compact open subgroup C of K , the operator

$$A^C := \int_{U_{\bar{P}}(0)} \lambda(\bar{u})^{-1} d\bar{u} : I_{\bar{P} \cap K}^K(V_\sigma)^C \rightarrow I_{\bar{P} \cap K}^K(V_\sigma)^C,$$

which is a map of finite-dimensional vector spaces, is bijective. Let $z = \chi(h_\alpha^\vee)$ be large enough such that $J_{\bar{P}|P}(\sigma \otimes \chi)$ is invertible. Let $f \in I_{\bar{P} \cap K}^K(V_\sigma)^C$, but view it as an element of $I_{\bar{P}}^G(\sigma \otimes \chi)$. Let $g \in K$. Then

$$\int_{U_{\bar{P}}} \left[\int_{U_{\bar{P}}(0)} f(\bar{u}_1 \bar{u}_2 g) d\bar{u}_1 \right] d\bar{u}_2 = \text{vol}(U_{\bar{P}}(0)) \int_{U_{\bar{P}}} f(\bar{u} g) d\bar{u},$$

hence

$$\int_{U_{\bar{P}}} \lambda(\bar{u})^{-1} d\bar{u} \circ \int_{U_{\bar{P}}(0)} \lambda(\bar{u})^{-1} d\bar{u} = \text{vol}(U_{\bar{P}}(0)) J_{\bar{P}|P}(\sigma \otimes \chi).$$

Since $J_{\bar{P}|P}(\sigma \otimes \chi)$ is invertible, $\int_{U_{\bar{P}}(0)} \lambda(\bar{u})^{-1} d\bar{u}$ is injective, hence A^C is bijective since it is a map of finite dimensional vector spaces. We can do this for every C , so A is invertible. $\square$

We can use the results above to show that $\mu(\sigma, z) = \mu(\sigma \otimes \chi) = j(\sigma \otimes \chi)^{-1}$ is not zero at $z = \infty$ and $z = 0$.

Proposition 4.4.14. *The function $\mu(\sigma, z)$ is not zero at $z = \infty$ and $z = 0$, i.e.*

$$\lim_{z \rightarrow \infty} j(\sigma \otimes \chi) \neq \infty \text{ and } \lim_{z \rightarrow 0} j(\sigma \otimes \chi) \neq \infty.$$

Proof. We will first show that $\lim_{z \rightarrow \infty} j(\sigma \otimes \chi) \neq \infty$. Let $f_\chi \in I_{\bar{P}}^G(\sigma \otimes \chi)$ be determined by $\text{supp}(f_\chi) = \bar{P}U_{P,r}$, $f_\chi(1) = v$, and f_χ is right $U_{P,r}$ -invariant. Then

$$J_{P|\bar{P}}(\sigma \otimes \chi) f_\chi(1) = \int_{U_P} f_\chi(u) du = \int_{U_{P,r}} f_\chi(u) du = \text{vol}(U_{P,r}) v,$$

which is independent of χ . Let $z = \chi(h_\alpha^\vee)$ be large enough such that $J_{\bar{P}|P}(\sigma \otimes \chi)$ is invertible, and write $g_\chi = J_{\bar{P}|P}(\sigma \otimes \chi)^{-1} f_\chi \in I_{\bar{P}}^G(\sigma \otimes \chi)$. Then

$$\begin{aligned} j(\sigma \otimes \chi) g_\chi(1) &= J_{P|\bar{P}}(\sigma \otimes \chi) \circ J_{\bar{P}|P}(\sigma \otimes \chi) g_\chi(1) = J_{P|\bar{P}}(\sigma \otimes \chi) f_\chi(1) \\ &= \text{vol}(U_{P,r}) v. \end{aligned}$$

By Proposition 4.4.11 and Lemma 4.4.13, $\lim_{z \rightarrow \infty} J_{\bar{P}|P}(\sigma \otimes \chi)$ exists and is invertible. Hence, $\lim_{z \rightarrow \infty} g_\chi = g \in I_{\bar{P} \cap K}^K(V_\sigma)$ exists. Hence,

$$\lim_{z \rightarrow \infty} j(\sigma \otimes \chi) g_\chi(1) = \lim_{z \rightarrow \infty} j(\sigma \otimes \chi) g(1) = \text{vol}(U_{P,r}) v,$$

so $\lim_{z \rightarrow \infty} j(\sigma \otimes \chi) \neq \infty$.

So far, we showed that $\mu(\sigma, \infty) \neq 0$. By Lemma 4.4.3, $\sigma = \tilde{\sigma} \otimes \tilde{\chi}$ for some unitary $\tilde{\sigma} \in \text{Irr}(M)_{[M, \sigma]}$ and $\tilde{\chi} \in \text{Hom}(M, \mathbb{R}_{>0})$. By (4.4.2), $\mu(\sigma, 0) = \mu(\tilde{\sigma}, 0)$ and $\mu(\sigma, \infty) = \mu(\tilde{\sigma}, \infty)$. By Lemma 4.4.6, $\mu(\tilde{\sigma}, \infty) \neq 0$ implies that $\mu(\tilde{\sigma}, 0) \neq 0$, i.e. $\mu(\sigma, 0) \neq 0$. $\square$

4.4.4 Zeros of the μ -function

In this subsection, we analyze the zeros of the μ -function. Recall that since M is maximal, we have $[N_G(M) : M] \leq 2$, so we will distinguish between two cases. In the case where $N_G(M)/M$ is trivial, it turns out that μ is constant. In the case where $N_G(M)/M$ is not trivial, we will prove that μ has at most two zeros in $\mathbb{C}^\times$.

Lemma 4.4.15. *If $\mu(\sigma, z)$ has no zeros on $\mathbb{C}^\times$, then $\mu(\sigma, z)$ is constant.*

Proof. Take a unitary $\tilde{\sigma} \in \text{Irr}(M)_{[M, \sigma]}$. Recall from Lemma 4.4.6 that we can write

$$\mu(\tilde{\sigma}, z) = c_\mu \prod_i (z - \lambda_i)^{n_i} (z^{-1} - \bar{\lambda}_i)^{n_i}.$$

Suppose $\mu(\tilde{\sigma}, z)$ has no zeros in $\mathbb{C}^\times$, then there are no factors with n_i positive. So there can only be factors with n_i negative. However, by Proposition 4.4.14, $\mu(\tilde{\sigma}, \infty) \neq 0$, so $\mu(\tilde{\sigma}, z)$ is constant, hence $\mu(\sigma, z)$ is also constant by (4.4.2). $\square$

Lemma 4.4.16. *If $N_G(M)/M$ is trivial, then $\mu(\sigma, z)$ is constant.*

Proof. First let us show that if $N_G(M)/M$ is trivial, then the parabolic subgroups P and $\bar{P}$ are not conjugate. Suppose that $N_G(M) = M$ and that P and $\bar{P}$ are conjugate, so there exists $g \in G$ such that $gPg^{-1} = \bar{P}$. Write $P = MU_P$, $\bar{P} = MU_{\bar{P}}$ then

$$gMU_Pg^{-1} = gMg^{-1}gU_Pg^{-1} = MU_{\bar{P}}.$$

A Levi factor of $\bar{P}$ is unique up to conjugation by elements of $U_{\bar{P}}$, see [Spr09, Proposition 16.1.1]. Thus $gMg^{-1} = \bar{u}M\bar{u}^{-1}$ for some $\bar{u} \in U_{\bar{P}}$. So $g^{-1}\bar{u} \in N_G(M) = M$, i.e. $g = \bar{u}m$ for some $m \in M$. But then $g \in \bar{P}$ and $P = g^{-1}\bar{P}g = \bar{P}$, a contradiction since P and $\bar{P}$ are opposite parabolic subgroups.

Now suppose that $\mu(\sigma \otimes \chi) = 0$ for some χ . This corresponds to $J_{\bar{P}|P}$ and/or $J_{P|\bar{P}}$ being singular at $\sigma \otimes \chi$. However, this can not happen if P and $\bar{P}$ are not conjugate, see [Wal03, Corollaire IV.1.2]. Therefore, $\mu(\sigma, z)$ has no zeros in $\mathbb{C}^\times$, and by Lemma 4.4.15, it is constant. $\square$

So from now on, let us assume that $[N_G(M) : M] = 2$ and that μ has a zero, otherwise there is nothing left to prove. Let s_α be the unique nontrivial element of $N_G(M)/M$, with representative $w \in N_G(M)$. Then s_α acts on $\text{Irr}(M)$ via $(s_\alpha \cdot \pi)(m) = \pi(w^{-1}mw)$.

Proposition 4.4.17. *Suppose that $\mu(\sigma, z)$ is not constant. Then there exists a unitary $\sigma_0 \in \text{Irr}(M)_{[M, \sigma]}$ such that the following statements hold:*

- (i) $\mu(\sigma_0) = 0$;
- (ii) $\mu(\sigma_0, z) = 0$ implies that $z = 1$ or $z = -1$;
- (iii) if $\mu(\sigma_0, z) = 0$, the zero at z is of order 2.

Proof. Since $\mu(\sigma, z)$ is not constant, $\mu(\tau) = 0$ for some $\tau \in \text{Irr}(M)_{[M, \sigma]}$, by Lemma 4.4.15. Moreover, by Lemma 4.4.16, $N_G(M)/M$ has a unique nontrivial element s_α . Since $\mu(\tau) = 0$, it follows that $s_\alpha \tau \cong \tau$, see [Wal03, p. 283]. By Lemma 4.4.3, there is a bijection

$$\text{Irr}(M)_{[M, \sigma]}^{\text{unit}} \times \text{Hom}(M, \mathbb{R}_{>0}) \rightarrow \text{Irr}(M)_{[M, \sigma]} : (\pi, \chi) \mapsto \pi \otimes \chi.$$

This bijection is s_α -equivariant by construction,

$$s_\alpha \cdot (\pi, \chi) = (s_\alpha \pi, s_\alpha \chi) \mapsto s_\alpha \pi \otimes s_\alpha \chi = s_\alpha \cdot (\pi \otimes \chi).$$

So $\tau = \sigma_0 \otimes \chi_0$ for a unitary representation σ_0 and $\chi_0 \in \text{Hom}(M, \mathbb{R}_{>0})$, with $s_\alpha \sigma_0 \cong \sigma_0$ and $s_\alpha \chi_0 \cong \chi_0$.

We will first prove (ii) and use that result to prove (i). For $\mu(\sigma_0 \otimes \chi) = \mu(\sigma_0, z)$ to be zero, we must have $s_\alpha(\sigma_0 \otimes \chi) \cong \sigma_0 \otimes \chi$ [Wal03, p. 283]. We have $s_\alpha \sigma_0 \cong \sigma_0$, so $s_\alpha(\sigma_0 \otimes \chi) \cong \sigma_0 \otimes s_\alpha \chi$. Therefore, $\mu(\sigma_0 \otimes \chi) = 0$ implies that $\sigma_0 \cong \sigma_0 \otimes \chi(s_\alpha \chi)^{-1}$, so $\chi(s_\alpha \chi)^{-1}(h_\alpha^\vee) = 1$. Write $\chi(h_\alpha^\vee) = z$, then $(s_\alpha \chi)(h_\alpha^\vee) = z^{-1}$, so $\chi(s_\alpha \chi)^{-1}(h_\alpha^\vee) = z^2$. Hence, $\mu(\sigma_0 \otimes \chi) = \mu(\sigma_0, z) = 0$ implies that $z^2 = 1$, hence $z = \pm 1$. This proves (ii).

Recall that $\mu(\tau) = \mu(\sigma_0 \otimes \chi_0) = 0$, where $\chi_0 \in \text{Hom}(M, \mathbb{R}_{>0})$. By the above, we obtain that $\chi_0(h_\alpha^\vee) = 1$, hence $\mu(\sigma_0) = 0$. This proves (i).

It remains to prove (iii). Suppose $\mu(\sigma_0, z) = 0$, then $z = \pm 1$ by (ii). In particular $z \in S^1$, so by the proof of Lemma 4.4.5, there exists a unitary character $\chi_z \in X_{\text{nr}}(M)$ such that $\chi_z(h_\alpha^\vee) = z$. Then $\sigma_0 \otimes \chi_z$ is unitary with $\mu(\sigma_0 \otimes \chi_z) = 0$. This means that at least one of $J_{\bar{P}|P}(\sigma_0 \otimes \chi_z)$ and $J_{P|\bar{P}}(\sigma_0 \otimes \chi_z)$ is singular. Since $\sigma_0 \otimes \chi_z$ is unitary (in particular Hermitian), we have the adjointness relation from Theorem 4.3.6.a with $\overline{\sigma_0 \otimes \chi_z}^\vee$ replaced by $\sigma_0 \otimes \chi_z$. So

$$\langle J_{\bar{P}|P}(\sigma_0 \otimes \chi_z) f_1, f_2 \rangle = \langle f_1, J_{P|\bar{P}}(\sigma_0 \otimes \chi_z) f_2 \rangle$$

for all $f_1 \in I_P^G(\sigma_0 \otimes \chi_z)$, $f_2 \in I_{\bar{P}}^G(\sigma_0 \otimes \chi_z)$. Thus, singularity of one of the two intertwining operators $J_{\bar{P}|P}$ and $J_{P|\bar{P}}$ at $\sigma_0 \otimes \chi_z$ automatically implies singularity of the other one at $\sigma_0 \otimes \chi_z$. By [Wal03, Corollaire IV.1.2], both singularities have order 1. Since

$$\mu(\sigma_0 \otimes \chi)^{-1} = j(\sigma_0 \otimes \chi) = J_{P|\bar{P}}(\sigma_0 \otimes \chi) \otimes J_{\bar{P}|P}(\sigma_0 \otimes \chi),$$

we obtain that the zero of μ at $\sigma_0 \otimes \chi_z$ has order 2. This proves (iii). $\square$

4.4.5 Hermitian techniques to find the poles

In this subsection, we locate the poles of $\mu(\sigma, z)$ using techniques with Hermitian and unitary representations. We will prove the main result of this paper, which is a formula for $\mu(\sigma, z)$ in terms of poles and zeros. Recall that in Lemma 4.4.6, we have already written the function $\mu(\sigma, z)$ in a form that is closer to the desired result. We also showed in Lemma 4.4.15 that $\mu(\sigma, z)$ is constant if it has no zeros, which is satisfied when $[N_G(M) : M] = 1$, see Lemma 4.4.16.

Therefore, from now on, we assume that $[N_G(M) : M] = 2$ and that $\mu(\sigma, z)$ is not constant. So $N_G(M)/M$ has a unique nontrivial element s_α . For

simplicity, we assume in this section that $\sigma = \sigma_0$, where σ_0 is as in Proposition 4.4.17. So σ is unitary with $s_\alpha \sigma \cong \sigma$ and $\mu(\sigma) = 0$. There is possibly another zero of $\mu(\sigma, z)$ at $z = -1$ and each zero is of order 2. For each zero of μ , we will find two poles in $\mathbb{R}^\times$. To do this, it suffices to find one pole for each zero, because by Lemma 4.4.6, one obtains the other pole for free. We already showed in Proposition 4.4.14 that $\mu(\sigma, z)$ is not zero at $z = \infty$ and $z = 0$, so we can conclude that $\mu(\sigma, z)$ does not have any more poles in $\mathbb{C}^\times$.

We will use arguments concerning Hermitian and unitary representations to locate the poles. See [Ren10, §IV] for definitions and useful results about Hermitian and unitary representations. Let $\chi \in \text{Hom}(M, \mathbb{R}_{>0}) \subset X_{\text{nr}}(M)$, then $\bar{\chi} = \chi$ and $\bar{\chi}^\vee = \chi^\vee = \chi^{-1}$. By unitarity of σ , we have $\bar{\sigma} \otimes \bar{\chi}^\vee \cong \sigma \otimes \chi^{-1}$. So $\sigma \otimes \chi$ is only Hermitian if $\sigma \otimes \chi \cong \sigma \otimes \chi^{-1}$, or equivalently if $\chi^2 \in X_{\text{nr}}(M, \sigma)$. We will show that the induced representation $I_P^G(\sigma \otimes \chi)$ is Hermitian whenever $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$ and $\mu(\sigma \otimes \chi) \neq \infty$. This will allow us to construct a continuous family of Hermitian forms on a family of Hermitian representations, which will help us to show that somewhere in this family, a pole of $\mu(\sigma \otimes \chi)$ must occur.

Lemma 4.4.18. *Let L be a Levi subgroup of G and $\tau \in \text{Irr}(L)$, $w \in N_G(L)$. Let $Q = LV$ be a parabolic subgroup of G . Then $I_Q^G(\tau)$ and $I_Q^G(w \cdot \tau)$ have the same irreducible subquotients with the same multiplicities. In particular, if $I_Q^G(\tau)$ is irreducible, then $I_Q^G(\tau) \cong I_Q^G(w \cdot \tau)$.*

Proof. This follows from [BZ77, Theorem 2.9, Remark 2.10]. See also [Ren10, Théorème VI.5.4]. $\square$

Lemma 4.4.19. *Suppose that $s_\alpha(\chi) = \chi^{-1}$ and that $I_P^G(\sigma \otimes \chi)$ is irreducible. Then $I_P^G(\sigma \otimes \chi)$ is Hermitian.*

Proof. Since $s_\alpha(\sigma \otimes \chi) \cong \sigma \otimes \chi^{-1}$, we have by Lemma 4.4.18 that $I_P^G(\sigma \otimes \chi) \cong I_P^G(\sigma \otimes \chi^{-1})$. Taking the complex conjugate representation commutes with parabolic induction. By [Ren10, §VI.1.2], taking the contragredient representation also commutes with parabolic induction. So we get

$$I_P^G(\sigma \otimes \chi) \cong I_P^G(\sigma \otimes \chi^{-1}) \cong I_P^G(\overline{\sigma \otimes \chi}^\vee) \cong \overline{I_P^G(\sigma \otimes \chi)}^\vee,$$

where the middle isomorphism follows from unitarity of σ , so $\overline{\sigma \otimes \chi}^\vee \cong \sigma \otimes \chi^{-1}$. This completes the proof that $I_P^G(\sigma \otimes \chi)$ is Hermitian. $\square$

The condition $s_\alpha \chi = \chi^{-1}$ from Lemma 4.4.19 is satisfied by unramified characters $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$, as shown below.

Lemma 4.4.20. *If $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$, then $s_\alpha \chi = \chi^{-1}$.*

Proof. The unramified character χ factors through M/M^1 , so

$$\chi : M/M^1 Z(G) \rightarrow \mathbb{R}_{>0}.$$

Because M is a maximal Levi subgroup, we have

$$M/M^1 \cong \mathbb{Z}^r \text{ and } Z(G)/Z(G)_{\text{cpt}} \cong \mathbb{Z}^{r-1}.$$

This is because the rank r of M/M^1 equals the dimension of the maximal split torus in M , which is one higher than the dimension of the maximal split torus in G , i.e. the rank of $Z(G)/Z(G)_{\text{cpt}}$, which is the same as the rank $r - 1$ of G/G^1 . Thus, we obtain the quotient

$$M/M^1 Z(G) = (M/M^1)/(Z(G)/Z(G)_{\text{cpt}}) \cong \mathbb{Z} \times H,$$

with H a finite group. So we can consider $\chi : \mathbb{Z} \times H \rightarrow \mathbb{R}_{>0}$, which must be trivial on H since it is a finite group. So $\chi : \mathbb{Z} \rightarrow \mathbb{R}_{>0}$ is of the form $n \mapsto x^n$ where $x \in \mathbb{R}_{>0}$. Now s_α acts on $\mathbb{Z}$ as a nontrivial group homomorphism and an involution, so the only possibility is multiplication by -1 . This induces the action on χ , which we conclude is given by $s_\alpha \chi = \chi^{-1}$. $\square$

Under the assumptions of Lemma 4.4.19, we will now construct a Hermitian form on $I_P^G(\sigma \otimes \chi)$, using the intertwining operator $J_{\bar{P}|P}(\sigma \otimes \chi)$. Write $\lambda(s_\alpha)$ for translation by s_α . Fix an isomorphism of M -representations $\varphi_\sigma : \sigma \rightarrow s_\alpha \cdot \sigma$. Moreover, let

$$\begin{aligned} J'_{\bar{P}|P}(\sigma \otimes \chi) &:= (\chi(h_\alpha^\vee) - 1) J_{\bar{P}|P}(\sigma \otimes \chi), \\ J'_{P|\bar{P}}(\sigma \otimes \chi) &:= (\chi^{-1}(h_\alpha^\vee) - 1) J_{P|\bar{P}}(\sigma \otimes \chi). \end{aligned}$$

Then both $J'_{\bar{P}|P}(\sigma \otimes \chi)$ and $J'_{P|\bar{P}}(\sigma \otimes \chi)$ do not have a pole at $\chi = 1$. They are invertible for χ in a neighbourhood of $\chi = 1$ in $X_{\text{nr}}(M)$. They are each other's inverse, up to a scalar which depends rationally on χ . Consider, for $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$ the composition

$$\begin{aligned} I_P^G(\sigma \otimes \chi) &\xrightarrow{J'_{\bar{P}|P}(\sigma \otimes \chi)} I_{\bar{P}}^G(\sigma \otimes \chi) \xrightarrow{\lambda(s_\alpha)} I_{\bar{P}}^G(s_\alpha \cdot \sigma \otimes \chi^{-1}) \\ &\xrightarrow{I_{\bar{P}}^G(\varphi_\sigma^{-1} \otimes \text{id})} I_{\bar{P}}^G(\sigma \otimes \chi^{-1}) \xrightarrow{\sim} I_{\bar{P}}^G(\overline{\sigma \otimes \chi})^\vee \xrightarrow{\sim} \overline{I_P^G(\sigma \otimes \chi)}^\vee. \end{aligned}$$

The last two isomorphisms come from Lemma 4.4.19. Using the map above, we take

$$\begin{aligned} I_P^G(\sigma \otimes \chi) \times I_P^G(\sigma \otimes \chi) &\xrightarrow{\sim} I_P^G(\sigma \otimes \chi) \times \overline{I_P^G(\sigma \otimes \chi)}^\vee \rightarrow \mathbb{C} \\ (v, v') &\mapsto \langle v, v' \rangle_\chi. \end{aligned} \tag{4.4.14}$$

This way, we obtain a sesquilinear form, but it is not necessarily Hermitian. Later, we will do some more work to construct a Hermitian form using (4.4.14).

For now, with (4.4.14), we have a family of nondegenerate sesquilinear forms on the family of representations $I_P^G(\sigma \otimes \chi)$, $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$. It only fails when $J'_{\bar{P}|P}(\sigma \otimes \chi)$ is not invertible. The maps depend rationally on χ , as we will prove below.

Lemma 4.4.21. *The sesquilinear forms as constructed in (4.4.14) are G -invariant and depend rationally on $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$.*

Proof. To show that the sesquilinear forms depend rationally on χ , let us consider each individual map in the composition. It turns out that most of the maps are independent of χ , but let us still describe each of them explicitly.

First of all, we consider the map

$$J'_{\bar{P}|P}(\sigma \otimes \chi) = (\chi(h_\alpha^\vee) - 1)J_{\bar{P}|P} : I_P^G(\sigma \otimes \chi) \rightarrow I_{\bar{P}}^G(\sigma \otimes \chi).$$

We may identify $I_P^G(\sigma \otimes \chi) \cong I_{K \cap P}^K(\sigma)$ and $I_{\bar{P}}^G \cong I_{K \cap \bar{P}}^K(\sigma)$, to consider it as a map

$$J'_{\bar{P}|P}(\sigma \otimes \chi) : I_{K \cap P}^K(\sigma) \rightarrow I_{K \cap \bar{P}}^K(\sigma),$$

and it makes sense to ask if this map is rational in the variable χ , see [Wal03, §IV.1] for details. By [Wal03, Théorème IV.1.1], $J_{\bar{P}|P}$ is rational in χ . Moreover $\chi(h_\alpha^\vee)$ is rational in χ , so it follows that indeed $J'_{\bar{P}|P}(\sigma \otimes \chi) = (\chi(h_\alpha^\vee) - 1)J_{\bar{P}|P}(\sigma \otimes \chi)$ is rational in χ .

The next step is the map

$$\lambda(s_\alpha) : I_{P \cap K}^K(\sigma) \cong I_P^G(\sigma \otimes \chi) \rightarrow I_P^G(s_\alpha \cdot \sigma \otimes \chi^{-1}) \cong I_{P \cap K}^K(s_\alpha \cdot \sigma).$$

It is given by $f \mapsto [g \mapsto f(s_\alpha^{-1}g)]$ which is independent of χ , so rational in χ .

Next up is the map

$$I_P^G(\varphi_\sigma^{-1} \otimes \text{id}) : I_{P \cap K}^K(s_\alpha \cdot \sigma) \cong I_P^G(s_\alpha \cdot \sigma \otimes \chi^{-1}) \rightarrow I_P^G(\sigma \otimes \chi^{-1}) \cong I_{P \cap K}^K(\sigma).$$

It is given by $f \mapsto [g \mapsto (\varphi_\sigma^{-1} \otimes \text{id})f(g)]$ which is again independent of χ .

Then we have the map

$$I_{P \cap K}^K(\sigma) \cong I_P^G(\sigma \otimes \chi^{-1}) \xrightarrow{\sim} I_P^G(\overline{\sigma \otimes \chi}^\vee) \cong I_{P \cap K}^K(\bar{\sigma}^\vee).$$

The isomorphism in the middle comes from unitarity of σ and the assumption that $\chi \in \text{Hom}(M, \mathbb{R}_{>0})$, so $\bar{\chi}^\vee \cong \chi^{-1}$. On the level of vector spaces, this map comes from the pairing $V_\sigma \times V_\sigma \rightarrow \mathbb{C}$ inducing the isomorphism $V_\sigma \cong \overline{V_\sigma}^\vee$, which is independent of χ .

The next map is

$$I_{P \cap K}^K(\bar{\sigma}^\vee) \cong I_P^G(\overline{\sigma \otimes \chi}^\vee) \xrightarrow{\sim} I_P^G(\overline{\sigma \otimes \chi})^\vee \cong I_{P \cap K}^K(\bar{\sigma})^\vee,$$

i.e. the map that describes how taking the contragredient representation commutes with parabolic induction. We have an M -invariant pairing $\bar{\sigma} \otimes \bar{\chi} \times \overline{\sigma \otimes \chi}^\vee \rightarrow \mathbb{C}$. We use it to build a K -invariant pairing

$$I_{P \cap K}^K(\overline{\sigma \otimes \chi}) \times I_{P \cap K}^K(\overline{\sigma \otimes \chi}^\vee) \rightarrow \mathbb{C}, (f_1, f_2) \mapsto \int_K \langle f_1(k), f_2(k) \rangle dk,$$

which gives an isomorphism $I_{P \cap K}^K(\bar{\sigma}^\vee) \cong I_{P \cap K}^K(\overline{\sigma \otimes \chi}^\vee) \cong I_{P \cap K}^K(\overline{\sigma \otimes \chi})^\vee \cong I_{P \cap K}^K(\bar{\sigma})^\vee$ independent of χ . By the Iwasawa decomposition $G = PK$, $\int_K$ is up to a positive scalar factor the same as $\int_{P \backslash G}$, hence the pairing is also G -invariant, and it induces the isomorphism $I_P^G(\overline{\sigma \otimes \chi}^\vee) \cong I_P^G(\overline{\sigma \otimes \chi})^\vee$, independent of χ .

Next, we need to consider the map

$$I_{P \cap K}^K(\bar{\sigma})^\vee \cong I_P^G(\overline{\sigma \otimes \chi})^\vee \xrightarrow{\sim} \overline{I_P^G(\sigma \otimes \chi)}^\vee \cong \overline{I_{P \cap K}^K(\sigma)}^\vee,$$

which shows that taking the complex conjugate representation commutes with parabolic induction. Again, this map is independent of χ .

Thus, the forms $\langle \cdot, \cdot \rangle_\chi$ depend rationally on χ . The sesquilinear forms are G -invariant because in the construction in (4.4.14), the first arrow is a composition of homomorphisms of G -representations (i.e. the homomorphisms described above), so it is G -equivariant, and the second arrow is G -invariant by definition. $\square$

We use the sesquilinear forms in (4.4.14) to build a family of Hermitian forms on the representations $I_P^G(\sigma \otimes \chi)$, which is positive definite on $I_P^G(\sigma)$, and nondegenerate if $\mu(\sigma \otimes \chi) \neq \infty$. We do this by multiplying with a suitable function $f(\chi)$, which depends continuously on χ .

Lemma 4.4.22. *Suppose $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$ and $\mu(\sigma \otimes \chi) \neq \infty$. Then $I_P^G(\sigma \otimes \chi)$ is irreducible, and on this representation we have the sesquilinear form as constructed in (4.4.14). We can multiply it by $f(\chi) \in S^1$, depending continuously on χ , such that*

$$\langle v, v' \rangle_{I_P^G(\sigma \otimes \chi)} := f(\chi) \langle v, v' \rangle_\chi \quad (4.4.15)$$

is a G -invariant Hermitian form on $I_P^G(\sigma \otimes \chi)$ which is nondegenerate, and positive definite for $\chi = 1$.

Proof. First we consider the case $\chi = 1$. Since σ is unitary, $I_P^G(\sigma)$ is also unitary, see [Ren10, §IV.2.3]. The zero $\mu(\sigma) = 0$ corresponds to a pole of order 1 of $J_{\bar{P}|P}(\sigma \otimes \chi)$ at $\chi = 1$. The construction of $J'_{\bar{P}|P}(\sigma \otimes \chi)$ gets rid of the pole at $\chi = 1$, therefore $J'_{\bar{P}|P}(\sigma)$ is invertible. Thus, the construction in (4.4.14) gives us a G -invariant sesquilinear form $\langle \cdot, \cdot \rangle_{\chi=1}$ on $I_P^G(\sigma)$.

By [Tan17, Proposition 6.3], $I_P^G(\sigma)$ is irreducible. By irreducibility of $I_P^G(\sigma)$, there is up to scalars only one Hermitian form on $I_P^G(\sigma)$, see [Ren10, p. 112]. By unitarity of $I_P^G(\sigma)$, there exists a positive definite Hermitian form $\langle \cdot, \cdot \rangle'_{\chi=1}$ on $I_P^G(\sigma)$. Then $\langle \cdot, \cdot \rangle_{\chi=1}$ must be a complex scalar multiple of $\langle \cdot, \cdot \rangle'_{\chi=1}$, since it is G -invariant, sesquilinear and $I_P^G(\sigma)$ is irreducible. We may pick $f(1) \in S^1$ such that $f(1)\langle \cdot, \cdot \rangle_{\chi=1}$ equals $\langle \cdot, \cdot \rangle'_{\chi=1}$ multiplied by a positive real scalar. In this way, we obtain a Hermitian form on $I_P^G(\sigma)$ which is positive definite.

Now we consider $\chi \neq 1$. For $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$, we have that $s_\alpha(\chi) = \chi^{-1}$. For $\chi \neq 1$, we have $\mu(\sigma \otimes \chi) \neq 0$ by Proposition 4.4.17 and $s_\alpha(\sigma \otimes \chi) \cong \sigma \otimes \chi^{-1} \not\cong \sigma \otimes \chi$. Hence, by [Tan17, Proposition 6.2], $I_P^G(\sigma \otimes \chi)$ is irreducible. Since $\mu(\sigma \otimes \chi) \neq 0$ and $\mu(\sigma \otimes \chi) \neq \infty$, we have that $J'_{\bar{P}|P}(\sigma \otimes \chi)$ is invertible. Therefore, we have the G -invariant sesquilinear form $\langle \cdot, \cdot \rangle_\chi$ on $I_P^G(\sigma \otimes \chi)$, as constructed in (4.4.14).

By Lemma 4.4.19, there exists a nondegenerate G -invariant Hermitian form $\langle \cdot, \cdot \rangle'_\chi$ on $I_P^G(\sigma \otimes \chi)$, which is unique up to real scalars. For every χ , fix one such $\langle \cdot, \cdot \rangle'_\chi$ and scale it, so that we can write

$$\langle \cdot, \cdot \rangle_\chi = e^{i\phi(\chi)} \langle \cdot, \cdot \rangle'_\chi.$$

Then the choice of $\langle \cdot, \cdot \rangle'_\chi$ is unique up to multiplication by ± 1 . Hence, $e^{i\phi(\chi)}$ is also unique up to multiplication by ± 1 . Moreover, since $\langle \cdot, \cdot \rangle_\chi$ depends continuously on χ ,

$$|\langle \cdot, \cdot \rangle_\chi| = |e^{i\phi(\chi)}| |\langle \cdot, \cdot \rangle'_\chi| = |\langle \cdot, \cdot \rangle'_\chi|$$

depends continuously on χ . Recall that we already fixed $f(1) = e^{-i\phi(1)}$ such that $f(1)\langle \cdot, \cdot \rangle_{\chi=1} = \langle \cdot, \cdot \rangle'_{\chi=1}$ is positive definite. Consider $0 \neq v \in I_{P \cap K}^K(V_\sigma)$, then $\langle v, v \rangle'_{\chi=1} > 0$. Consider a neighbourhood of 1 such that for all χ in this neighbourhood, $\langle v, v \rangle'_\chi \neq 0$. This is possible since $|\langle \cdot, \cdot \rangle'_\chi|$ depends continuously on χ . The $\langle \cdot, \cdot \rangle'_\chi$ were chosen uniquely up to a sign, so it is possible to choose them in such a way that $\langle v, v \rangle'_\chi > 0$ for all χ in this neighbourhood of 1. In this way, $\langle \cdot, \cdot \rangle'_\chi$ depends continuously on χ in this neighbourhood. In fact, for all χ we can pick the sign of $\langle \cdot, \cdot \rangle'_\chi$ such that the Hermitian forms will depend continuously on χ . Then also $f(\chi) = e^{-i\phi(\chi)}$ depends continuously on χ , and we are done. $\square$

Lemma 4.4.23. *Let (π, V) be a smooth Hermitian representation of G . Let C be a compact open subgroup of G and let V^C be the space of C -fixed vectors in V . Then the nondegenerate G -invariant Hermitian form on V restricts to a nondegenerate C -invariant Hermitian form on V^C .*

Proof. Since C is compact, we can write

$$V = V^C \oplus V_C$$

where V_C is spanned by the vectors $c \cdot v - v$ for $c \in C$, $v \in V$. Since (π, V) is Hermitian, we have a G -invariant nondegenerate Hermitian form on V ,

$$\langle \cdot, \cdot \rangle : V \times V = (V^C \oplus V_C) \times (V^C \oplus V_C) \rightarrow \mathbb{C}.$$

Consider the projection operator

$$p_C := \frac{1}{|C|} \int_C \pi(c) dc = \begin{cases} 1 & \text{on } V^C \\ 0 & \text{on } V_C. \end{cases}$$

Since the Hermitian form on V is G -invariant, in particular C -invariant, we have that

$$\langle c \cdot v, v' \rangle = \langle v, c^{-1} \cdot v' \rangle$$

for all $v, v' \in V$, $c \in C$. It follows that for all $v \in V^C$, $v' \in V_C$,

$$\langle v, v' \rangle = \langle p_C v, v' \rangle = \langle v, p_C v' \rangle = \langle v, 0 \rangle = 0.$$

So the pairing between V^C and V_C is zero, but the form on the whole space V is nondegenerate, therefore the form restricted to V^C must be nondegenerate. $\square$

For a Hermitian form on a finite dimensional vector space V , we can consider its signature $(\dim(V^+), \dim(V^0), \dim(V^-))$, where $V = V^+ \oplus V^0 \oplus V^-$ and the form is positive definite on V^+ , zero on V^0 , negative definite on V^- . For Hermitian forms on admissible representations, we can consider the signature by restricting to open compact subgroups, which will be done to prove the next result.

Lemma 4.4.24. *Suppose $\mu(\sigma) = 0$, $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$. If $\mu(\sigma \otimes \chi^t)$ does not have a pole for $t \in [0, \tau]$, $\tau \in \mathbb{R}_{>0}$, then $I_P^G(\sigma \otimes \chi^\tau)$ is unitary.*

Proof. By Lemma 4.4.22, for each $t \in [0, \tau]$, the representation $I_P^G(\sigma \otimes \chi^t)$ is Hermitian, and we have the Hermitian form as constructed in (4.4.15). The representations are realized on the same vector space, say $I_{P \cap K}^K(V_\sigma)$. By Lemma 4.4.22, the Hermitian form on $I_P^G(\sigma)$ is positive definite, i.e.

$\langle f, f \rangle_{I_P^G(\sigma)} > 0$. The function $t \mapsto \langle f, f \rangle_{I_P^G(\sigma \otimes \chi^t)}$ is continuous for $f \in I_{P \cap K}^K(V_\sigma)$, see Lemma 4.4.21 and Lemma 4.4.22. Let $C \subset K$ be a compact open subgroup fixing f . Recall that $I_P^G(\sigma \otimes \chi^t)$ is irreducible, hence admissible by [FP22, Theorem 6.3], therefore $\dim I_{P \cap K}^K(V_\sigma)^C$ is finite. By Lemma 4.4.23, the G -invariant nondegenerate Hermitian form on $I_P^G(\sigma \otimes \chi^t)$ constructed in (4.4.15) restricts to a C -invariant nondegenerate Hermitian form on $I_{P \cap K}^K(V_\sigma)^C$. We have a continuous family of such forms parametrized by $t \in [0, \tau]$. For $t = 0$, recall that we have a positive definite form, so the form has signature $(\dim I_{P \cap K}^K(V_\sigma)^C, 0, 0)$. For any $t \in [0, \tau]$, the form is nondegenerate, so with signature $(p_t, 0, q_t)$ with $p_t + q_t = \dim I_{P \cap K}^K(V_\sigma)^C$. If for some t we have $q_t > 0$, then since the forms vary continuously, we must have that for some t' the form is degenerate, a contradiction. So all the Hermitian forms on $I_{P \cap K}^K(V_\sigma)^C$ for $t \in [0, \tau]$ are positive definite. We can do this for all f and C fixing f , hence all the representations $I_P^G(\sigma \otimes \chi^t)$ for $t \in [0, \tau]$ are unitary. $\square$

We have just shown that if $\mu(\sigma \otimes \chi^t)$ does not have a pole for any $t \in \mathbb{R}_{\geq 0}$, then all representations $I_P^G(\sigma \otimes \chi^t)$ are unitary. We want to show that this is not possible, so that we obtain a pole for some t .

For the next result, we consider the maximal C^* -algebra $C^*(G)$ of G , which is a completion of $\mathcal{H}(G)$ such that the irreducible smooth unitary representations of G are in bijection with the irreducible representations of $C^*(G)$. This bijection is given by

$$\text{Irr}(G)^{\text{unit}} \rightarrow \text{Irr}(C^*(G)), (\pi, V) \mapsto (\pi, \mathcal{V}),$$

where $\mathcal{V}$ denotes the Hilbert space completion of V , and the action of $C^*(G)$ on $\mathcal{V}$ is (for functions $f \in \mathcal{H}(G)$) given by

$$\pi(f)(v) = \int_G f(g) \pi(g) v dg. \quad (4.4.16)$$

Every unitary representation (π, V) of G gives a homomorphism of C^* -algebras $\pi : C^*(G) \rightarrow \mathcal{B}(\mathcal{V})$, where $\mathcal{B}(\mathcal{V})$ is the C^* -algebra of bounded linear operators on $\mathcal{V}$. Since it is a homomorphism of C^* -algebras, it must be norm-decreasing, see [Mur90, Theorem 2.1.7].

Lemma 4.4.25. *Suppose $\mu(\sigma) = 0$ and let $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$ such that $\chi(h_\alpha^\vee) > 1$. Then $\mu(\sigma \otimes \chi^t)$ has a pole for some $t \in \mathbb{R}_{>0}$.*

Proof. Suppose not, then by Lemma 4.4.24, $\pi^t := I_P^G(\sigma \otimes \chi^t)$ is a unitary G -representation for all $t \in \mathbb{R}_{\geq 0}$. Let $v \in I_{P \cap K}^K(V_\sigma) \cong I_P^G(V_\sigma)$ such that $\text{supp}(v) = (P \cap K) \cdot U_{-\alpha, r}$ for some r , and such that v is right $U_{-\alpha, r}$ -invariant. Then $v(\bar{u}) = v(1)$ for all $\bar{u} \in U_{-\alpha, r}$. Let C be a compact open subgroup of

G fixing v (this is possible because $I_P^G(V_\sigma)$ is smooth) and put $f = 1_{h_\alpha^\vee C}$. Then $f \in \mathcal{H}(G) \subset C^*(G)$, where $C^*(G)$ is the maximal C^* -algebra of G , so $\|f\|_{C^*(G)} \in \mathbb{R}_{\geq 0}$. We can view π^t as a representation of $C^*(G)$, as written in (4.4.16). We have

$$\begin{aligned} \pi^t(f)v &= \int_G f(g)\pi^t(g)v dg = \int_{h_\alpha^\vee C} \pi^t(g)v dg = \int_{h_\alpha^\vee C} (\sigma \otimes \chi^t)(g)v dg \\ &= \int_{h_\alpha^\vee C} (\sigma \otimes \chi^t)(h_\alpha^\vee)v dg = \text{vol}(h_\alpha^\vee C) \cdot (\sigma \otimes \chi^t)(h_\alpha^\vee)v, \end{aligned}$$

where the second to last equality holds because C is a compact open subgroup fixing v , so the only nontrivial contribution of the action of $h_\alpha^\vee C$ on v comes from $h_\alpha^\vee$. Note that $\pi^t(f)v$ is again right $U_{-\alpha,r}$ -invariant, and supported in $(P \cap K) \cdot U_{-\alpha,r}$.

Since $\pi^t(f)v$ lies in $I_{P \cap K}^K(V_\sigma)$, it also lies in the Hilbert space completion $\mathcal{V}$ of $I_{P \cap K}^K(V_\sigma)$. Let $\mathcal{B}(\mathcal{V})$ be the C^* -algebra of bounded linear operators on $\mathcal{V}$, with the operator norm denoted by $\|\cdot\|_{\text{op}}$. The operator norm of $\pi^t(f)$ is given by

$$\|\pi^t(f)\|_{\text{op}} = \sup_{v' \neq 0} \frac{\|\pi^t(f)v'\|_{\mathcal{V}}}{\|v'\|_{\mathcal{V}}}, \text{ where } \|v'\|_{\mathcal{V}}^2 = \int_K \|v'(k)\|^2 dk.$$

Let v with $\text{supp}(v) = (P \cap K) \cdot U_{-\alpha,r}$ as above. Then

$$\begin{aligned} \|v\|_{\mathcal{V}}^2 &= \int_K \|v(k)\|^2 dk = \int_{(P \cap K) \cdot U_{-\alpha,r}} \|v(k)\|^2 dk \\ &= \text{vol}(U_{-\alpha,r}) \int_{P \cap K} \|v(k)\|^2 dk = \text{vol}(U_{-\alpha,r}) \int_{P \cap K} \|\sigma(k)v(1)\|^2 dk \\ &= \text{vol}(U_{-\alpha,r}) \text{vol}(P \cap K) \|v(1)\|^2, \end{aligned}$$

since $v \in I_{P \cap K}^K(V_\sigma)$ is right $U_{-\alpha,r}$ -invariant and σ is unitary. Similarly, we have

$$\begin{aligned} \|\pi^t(f)v\|_{\mathcal{V}}^2 &= \int_K \|\pi^t(f)v(k)\|^2 dk = \int_{(P \cap K) \cdot U_{-\alpha,r}} \|\pi^t(f)v(k)\|^2 dk \\ &= \text{vol}(U_{-\alpha,r}) \text{vol}(h_\alpha^\vee C)^2 \int_{P \cap K} \|(\sigma \otimes \chi^t)(h_\alpha^\vee)v(k)\|^2 dk \\ &= \text{vol}(U_{-\alpha,r}) \text{vol}(h_\alpha^\vee C)^2 (\chi^t(h_\alpha^\vee))^2 \int_{P \cap K} \|\sigma(h_\alpha^\vee)\sigma(k)v(1)\|^2 dk \\ &= \text{vol}(U_{-\alpha,r}) \text{vol}(h_\alpha^\vee C)^2 (\chi^t(h_\alpha^\vee))^2 \text{vol}(P \cap K) \|v(1)\|^2 \\ &= \text{vol}(h_\alpha^\vee C)^2 (\chi^t(h_\alpha^\vee))^2 \|v\|_{\mathcal{V}}^2. \end{aligned}$$

Recall that since π^t is unitary, it gives a homomorphism of C^* -algebras $\pi^t : C^*(G) \rightarrow \mathcal{B}(\mathcal{V})$, which is norm-decreasing. Since $\chi(h_\alpha^\vee) > 1$, we have for $t \rightarrow \infty$ that

$$\|\pi^t(f)v\|_{\mathcal{V}}/\|v\|_{\mathcal{V}} = \text{vol}(h_\alpha^\vee C) \chi^t(h_\alpha^\vee) \rightarrow \infty, \text{ hence } \lim_{t \rightarrow \infty} \|\pi^t(f)\|_{\text{op}} = \infty$$

which contradicts $\|\pi^t(f)\|_{\text{op}} \leq \|f\|_{C^*(G)}$. So $\mu(\sigma \otimes \chi^t) = \infty$ for some $t > 0$. $\square$

Now we are ready to prove the main result. We drop the assumption $\sigma = \sigma_0$ that was used to prove the results in this subsection so far.

Theorem 4.4.26. *There exists a unitary $\sigma_0 \in \text{Irr}(M)_{[M, \sigma]}$ such that*

$$\mu(\sigma_0 \otimes \chi) = \mu(\sigma_0, z) = c_\mu \cdot \frac{(1-z)(1-z^{-1})}{(1-qz)(1-qz^{-1})} \frac{(1+z)(1+z^{-1})}{(1+q'z)(1+q'z^{-1})},$$

where $c_\mu \in \mathbb{R}_{>0}$ and $q \geq 1$, $q' \geq 1$.

Proof. Suppose that $\mu(\sigma, z)$ is constant, then we can take $q = q' = 1$. Suppose that $\mu(\sigma, z)$ is not constant, then there exists a unitary $\sigma_0 \in \text{Irr}(M)_{[M, \sigma]}$ as in Proposition 4.4.17. By Lemma 4.4.6, $\mu(\sigma_0, z)$ has the form

$$\mu(\sigma_0, z) = c_\mu \prod_i (z - \lambda_i)^{n_i} (z^{-1} - \overline{\lambda_i})^{n_i},$$

where $\lambda_i \in \mathbb{C}$, $n_i \in \mathbb{Z}$, $c_\mu \in \mathbb{R}_{>0}$. By Proposition 4.4.14, $\mu(\sigma_0, z)$ is not zero at $z = 0$ and $z = \infty$. By Proposition 4.4.17, the zero at $z = 1$ is of order 2. So $\mu(\sigma_0, z)$ has a factor $(1-z)(1-z^{-1})$ in the numerator. By Lemma 4.4.25, $\mu(\sigma_0) = 0$ implies that $\mu(\sigma_0 \otimes \chi^t)$ has a pole for some $t \in \mathbb{R}_{>0}$, $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$ with $\chi(h_\alpha^\vee) > 1$. So $\mu(\sigma_0, z)$ has a pole at $q = \chi^t(h_\alpha^\vee) \in \mathbb{R}_{>1}$ and μ has a factor $(1-qz)(1-qz^{-1})$ in the denominator. The only other possible zero of $\mu(\sigma_0, z)$ is at $z = -1$, and again of multiplicity 2, so μ has a factor $(1+z)(1+z^{-1})$ in the numerator (if μ is not zero at $z = -1$, we can cancel out this factor by taking $q' = 1$ in the formula). Suppose that $\mu(\sigma_0, -1) = 0$ and let χ' be a unitary character such that $z = \chi'(h_\alpha^\vee) = -1$. Then $\mu(\sigma_0 \otimes \chi') = 0$, and by Lemma 4.4.25, $\mu((\sigma_0 \otimes \chi') \otimes \chi^t) = \mu(\sigma_0 \otimes \chi' \chi^t)$ has a pole for some $\chi \in \text{Hom}(M/Z(G), \mathbb{R}_{>0})$, $t \in \mathbb{R}_{>0}$. This implies that $\mu(\sigma_0, z)$ has a pole at $z = \chi'(h_\alpha^\vee) \chi^t(h_\alpha^\vee) = -1 \cdot q'$ with $q' \in \mathbb{R}_{>1}$. So μ has a factor $(1+q'z)(1+q'z^{-1})$ in the denominator.

It remains to show that $\mu(\sigma_0, z)$ does not have any additional factors. More factors in the numerator would mean that $\mu(\sigma_0, z)$ has extra zeros in $\mathbb{C}^\times$, which is not possible. More factors in the denominator would mean that $\mu(\sigma_0, \infty) = \mu(\sigma_0, 0) = 0$, which is not possible by Proposition 4.4.14. $\square$

Chapter 5

Bernstein blocks for principal series representations of BD covers

5.1 Overview and results

Let G be a Brylinski-Deligne cover of a split reductive p -adic group. In this chapter, we wish to describe the principal series components in the Bernstein decomposition of $\text{Rep}(G)$. A principal series component is a Bernstein block in $\text{Rep}(G)$ coming from a supercuspidal representation of a minimal Levi subgroup of G . Since G is a cover of a split group G' , a minimal Levi subgroup of G is necessarily the inverse image of a maximal torus of G' . So let T be a maximal torus of G . It is a Brylinski-Deligne cover of a split maximal torus T' of G' . Let σ be an irreducible representation of T , denote by $\mathfrak{s}_T$ its inertial equivalence class for T and denote by $\mathfrak{s}_G$ its inertial equivalence class for G . Let $T_{\text{cpt}} = T(\mathcal{O}_F)$ be the maximal compact subgroup of T . Then by the Bernstein decomposition (Proposition 2.2.12), we have equivalences of categories

$$\begin{aligned}\text{Rep}_{\mathfrak{s}_T}(T) &\simeq \text{End}_T(\text{ind}_{T_{\text{cpt}}}^T(\sigma|_{T_{\text{cpt}}})) - \text{Mod} \text{ and} \\ \text{Rep}_{\mathfrak{s}_G}(G) &\simeq \text{End}_G(I_P^G(\text{ind}_{T_{\text{cpt}}}^T(\sigma|_{T_{\text{cpt}}})) - \text{Mod}.\end{aligned}$$

In this chapter, we describe the structure of these endomorphism algebras, applying the results in [Sol22]. These methods were written for reductive groups, but using the results in Chapter 4, they generalize to covering groups.

In fact, it turns out that $\sigma|_{T_{\text{cpt}}}$ decomposes without multiplicities larger than one, so we may work with smaller progenerators $\text{ind}_{T_{\text{cpt}}}^T(\sigma_1)$ for $\text{Rep}_{\mathfrak{s}_T}(T)$ and $I_P^G(\text{ind}_{T_{\text{cpt}}}^T(\sigma_1))$ for $\text{Rep}_{\mathfrak{s}_G}(G)$, where σ_1 is an irreducible subrepresentation of $\sigma|_{T_{\text{cpt}}}$. The main result of this chapter is the following.

Theorem 5.1.1. *In the setting described above, there are equivalences of categories*

$$\begin{aligned}\mathrm{Rep}_{\mathfrak{s}_T}(T) &\simeq \mathrm{End}_T(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\sigma_1)) - \mathrm{Mod} \text{ and} \\ \mathrm{Rep}_{\mathfrak{s}_G}(G) &\simeq \mathrm{End}_G(I_P^G(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\sigma_1))) - \mathrm{Mod}.\end{aligned}$$

The endomorphism algebras have the following structure.

- (i) $\mathrm{End}_T(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\sigma_1)) \cong \mathbb{C}[\mathcal{O}]$ is the ring of regular functions on the complex algebraic torus $\mathcal{O} = \sigma X_{\mathrm{nr}}(T) \cong X_{\mathrm{nr}}(Z(T))$.
- (ii) $\mathrm{End}_G(I_P^G(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\sigma_1))) \cong \mathcal{H}(\mathcal{O}, G) \rtimes \mathbb{C}[R(\mathcal{O}), \mathfrak{h}]$ where $\mathcal{H}(\mathcal{O}, G)$ is an affine Hecke algebra with known parameters containing $\mathbb{C}[\mathcal{O}]$ as a subalgebra, $R(\mathcal{O})$ is a finite group and $\mathfrak{h}$ is a cocycle that may be nontrivial.

Let us elaborate on the structure of $\mathcal{H}(\mathcal{O}, G) \rtimes \mathbb{C}[R(\mathcal{O}), \mathfrak{h}]$. It has a $\mathbb{C}[\mathcal{O}]$ -basis indexed by

$$W(T, \mathcal{O}) := \{w \in W = N_G(T)/T : w \cdot \sigma \in \mathcal{O}\} = W(\Sigma_{\mathcal{O}, \mu}) \rtimes R(\mathcal{O})$$

where $\Sigma_{\mathcal{O}, \mu}$ is the set of reduced roots α of (G, T) such that the Harish-Chandra function μ_α has a zero on $\mathcal{O}$. It determines a root system, and the parabolic subgroup P determines a basis $\Delta_{\mathcal{O}, \mu}$. The underlying vector space of $\mathcal{H}(\mathcal{O}, G)$ is $\mathbb{C}[\mathcal{O}] \otimes_{\mathbb{C}} \mathbb{C}[W(\Sigma_{\mathcal{O}, \mu})]$ and the multiplication rules are as follows:

- $\mathbb{C}[Z(T)/Z(T)_{\mathrm{cpt}}] \cong \mathbb{C}[\mathcal{O}]$ is embedded as a commutative subalgebra;
- $\mathbb{C}[W(\Sigma_{\mathcal{O}, \mu})] = \langle \theta_w : w \in W(\Sigma_{\mathcal{O}, \mu}) \rangle$ is embedded as the Iwahori-Hecke algebra $\mathcal{H}(W(\Sigma_{\mathcal{O}, \mu}), q_F)$, i.e.

$$\begin{aligned}\theta_w \theta_v &= \theta_{wv} \text{ if } \ell_{\mathcal{O}}(w) + \ell_{\mathcal{O}}(v) = \ell_{\mathcal{O}}(wv) \\ (\theta_{s_\alpha} + 1)(\theta_{s_\alpha} - q_F) &= 0 \text{ if } \alpha \in \Delta_{\mathcal{O}, \mu}\end{aligned}$$

- for $\alpha \in \Delta_{\mathcal{O}, \mu}$ and $t \in Z(T)/Z(T)_{\mathrm{cpt}}$ corresponding to the element $X_t \in \mathbb{C}[Z(T)/Z(T)_{\mathrm{cpt}}]$,

$$X_t \theta_{s_\alpha} - \theta_{s_\alpha} X_{s_\alpha(t)} = (q_F - 1 + X_\alpha^{-1}(q_\alpha - q_{\alpha*})) \frac{X_t - X_{s_\alpha(t)}}{1 - X_\alpha^{-2}},$$

where $\mathbb{C}[\mathcal{O}] \ni X_\alpha : \chi \mapsto \chi(h_\alpha^\vee)$ with $h_\alpha^\vee$ as in Section 4.4.1 and $q_\alpha, q_{\alpha*}$ are the poles of μ_α . Depending on the root α , they are either given by $q_\alpha = q_F$ and $q_{\alpha*} = 1$ or by $q_\alpha = q_{\alpha*} = q_F^{1/2}$. So the equation above simplifies to either

$$X_t \theta_{s_\alpha} - \theta_{s_\alpha} X_{s_\alpha(t)} = (q_F - 1) \frac{X_t - X_{s_\alpha(t)}}{1 - X_\alpha^{-1}}$$

or to

$$X_t \theta_{s_\alpha} - \theta_{s_\alpha} X_{s_\alpha(t)} = (q_F - 1) \frac{X_t - X_{s_\alpha(t)}}{1 - X_\alpha^{-2}},$$

depending on α .

There is a 2-cocycle $\natural : W(T, \mathcal{O})^2 \rightarrow \mathbb{C}[\mathcal{O}]^\times$ describing how the elements in $\mathbb{C}[R(\mathcal{O}), \natural]$ multiply with the elements in $\mathcal{H}(\mathcal{O}, G)$ [Sol22, Theorem 10.6(b)].

Part (i) of Theorem 5.1.1 is proved in Section 5.2.1. In Section 5.2.2, we show with an example that Theorem 5.1.1 in general does not hold for covering groups that are not BD covering groups, in the sense that in general one can not work with the smaller progenitors for $\text{Rep}_{\mathfrak{s}_T}(T)$ and $\text{Rep}_{\mathfrak{s}_G}(G)$. Part (ii) of Theorem 5.1.1, together with the structure of $\mathcal{H}(\mathcal{O}, G)$ as given above, is discussed in Section 5.3. In Section 5.3.3, we show with an example that the cocycle $\natural : W(T, \mathcal{O})^2 \rightarrow \mathbb{C}[\mathcal{O}]^\times$ in $\mathcal{H}(\mathcal{O}, G) \rtimes \mathbb{C}[R(\mathcal{O}), \natural]$ can be nontrivial.

5.2 Endomorphism algebras for covers of tori

In this section, we discuss the structure of $\text{Rep}_{\mathfrak{s}_T}(T)$. For BD covers of tori, we give a general (nice) description. We show with an example that for non-BD covers, the description is not as nice.

5.2.1 The endomorphism algebra for a BD torus

Consider a Brylinski-Deligne cover

$$1 \rightarrow \mu_n \rightarrow T \rightarrow T' \rightarrow 1,$$

let σ be an irreducible representation of T and denote by $\mathfrak{s}_T$ its inertial equivalence class for T . Let $T_{\text{cpt}} = T(\mathcal{O}_F)$ be the maximal compact subgroup of T . Recall from Proposition 2.2.12 that

$$\text{Rep}_{\mathfrak{s}_T}(T) \simeq \text{End}_T(\text{ind}_{T_{\text{cpt}}}^T(\sigma|_{T_{\text{cpt}}})) - \text{Mod}.$$

In this section, we wish to describe the structure of $\text{End}_T(\text{ind}_{T_{\text{cpt}}}^T(\sigma|_{T_{\text{cpt}}}))$, using the method in [Sol22, §2].

As explained in [Wei09], there is a one-to-one correspondence between the irreducible representations of T and the central characters of T . This correspondence is given explicitly as follows. Let $\chi : Z(T) \rightarrow \mathbb{C}^\times$ be a genuine central character, where *genuine* means that its restriction to μ_n is a faithful unitary character. We can extend it to a character $\tilde{\chi}$ of a maximal abelian subgroup A of T . By Lemma 2.1.20, we may take $A = Z(T)T_{\text{cpt}}$. Then $\pi_\chi = \text{ind}_A^T \tilde{\chi}$ is the unique irreducible genuine representation of T with central character χ . This gives us a bijection

$$\begin{aligned} \{\text{genuine characters of } Z(T)\} &\leftrightarrow \{\text{irreducible genuine representations of } T\} \\ \chi &\mapsto \pi_\chi. \end{aligned} \tag{5.2.1}$$

For such a representation π_χ of T , the pair $(T_{\text{cpt}}, \tilde{\chi}|_{T_{\text{cpt}}})$ is a $[T, \pi_\chi]_T$ type [Wan24, Lemma 3.5].

We follow [Sol22, §2] and want to determine the groups $T_{\pi_\chi}^2$, $T_{\pi_\chi}^3$ and $T_{\pi_\chi}^4$ for an irreducible genuine representation $\pi_\chi = \text{ind}_A^T \tilde{\chi}$ of T . Let us start with the group

$$T_{\pi_\chi}^2 := \bigcap_{\chi' \in X_{\text{nr}}(T, \pi_\chi)} \ker \chi',$$

where

$$X_{\text{nr}}(T, \pi_\chi) = \{\chi' \in X_{\text{nr}}(T) : \pi_\chi \otimes \chi' \cong \pi_\chi\}.$$

Because of the bijection 5.2.1, for $\pi_\chi \otimes \chi'$ to be isomorphic to π_χ , it is sufficient and necessary for them to have the same central character. It follows that

$$X_{\text{nr}}(T, \pi_\chi) = \text{Hom}(T/A, \mathbb{C}^\times), \text{ hence } T_{\pi_\chi}^2 = A. \quad (5.2.2)$$

To determine the groups $T_{\pi_\chi}^3$ and $T_{\pi_\chi}^4$, we need to fix an irreducible subrepresentation (σ_1, E_1) of $(\pi_\chi, E)|_{T_{\text{cpt}}}$ such that the stabilizer of the subspace $E_1 \subset E$ in T is maximal. Then by definition,

$$T_{\pi_\chi}^3 := \{t \in T : \pi_\chi(t)E_1 = E_1\} \text{ and } T_{\pi_\chi}^4 := \{t \in T : t \cdot \sigma_1 \cong \sigma_1\}.$$

There is a sequence of inclusions $T_{\text{cpt}} \subset T_{\pi_\chi}^2 \subset T_{\pi_\chi}^3 \subset T_{\pi_\chi}^4 \subset T$ and the groups are all normal subgroups of T . Denote by $\mu_{\pi_\chi, 1}$ the multiplicity of σ_1 in $\pi_\chi|_{T_{\text{cpt}}}$, then by [Roc09, Lemma 1.6.3.1],

$$[T_{\pi_\chi}^4 : T_{\pi_\chi}^3] = [T_{\pi_\chi}^3 : T_{\pi_\chi}^2] = \mu_{\pi_\chi, 1}. \quad (5.2.3)$$

In our case, we may pick σ_1 to be the character $\tilde{\chi}|_{T_{\text{cpt}}}$.

Lemma 5.2.1. *Let χ be a genuine character of $Z(T)$, let $\tilde{\chi}$ be its extension to $A = Z(T)T_{\text{cpt}}$ and let $\pi_\chi = \text{ind}_A^T \tilde{\chi}$ be the corresponding irreducible representation of T . Then the multiplicity $\mu_{\pi_\chi, 1}$ of $\tilde{\chi}|_{T_{\text{cpt}}}$ in $\pi_\chi|_{T_{\text{cpt}}}$ equals one.*

Proof. We want to know the multiplicity of $\tilde{\chi}|_{T_{\text{cpt}}}$ in $\pi_\chi|_{T_{\text{cpt}}}$. The representation $\pi_\chi = \text{ind}_A^T \tilde{\chi}$ is constructed by algebraic induction [Wei09, Theorem 3.1], since A has finite index ℓ in T . Let $t_1, \dots, t_\ell$ be representatives in T of the elements in the group T/A . For $t \in T$, denote its image in T/A by $\bar{t}$. The representation π_χ is given on the vector space $E = \bigoplus_{i=1}^\ell \bar{t}_i \mathbb{C}$, on which T acts via

$$t \cdot \sum_{i=1}^\ell \bar{t}_i v_i = \sum_{i=1}^\ell \overline{t_{j(i)}} \tilde{\chi}(a_i) v_i, \quad (5.2.4)$$

where $\overline{t_{j(i)}} = \overline{tt_i}$, and $a_i \in A$ such that $tt_i = t_{j(i)}a_i$. Note that T/A is abelian since it is a quotient of T' , so $t_{j(i)} = t_i$. Now we want to describe the restriction to T_{cpt} , so let $t \in T_{\text{cpt}}$. Then for every i , we can write $tt_i = t_i \varepsilon_i t$ where $\varepsilon_i = [t, t_i] \in \mu_n$, $\varepsilon_i t \in A$. Therefore

$$t \cdot \sum_{i=1}^\ell \bar{t}_i v_i = \sum_{i=1}^\ell \bar{t}_i \tilde{\chi}(\varepsilon_i) \tilde{\chi}(t) v_i.$$

Hence, $\pi_\chi|_{T_{\text{cpt}}}$ is the direct sum of the characters

$$\chi_i : t \mapsto \tilde{\chi}([t, t_i]) \tilde{\chi}(t), \quad 1 \leq i \leq \ell.$$

We want to know how many of these are isomorphic to $\tilde{\chi}|_{T_{\text{cpt}}}$. Since $\tilde{\chi}$ is a genuine character, χ_i is isomorphic to $\tilde{\chi}|_{T_{\text{cpt}}}$ if and only if $[t, t_i] = 1$ for all $t \in T_{\text{cpt}}$. Recall that $A = Z(T)T_{\text{cpt}}$ is the maximal abelian subgroup of T

containing T_{cpt} , and that the t_i are representatives of the elements of T/A . Hence, $[t, t_i] = 1$ for all $t \in T_{\text{cpt}}$ implies that $t_i \in A$, i.e. $\bar{t}_i = 1$ in A . It follows that only one of the irreducible subrepresentations of $\pi_\chi|_{T_{\text{cpt}}}$ is isomorphic to $\tilde{\chi}|_{T_{\text{cpt}}}$. So the multiplicity $\mu_{\pi_\chi, 1} = 1$. $\square$

From now on, we write $\sigma := \pi_\chi$. By (5.2.2), Lemma 5.2.1 and (5.2.3), we may conclude that $T_\sigma^4 = T_\sigma^3 = T_\sigma^2 = A =: T_\sigma$. Since we have $\mu_{\sigma, 1} = 1$, we are in the situation of [Sol22, Working hypothesis 10.2] and we may work with a smaller progenerator of $\text{Rep}_{s_T}(T)$. Namely, consider the irreducible subrepresentation $\sigma_1 := \tilde{\chi}|_{T_{\text{cpt}}}$ of $\sigma|_{T_{\text{cpt}}}$. Then $\text{End}_T(\text{ind}_{T_{\text{cpt}}}^T(\sigma_1))$ is also a progenerator for $\text{Rep}_{s_T}(T)$ and we have an equivalence of categories

$$\text{Rep}_{s_T}(T) \simeq \text{End}_T(\text{ind}_{T_{\text{cpt}}}^T(\sigma_1)) - \text{Mod}.$$

Write $\mathcal{O} = \sigma X_{\text{nr}}(T)$, then the bijection $X_{\text{nr}}(T)/X_{\text{nr}}(T, \sigma) \rightarrow \mathcal{O}$ gives $\mathcal{O}$ the structure of a complex variety, which can also be identified with $X_{\text{nr}}(Z(T))$. Let $\mathbb{C}[T_\sigma/T_{\text{cpt}}]$ be the group algebra of $T_\sigma/T_{\text{cpt}} = Z(T)/Z(T)_{\text{cpt}}$ and let $\mathbb{C}[\mathcal{O}]$ be the ring of regular functions on $\mathcal{O}$. Then by [Sol22, §10.1],

$$\text{End}_T(\text{ind}_{T_{\text{cpt}}}^T(\sigma_1)) \cong \mathbb{C}[T_\sigma/T_{\text{cpt}}] \cong \mathbb{C}[\mathcal{O}].$$

Recall that $Y_{Q,n}$ is the cocharacter lattice of $Z(T)$. So $\mathbb{C}[\mathcal{O}] \cong \mathbb{C}[Y_{Q,n}]$, which is the Hecke algebra $\mathcal{H}(T, \sigma_1)$ arising from the type $(T_{\text{cpt}}, \sigma_1)$ for $[T, \sigma]_T$ [Wan24].

5.2.2 Example of the endomorphism algebra for a non-BD covering torus

In this section, we will give an example of a covering torus $\tilde{T}$ of T' and a representation σ of $\tilde{T}$, such that the multiplicity $\mu_{\sigma, 1}$ is greater than 1. By the results in the previous section, it will be necessary that $\tilde{T}$ is not a Brylinski-Deligne cover. We use the tilde notation $\tilde{T}$ to emphasize that $\tilde{T}$ is a covering group of T' that is not a BD extension.

Let $T' = F^\times \times F^\times$ and consider a central extension

$$1 \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow \tilde{T} \rightarrow T' \rightarrow 1,$$

which is defined as follows. Let $y_1, y_2 : F^\times \rightarrow T'$ be the inclusion maps into the first and second copy of $F^\times$ in T' . As a set, $\tilde{T} = (\mathbb{Z}/n\mathbb{Z}) \times T'$. Write $s : T' \rightarrow \{0\} \times T' \subset \tilde{T}, t \mapsto (0, t)$. Let the group law on $\tilde{T}$ be determined by

$$\begin{aligned} s(y_1(a))s(y_1(a')) &= s(y_1(a)y_1(a')); \\ s(y_2(b))s(y_2(b')) &= s(y_2(b)y_2(b')); \\ s(y_1(a))s(y_2(b)) &= (v(a)v(b) \pmod{n}) \cdot s(y_1(a)y_2(b)); \\ s(y_2(b))s(y_1(a)) &= s(y_2(b)y_1(a)) \end{aligned}$$

for all $a, a', b, b' \in F^\times$, where $v : F^\times \rightarrow \mathbb{Z}$ is the valuation on F . Then the commutator $[\cdot, \cdot] : T' \times T' \rightarrow \mathbb{Z}/n\mathbb{Z}$ is given by:

$$\begin{aligned} [y_1(a), y_2(b)] &= \mathbf{s}(y_1(a))\mathbf{s}(y_2(b))\mathbf{s}(y_1(a))^{-1}\mathbf{s}(y_2(b))^{-1} \\ &= v(a)v(b) \pmod{n} \in \mathbb{Z}/n\mathbb{Z}; \\ [y_1(a), y_1(a')] &= [y_2(b), y_2(b')] = 1 \end{aligned}$$

for all $a, a', b, b' \in F^\times$.

The center of $\tilde{T}$ is generated by $\mathbb{Z}/n\mathbb{Z}$ and the $\mathbf{s}(y_1(a))$, $\mathbf{s}(y_2(b))$ where $n \mid v(a)$ and $n \mid v(b)$. Write Z for the subgroup of both T' and $\tilde{T}$, which is generated by the $y_1(a)$, $y_2(b)$ where $n \mid v(a)$ and $n \mid v(b)$. The group $T'_{\text{cpt}} = T'(\mathcal{O}_F^\times)$ is a subgroup of $\tilde{T}$ contained in Z . We have a short exact sequence

$$\mathbb{Z}/n\mathbb{Z} \hookrightarrow \tilde{T}/Z \twoheadrightarrow T'/Z \cong (T'/T'_{\text{cpt}})/(Z/T'_{\text{cpt}}) \cong (F^\times/\mathcal{O}_F^\times)^2/(\pi^n)^2 \cong (\mathbb{Z}/n\mathbb{Z})^2.$$

So the quotient group $H := \tilde{T}/Z$ is a central extension of $(\mathbb{Z}/n\mathbb{Z})^2$ by $\mathbb{Z}/n\mathbb{Z}$. In fact, H is isomorphic to the finite Heisenberg group $H_3(\mathbb{Z}/n\mathbb{Z})$ of upper unitriangular 3×3 matrices over $\mathbb{Z}/n\mathbb{Z}$. The center of $\tilde{T}/Z$ is $\mathbb{Z}/n\mathbb{Z}$. Let $\varepsilon : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{C}^\times$ be a genuine character, then H has a unique irreducible representation of dimension n with central character ε . This lifts to an irreducible representation π_ε of $\tilde{T}$ which is trivial on T'_{cpt} . The restriction of π_ε to $\tilde{T}_{\text{cpt}} = \mathbb{Z}/n\mathbb{Z} \times T'_{\text{cpt}}$ becomes n copies of $\varepsilon \times \text{triv}$. So the multiplicity of $\varepsilon \times \text{triv}$ in $\pi_\varepsilon|_{\tilde{T}_{\text{cpt}}}$ is $\mu_{\pi_\varepsilon, 1} = n$.

Let us work this out in more detail. The quotient group $H := \tilde{T}/Z$ fits in an exact sequence

$$0 \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow H \rightarrow (\mathbb{Z}/n\mathbb{Z})^2 \rightarrow 0.$$

As a set, $H = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Denote its elements by (c, a, b) . The group operation on H is then given by:

$$(c, a, b)(c', a', b') = (c + c' + ab', a + a', b + b').$$

The center of H equals the first copy of $\mathbb{Z}/n\mathbb{Z}$ in H . Let $\varepsilon : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{C}^\times$ be a genuine character, $\varepsilon : 1 \mapsto \zeta_n$ where ζ_n is a primitive n -th root of unity. We want to build an irreducible representation of H of dimension n with central character ε . The group H is generated by the elements $z := (1, 0, 0)$, $x := (0, 1, 0)$, $y := (0, 0, 1)$. Consider the n -dimensional vector space $V := \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} v_i \mathbb{C}$ and let H act on it via

$$z : v_i \mapsto \zeta_n v_i; \quad x : v_i \mapsto \zeta_n^i v_i; \quad y : v_i \mapsto v_{i+1}. \quad (5.2.5)$$

In H , we have that $xyx^{-1}y^{-1} = z$, and indeed

$$v_i \xrightarrow{y^{-1}} v_{i-1} \xrightarrow{x^{-1}} \zeta_n^{-(i-1)} v_{i-1} \xrightarrow{y} \zeta_n^{-(i-1)} v_i \xrightarrow{x} \zeta_n^{i-(i-1)} v_i = \zeta_n v_i = z \cdot v_i.$$

So (5.2.5) is well-defined. It gives the unique n -dimensional representation of H with central character ε . It extends to an irreducible n -dimensional

representation of $\tilde{T}$, which we will denote by π_ε . For this representation, we want to determine the groups $\tilde{T}_{\pi_\varepsilon}^2, \tilde{T}_{\pi_\varepsilon}^3, \tilde{T}_{\pi_\varepsilon}^4$, like in [Sol22, §2].

Firstly, we have

$$\tilde{T}_{\pi_\varepsilon}^2 = \bigcap_{\chi \in X_{\text{nr}}(\tilde{T}, \pi_\varepsilon)} \ker \chi,$$

where

$$X_{\text{nr}}(\tilde{T}, \pi_\varepsilon) = \{\chi \in X_{\text{nr}}(\tilde{T}) : \pi_\varepsilon \otimes \chi \cong \pi_\varepsilon\}.$$

By [Wei09, Theorem 3.1], for $\chi \in X_{\text{nr}}(\tilde{T}) = \text{Hom}(\tilde{T}/\tilde{T}_{\text{cpt}}, \mathbb{C}^\times)$, we have that $\pi_\varepsilon \otimes \chi \cong \pi_\varepsilon$ if and only if the two representations have the same central character. Recall that $\tilde{T}_{\text{cpt}} \subset Z(\tilde{T})$. Hence, $X_{\text{nr}}(\tilde{T}, \pi_\varepsilon) = \text{Hom}(\tilde{T}/Z(\tilde{T}), \mathbb{C}^\times)$ and $\tilde{T}_{\pi_\varepsilon}^2 = Z(\tilde{T})$. This is the same as in the case of a BD covering torus.

Now, let us look at the group $\tilde{T}_{\pi_\varepsilon}^3$. Recall that $\pi_\varepsilon|_{\tilde{T}_{\text{cpt}}}$ is n copies of $\varepsilon \times \text{triv}$ on the vector space $V = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} v_i \mathbb{C}$. Consider the irreducible subrepresentation $(\varepsilon \times \text{triv}, v_1 \mathbb{C})$. We want to determine the group

$$\tilde{T}_{\pi_\varepsilon}^3 = \{t \in \tilde{T} : t \cdot v_1 \mathbb{C} = v_1 \mathbb{C}\}.$$

The representation π_ε factors through the quotient $H = \tilde{T}/Z$. The generators $z = (1, 0, 0)$, $x = (0, 1, 0)$ and $y = (0, 0, 1)$ of H satisfy $z \cdot v_1 \mathbb{C} = v_1 \mathbb{C}$, $x \cdot v_1 \mathbb{C} = v_1 \mathbb{C}$ and $y \cdot v_1 \mathbb{C} = v_2 \mathbb{C}$. So $\tilde{T}_{\pi_\varepsilon}^3$ is generated by Z and representatives of z and x in $\tilde{T}$. More precisely, $\tilde{T}_{\pi_\varepsilon}^3$ is generated by $Z(\tilde{T})$ and the first copy of $F^\times$ in T' . The index of $\tilde{T}_{\pi_\varepsilon}^2 = Z(\tilde{T})$ in $\tilde{T}_{\pi_\varepsilon}^3$ equals $n = \mu_{\pi_\varepsilon, 1}$, as required.

Lastly, we want to determine the group

$$\tilde{T}_{\pi_\varepsilon}^4 = \{t \in \tilde{T} : t \cdot (\varepsilon \times \text{triv}) \cong \varepsilon \times \text{triv}\}.$$

The representation $t \cdot (\varepsilon \times \text{triv})$ is given by $t \cdot (\varepsilon \times \text{triv})(t') = (\varepsilon \times \text{triv})(t^{-1}t't) = \pi_\varepsilon(t^{-1}t't)$ for all $t' \in \tilde{T}_{\text{cpt}} \cong \mathbb{Z}/n\mathbb{Z} \times T'_{\text{cpt}}$. By the arguments above, it only remains to check how $y = (0, 0, 1) \in H$ acts on $\varepsilon \times \text{triv}$. The group $\tilde{T}_{\text{cpt}}/T'_{\text{cpt}} \cong \mathbb{Z}/n\mathbb{Z}$ is generated by z . We have

$$v_i \xrightarrow{y} v_{i+1} \xrightarrow{z} \zeta_n v_{i+1} \xrightarrow{-y} \zeta_n v_i.$$

So $y \cdot (\varepsilon \times \text{triv}) \cong \varepsilon \times \text{triv}$, and it follows that $\tilde{T}_{\pi_\varepsilon}^4 = \tilde{T}$. The index of $\tilde{T}_{\pi_\varepsilon}^3$ in $\tilde{T}_{\pi_\varepsilon}^4$ is n , as required.

Then by [Sol22, (2.25)],

$$\text{End}_{\tilde{T}}(\text{ind}_{\tilde{T}_{\text{cpt}}}^{\tilde{T}}(\pi_\varepsilon|_{\tilde{T}_{\text{cpt}}})) \cong \mathbb{C}[X_{\text{nr}}(\tilde{T})] \rtimes \mathbb{C}[X_{\text{nr}}(\tilde{T}, \pi_\varepsilon), \mathfrak{h}]$$

where the cocycle $\mathfrak{h}$ is non-trivial because $\mu_{\pi_\varepsilon, 1} = n > 1$.

5.3 Endomorphism algebras for BD covers

In this section, we go back to the situation where G is a Brylinski-Deligne covering group with maximal BD torus T , as described at the start of this

chapter. Let $\chi : Z(T) \rightarrow \mathbb{C}^\times$ be a genuine central character of T , and let σ be the unique irreducible genuine representation of T with central character χ , as described in §5.2.1. Let σ_1 be an irreducible subrepresentation of $\sigma|_{T_{\text{cpt}}}$. By the results in §5.2.1, we have $\mu_{\sigma,1} = 1$ and $T_\sigma = Z(T)T_{\text{cpt}}$. The complex algebraic variety $\mathcal{O} = \sigma X_{\text{nr}}(T)$ can be identified with $X_{\text{nr}}(Z(T))$. Let $\mathbb{C}[T_\sigma/T_{\text{cpt}}]$ be the group algebra of $T_\sigma/T_{\text{cpt}} = Z(T)/Z(T)_{\text{cpt}}$ and let $\mathbb{C}[\mathcal{O}]$ be the ring of regular functions on $\mathcal{O}$. Then by [Sol22, §10.1],

$$\text{End}_T(\text{ind}_{T_{\text{cpt}}}^T(\sigma_1)) \cong \mathbb{C}[T_\sigma/T_{\text{cpt}}] \cong \mathbb{C}[\mathcal{O}].$$

The next step is now to describe the algebra $\text{End}_G(I_P^G(\text{ind}_{T_{\text{cpt}}}^T(\sigma_1)))$, which is a progenerator for the category $\text{Rep}_{\mathfrak{s}_G}(G)$. Let

$$W = N_G(T)/T = N_{G'}(T')/T'.$$

By [Sol22, Theorem 10.9], the algebra $\text{End}_G(I_P^G(\text{ind}_{T_{\text{cpt}}}^T(\sigma_1)))$ has a $\mathbb{C}[\mathcal{O}]$ -basis indexed by

$$W(T, \mathcal{O}) := \{w \in W : w \cdot \sigma \in \mathcal{O}\}.$$

This group has a decomposition $W(T, \mathcal{O}) = W(\Sigma_{\mathcal{O}, \mu}) \rtimes R(\mathcal{O})$, where $\Sigma_{\mathcal{O}, \mu}$ is the set of reduced roots α of (G, T) such that the Harish-Chandra function μ_α has a zero on $\mathcal{O}$. Moreover, there exists an affine Hecke algebra $\mathcal{H}(\mathcal{O}, G)$ with torus $\mathcal{O}$, and a 2-cocycle $\natural : W(T, \mathcal{O})^2 \rightarrow \mathbb{C}[\mathcal{O}]^\times$ such that

$$\text{End}_G(I_P^G(\text{ind}_{T_{\text{cpt}}}^T(\sigma_1))) \cong \mathcal{H}(\mathcal{O}, G) \rtimes \mathbb{C}[R(\mathcal{O}), \natural]. \quad (5.3.1)$$

Our aim is to determine the parameters of the affine Hecke algebra $\mathcal{H}(\mathcal{O}, G)$. One could also hope for an easier description of the 2-cocycle $\natural$ in this setting, but we will illustrate with an example in Section 5.3.3 that this is not achievable.

Let us explain in more detail how the algebra $\mathcal{H}(\mathcal{O}, G)$ is built. The set $\Sigma_{\mathcal{O}, \mu}$ of reduced roots α of (G, T) such that μ_α has a zero on $\mathcal{O}$, determines a reduced root system with basis $\Delta_{\mathcal{O}, \mu}$ determined by the parabolic subgroup $P = TU$. Recall that μ_α is computed in M_α , which is a Levi subgroup of G that contains the root subgroup U_α and which has T as a maximal Levi subgroup. For $\alpha \in \Sigma_{\mathcal{O}, \mu}$, let $h_\alpha^\vee$ be the unique generator of $(T_\sigma \cap M_\alpha^1)/T_{\text{cpt}} \cong \mathbb{Z}$ such that $|\alpha(h_\alpha^\vee)|_F > 1$. Let $X_\alpha : X_{\text{nr}}(T) \rightarrow \mathbb{C}$ be the function $X_\alpha : \chi' \mapsto \chi'(h_\alpha^\vee)$. There exists a unitary $\sigma_0 \in \sigma X_{\text{nr}}(T)$ such that for all $\alpha \in \Sigma_{\mathcal{O}, \mu}$, $s_\alpha \sigma_0 \cong \sigma_0$ and the function μ_α has the form

$$\begin{aligned} \mu_\alpha(\sigma_0 \otimes \cdot) &= \mu_\alpha(\sigma_0, X_\alpha) \\ &= c_\alpha \frac{(1 - X_\alpha)(1 - X_\alpha^{-1})}{(1 - q_\alpha X_\alpha)(1 - q_\alpha X_\alpha^{-1})} \frac{(1 + X_\alpha)(1 + X_\alpha^{-1})}{(1 + q_{\alpha*} X_\alpha)(1 + q_{\alpha*} X_\alpha^{-1})}, \end{aligned} \quad (5.3.2)$$

where $c_\alpha \in \mathbb{R}_{>0}$ and $q_\alpha, q_{\alpha*} \in \mathbb{R}_{\geq 1}$ with $1 \neq q_\alpha \geq q_{\alpha*}$. We replace σ by σ_0 . Let $\Sigma_{\mathcal{O}}^\vee := \{h_\alpha^\vee : \alpha \in \Sigma_{\mathcal{O}, \mu}\}$. There is a corresponding set of coroots $\Sigma_{\mathcal{O}} := \{\alpha^\sharp : \alpha \in \Sigma_{\mathcal{O}, \mu}\}$ such that $(\Sigma_{\mathcal{O}}^\vee, T_\sigma/T_{\text{cpt}}, \Sigma_{\mathcal{O}}, (T_\sigma/T_{\text{cpt}})^\vee)$ is a root datum with Weyl group $W(\Sigma_{\mathcal{O}, \mu})$ [Sol22, Proposition 3.1]. It has a basis $\Delta_{\mathcal{O}}^\vee$

determined by P . To this based root datum we associate the affine Hecke algebra

$$\mathcal{H}(\mathcal{O}, G) = \mathcal{H}(\Sigma_{\mathcal{O}}^{\vee}, T_{\sigma}/T_{\text{cpt}}, \Sigma_{\mathcal{O}}, (T_{\sigma}/T_{\text{cpt}})^{\vee}, \lambda, \lambda^*, q_F),$$

where q_F is the cardinality of the residue field of F . The labels λ, λ^* are given by

$$\lambda(h_{\alpha}^{\vee}) = \log(q_{\alpha}q_{\alpha^*})/\log(q_F) \text{ and } \lambda^*(h_{\alpha}^{\vee}) = \log(q_{\alpha}q_{\alpha^*}^{-1})/\log(q_F), \quad (5.3.3)$$

where q_{α} and q_{α^*} come from the poles of μ_{α} in (5.3.2). The underlying vector space of $\mathcal{H}(\mathcal{O}, G)$ is $\mathbb{C}[T_{\sigma}/T_{\text{cpt}}] \otimes_{\mathbb{C}} \mathbb{C}[W(\Sigma_{\mathcal{O}, \mu})]$ and the multiplication rules are as in [Sol25, §1]:

- $\mathbb{C}[T_{\sigma}/T_{\text{cpt}}] \cong \mathbb{C}[\mathcal{O}]$ is embedded as a commutative subalgebra;
- $\mathbb{C}[W(\Sigma_{\mathcal{O}, \mu})] = \langle \theta_w : w \in W(\Sigma_{\mathcal{O}, \mu}) \rangle$ is embedded as the Iwahori-Hecke algebra $\mathcal{H}(W(\Sigma_{\mathcal{O}, \mu}), q_F^{\lambda})$, i.e.

$$\theta_w \theta_v = \theta_{wv} \text{ if } \ell_{\mathcal{O}}(w) + \ell_{\mathcal{O}}(v) = \ell_{\mathcal{O}}(wv)$$

$$(\theta_{s_{\alpha}} + 1)(\theta_{s_{\alpha}} - q_F^{\lambda(\alpha)}) = (\theta_{s_{\alpha}} + 1)(\theta_{s_{\alpha}} - q_{\alpha}q_{\alpha^*}) = 0 \text{ if } \alpha \in \Delta_{\mathcal{O}, \mu}$$

- for $\alpha \in \Delta_{\mathcal{O}, \mu}$ and $t \in T_{\sigma}/T_{\text{cpt}}$ corresponding to $X_t \in \mathbb{C}[T_{\sigma}/T_{\text{cpt}}]$,

$$X_t \theta_{s_{\alpha}} - \theta_{s_{\alpha}} X_{s_{\alpha}(t)} = (q_{\alpha}q_{\alpha^*} - 1 + X_{\alpha}^{-1}(q_{\alpha} - q_{\alpha^*})) \frac{X_t - X_{s_{\alpha}(t)}}{1 - X_{\alpha}^{-2}}.$$

The poles of μ_{α} correspond to the non-unitary $\sigma \otimes \chi' \in \mathcal{O}$ for which $I_{P \cap M_{\alpha}}^{M_{\alpha}}(\sigma \otimes \chi')$ is reducible [Sol25, Corollary 1.3]. For each $\alpha \in \Sigma_{\mathcal{O}, \mu}$, the parameters q_{α} and q_{α^*} may be computed in the group M_{α} , i.e. in the affine Hecke algebra

$$\begin{aligned} \text{End}_{M_{\alpha}}(I_{TU_{\alpha}}^{M_{\alpha}}(\text{ind}_{T_{\text{cpt}}}^T(\sigma_1))) &\cong \mathcal{H}(\mathcal{O}, M_{\alpha}) \\ &= \mathcal{H}(\{\pm h_{\alpha}^{\vee}\}, T_{\sigma}/T_{\text{cpt}}, \{\pm \alpha^{\sharp}\}, (T_{\sigma}/T_{\text{cpt}})^{\vee}, \lambda_{\alpha}, \lambda_{\alpha}^*, q_F), \end{aligned} \quad (5.3.4)$$

where $\lambda_{\alpha} := \lambda|_{\{\pm h_{\alpha}^{\vee}\}}$ and $\lambda_{\alpha}^* := \lambda^*|_{\{\pm h_{\alpha}^{\vee}\}}$. The problem may be simplified even further, by reducing to the simply connected cover of the derived group of M_{α} . This will be discussed in the following section.

5.3.1 Reduction to covers of simply connected groups

In this section, our goal is to generalize the results of [Sol25, §2] to finite central (BD) extensions of reductive p -adic groups. First, we study the more general situation of finite central extensions of reductive groups; later we show how the results apply to BD extensions of reductive groups.

Let G' be a reductive p -adic group and let $\text{Rep}_{\mathfrak{s}'}(G')$ be a Bernstein block in $\text{Rep}(G')$, where $\mathfrak{s}'$ is an inertial equivalence class. In [Sol22], an affine Hecke algebra $\mathcal{H}' := \mathcal{H}(\mathcal{O}', G')$ is associated to $\mathfrak{s}'$, whose module category is shown to be closely related to $\text{Rep}_{\mathfrak{s}'}(G')$. In [Sol25, §2], the analysis of the parameters of $\mathcal{H}'$ was reduced to the case where $\mathcal{G}'$ is absolutely simple and simply connected. The results apply to any homomorphism $\eta' : \tilde{\mathcal{G}}' \rightarrow \mathcal{G}'$ between connected reductive F -groups such that:

- the kernel of $d\eta' : \mathrm{Lie}(\tilde{\mathcal{G}}') \rightarrow \mathrm{Lie}(\mathcal{G}')$ is central, and
- the cokernel of η' is a commutative F -group.

We fix such a map η' . It restricts to a central isogeny between the derived groups and induces a homomorphism on F -rational points.

For central extensions of reductive groups, we also have a notion of (central) isogeny.

Definition 5.3.1. Let G_1, G_2 be central extensions of connected reductive groups G'_1 and G'_2 , by finite abelian groups μ_1 and μ_2 , respectively. A *(central) isogeny* between G_1 and G_2 is a group homomorphism $\phi : G_1 \rightarrow G_2$ with $\phi(\mu_1) \subset \mu_2$, such that the map

$$\phi' : G'_1 \cong G_1/\mu_1 \rightarrow G_2/\mu_2 \cong G'_2$$

is the map on F -rational points coming from a (central) isogeny $\phi' : \mathcal{G}'_1 \rightarrow \mathcal{G}'_2$ between the reductive algebraic groups. The maps fit into a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_1 & \longrightarrow & G_1 & \longrightarrow & G'_1 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \phi & & \downarrow \phi' & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & G_2 & \longrightarrow & G'_2 & \longrightarrow & 1. \end{array}$$

Remark 5.3.2. In a similar way as in Definition 5.3.1, we may extend notions of properties of maps between reductive groups G'_1 and G'_2 to notions of properties of maps between covering groups G_1 and G_2 . The condition will always be that the image of μ_1 lies in μ_2 , and that the map on the quotients $G'_1 \cong G_1/\mu_1 \rightarrow G_2/\mu_2 \cong G'_2$ has the desired property as a map between reductive groups.

Note that given an isogeny $\phi' : \mathcal{G}'_1 \rightarrow \mathcal{G}'_2$, it is in general not always possible to extend it to an isogeny between the extensions G_1 and G_2 . But when it is possible and the isogeny is central, we have a pullback functor $\phi^* : \mathrm{Rep}(G_2) \rightarrow \mathrm{Rep}(G_1)$, satisfying the following result.

Lemma 5.3.3. *Let G_1, G_2 be groups as in Definition 5.3.1, with a central isogeny $\phi : G_1 \rightarrow G_2$. Let π be an irreducible (admissible) representation of G_2 . Then $\phi^*(\pi)$ is a finite direct sum of irreducible (admissible) representations of G_1 .*

Proof. This is [Sil79]. All the arguments generalize to covering groups with a central isogeny (cf. Definition 5.3.1) between them. $\square$

Recall that we fixed a map $\eta' : \tilde{\mathcal{G}}' \rightarrow \mathcal{G}'$ satisfying the conditions stated at the beginning of this section. Let $\tilde{G}$ be a central extension of $\tilde{\mathcal{G}}'$ by a finite abelian group $\tilde{\mu}$. In order for the results in [Sol25, §2] to generalize to our situation, it is required that the map $\eta' : \tilde{\mathcal{G}}' \rightarrow \mathcal{G}'$ lifts to a map $\eta : \tilde{G} \rightarrow G$, in the sense of Remark 5.3.2. So from now on, we assume that such a map η exists. Let G_{der} be the inverse image of G'_{der} in G (similar for $\tilde{G}$). Then η restricts to a central isogeny $\eta : \tilde{G}_{\mathrm{der}} \rightarrow G_{\mathrm{der}}$ in the sense of Definition 5.3.1. We have a pullback functor $\eta^* : \mathrm{Rep}(G) \rightarrow \mathrm{Rep}(\tilde{G})$.

Lemma 5.3.4. *Let $\pi \in \text{Irr}(G)$. Then $\eta^*(\pi)$ is a finite direct sum of irreducible $\tilde{G}$ -representations.*

Proof. By Lemma 5.3.3, the result holds for pullback along the map $\eta_{\text{der}} : \tilde{G}_{\text{der}} \rightarrow G_{\text{der}}$.

Now, let us show that the statement holds for pullback along the inclusion $G_{\text{der}} \rightarrow G$. See also the proof of [Tad92, Lemma 2.1]. Let $\mathcal{C}' := Z(\mathcal{G}')^0$ be the identity component of the center of $\mathcal{G}'$. There is a central isogeny $\mathcal{G}'_{\text{der}} \times \mathcal{C}' \rightarrow \mathcal{G}'$, inducing a map on F -rational points $G'_{\text{der}} \times C' \rightarrow G'$. Since C' is a subgroup of G' , it extends to a subgroup C of G , which is an extension of C' by μ . The group $G_{\text{der}} \times C$ is an extension of $G'_{\text{der}} \times C'$ by $\mu \times \mu$ and the central isogeny $\mathcal{G}'_{\text{der}} \times \mathcal{C}' \rightarrow \mathcal{G}'$ extends to a central isogeny $G_{\text{der}} \times C \rightarrow G$, in the sense of Definition 5.3.1. By Lemma 5.3.3, $\pi|_{G_{\text{der}}C}$ is a finite direct sum of irreducible $G_{\text{der}}C$ -representations. Then by the Schur Lemma, $\pi|_{G_{\text{der}}}$ is a finite direct sum of irreducible G_{der} -representations.

Thus, pullback along the composition $\tilde{G}_{\text{der}} \rightarrow G_{\text{der}} \rightarrow G$ satisfies the result. In other words, $\eta^*(\pi)|_{\tilde{G}_{\text{der}}}$ is a finite direct sum of irreducible $\tilde{G}_{\text{der}}$ -representations. Hence, $\eta^*(\pi)$ is a finite length representation. This implies that $\eta^*(\pi)$ is completely reducible, see [Sil79, Lemma 3]. $\square$

In this setting, one can follow the same strategy as in [Sol25, §2] to show that the Hecke algebra parameters for $\tilde{G}$ and G are related as described in [Sol25, Proposition 2.4]. The statements and the proofs are completely analogous, using the Harish-Chandra μ -function for covering groups discussed in [FS25]. Recall that the condition is that the map $\eta' : \tilde{G}' \rightarrow G'$ extends to a map $\eta : \tilde{G} \rightarrow G$ in the sense of Remark 5.3.2. Suppose that $\tilde{G}$ is the fiber product $\tilde{G} := G \times_{G'} \tilde{G}'$, which is a central extension of $\tilde{G}'$ by μ , fitting in a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu & \longrightarrow & \tilde{G} = G \times_{G'} \tilde{G}' & \longrightarrow & \tilde{G}' \longrightarrow 1 \\ & & \downarrow & & \downarrow \eta & & \downarrow \eta' \\ 1 & \longrightarrow & \mu & \longrightarrow & G & \longrightarrow & G' \longrightarrow 1. \end{array}$$

Then by construction, the map $\eta' : \tilde{G}' \rightarrow G'$ always lifts to a map $\eta : \tilde{G} \rightarrow G$ between the extensions. In this way, the calculation of the Hecke algebra parameters for G may be reduced to the calculation of the Hecke algebra parameters for a finite central extension of $\tilde{G}'$ by μ .

We may apply the above construction with the fiber product to the simply connected cover $\eta_{\text{sc}} : G'_{\text{sc}} \rightarrow G'_{\text{der}} \rightarrow G'$. This way, we may reduce to the case of a covering group of a simply connected group. In the split case, all such groups are described by BD covering groups [BD01, §0]. So if G is a covering group of a split group G' , then we can reduce to a BD covering group of the split simply connected group G'_{sc} .

When G is a BD extension of a split connected reductive group G' , the reduction to a BD extension of G'_{sc} can also be done in the following way. There are inclusions of the cocharacter lattices $Y_{\text{sc}} \hookrightarrow Y_{\text{der}} \hookrightarrow Y$. By restriction of

the quadratic form on Y to Y_{sc} and Y_{der} , we get BD extensions G_{sc} and G_{der} of G'_{sc} and G'_{der} respectively, fitting in a commutative diagram

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & \mu_n & \longrightarrow & G_{\text{sc}} & \longrightarrow & G'_{\text{sc}} & \longrightarrow & 1 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & \mu_n & \longrightarrow & G_{\text{der}} & \longrightarrow & G'_{\text{der}} & \longrightarrow & 1 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & \mu_n & \longrightarrow & G & \longrightarrow & G' & \longrightarrow & 1.
 \end{array}$$

In fact, G_{der} is just the inverse image of the derived subgroup G'_{der} of G' , so we naturally obtain the lower part of the commutative diagram. However, since G'_{sc} is not a subgroup of G' , the existence of the BD extension G_{sc} fitting in the commutative diagram really follows from the inclusion of cocharacter lattices $Y_{\text{sc}} \hookrightarrow Y$. The map $\eta'_{\text{sc}} : G'_{\text{sc}} \rightarrow G'$ satisfies the conditions of [Sol25, §2] and it extends to a map $\eta_{\text{sc}} : G_{\text{sc}} \rightarrow G$ (namely, the map in the commutative diagram). Hence, the calculation of the Hecke algebra parameters for G may be reduced to the calculation of the Hecke algebra parameters for G_{sc} (or for G_{der}), and they are related as in [Sol25, Proposition 2.4].

Concretely, recall that the Hecke algebra parameters are given by the poles of the Harish-Chandra μ -function. So suppose that we want to find the poles $q_\alpha, q_{\alpha*}$ of μ_α in (5.3.2). Let $\tilde{\sigma}$ be an irreducible constituent of $\eta_{\text{sc}}^*(\sigma)$ (see Lemma 5.3.4), which is an irreducible representation of the Levi subgroup $M_{\text{sc}} = \eta_{\text{sc}}^{-1}(M)$ of G_{sc} . Write $\tilde{O} = \tilde{\sigma} X_{\text{nr}}(M_{\text{sc}})$. Then by [Sol25, Corollary 2.3],

$$\Sigma_{\tilde{O}, \mu} = \eta_{\text{sc}}^*(\Sigma_{O, \mu}) = \{\tilde{\alpha} = \alpha \circ \eta_{\text{sc}} : \alpha \in \Sigma_{O, \mu}\}.$$

For each $\alpha \in \Sigma_{O, \mu}$, we can compare the parameters $q_\alpha, q_{\alpha*}$ (poles of μ_α) for G , with the poles $q_{\tilde{\alpha}}, q_{\tilde{\alpha}*}$ of $\mu_{\tilde{\alpha}}$ for G_{sc} . Namely, there exists $N_\alpha \in \{1/2, 1, 2\}$ such that $\eta_{\text{sc}}(h_\alpha^\vee) = (h_{\tilde{\alpha}}^\vee)^{1/N_\alpha}$. Then $q_\alpha, q_{\alpha*}$ and $q_{\tilde{\alpha}}, q_{\tilde{\alpha}*}$ are related according one of the following three options:

- (i) if $N_\alpha = 1$, then $q_\alpha = q_{\tilde{\alpha}}$ and $q_{\alpha*} = q_{\tilde{\alpha}*}$;
- (ii) if $N_\alpha = 2$, then $q_{\alpha*} = 1$ and $q_{\tilde{\alpha}} = q_{\tilde{\alpha}*} = q_\alpha^{1/2}$;
- (iii) if $N_\alpha = 1/2$, then $q_{\tilde{\alpha}*} = 1$ and $q_\alpha = q_{\alpha*} = q_{\tilde{\alpha}}^{1/2}$.

We will illustrate with an example how the parameters may be related.

Example 5.3.5. Let G be a BD extension of $G' = \text{PGL}_2(F)$ by μ_n . The simply connected cover of $\text{PGL}_2(F)$ equals $\text{SL}_2(F)$. Let G_{sc} be the BD extension of $G'_{\text{sc}} = \text{SL}_2(F)$ determined by restricting the bilinear form B_Q on Y to Y_{sc} . We have a map $\eta : G_{\text{sc}} \rightarrow G$ and we want to find N_α such that $\eta(h_\alpha^\vee) = (h_{\tilde{\alpha}}^\vee)^{1/N_\alpha}$.

We have $G^1 = G$ by Lemma 2.1.13, because G' is semisimple. Hence, the element $h_\alpha^\vee$ is the unique generator of $(Z(T)T_{\text{cpt}})/T_{\text{cpt}} = Z(T)/(Z(T) \cap T_{\text{cpt}}) \cong$

$\mathbb{Z}$ such that $|\alpha(h_\alpha^\vee)|_F > 1$. Recall that $Z(T)$ is described by $Y_{Q,n} = \{y \in Y : B_Q(y, z) \in n\mathbb{Z} \ \forall z \in Y\}$. Let e_1 be the generator of Y given by

$$e_1 : x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \in T' \subset \mathrm{PGL}_2(F)$$

and let $n_1 := B_Q(e_1, e_1)$. Then $Y_{Q,n} = (n/\mathrm{gcd}(n_1, n))Y$, so $n_2 e_1$ is a generator of $Y_{Q,n}$, where $n_2 := n/\mathrm{gcd}(n_1, n)$. Since $h_\alpha^\vee$ is an element of $Z(T)/(Z(T) \cap T_{\mathrm{cpt}})$ and $\mu_n \subset Z(T) \cap T_{\mathrm{cpt}}$, we may work with a representative of $h_\alpha^\vee$ in $\mathrm{PGL}_2(F)$, which we will take to be $e_1(\varpi^{-n_2})$.

Let $\tilde{T}'$ be the maximal torus of $G'_{\mathrm{sc}} = \mathrm{SL}_2(F)$. Let $\tilde{e}_1$ be the generator of Y_{sc} given by

$$\tilde{e}_1 : x \mapsto \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in \tilde{T}' \subset \mathrm{SL}_2(F).$$

Note that in $\mathrm{PGL}_2(F)$, we have

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix},$$

so in this way we may identify $Y_{\mathrm{sc}} = 2Y \subset Y$. We have

$$\begin{aligned} (Y_{\mathrm{sc}})_{Q,n} &= (2Y)_{Q,n} = \{y \in 2Y : B_Q(y, z) \in n\mathbb{Z} \ \forall z \in 2Y\} \\ &= 2\{y \in Y : B_Q(y, z) \in \frac{n}{4}\mathbb{Z} \ \forall z \in Y\} \\ &= 2n_3 Y, \end{aligned}$$

where

$$n_3 := \frac{n/\mathrm{gcd}(n, 4)}{\mathrm{gcd}(n_1, n/\mathrm{gcd}(n, 4))}.$$

Then $h_\alpha^\vee = \tilde{e}_1(\varpi^{-n_3})$, thus

$$h_\alpha^\vee = \begin{pmatrix} \varpi^{-n_2} & 0 \\ 0 & 1 \end{pmatrix} \text{ and } h_\alpha^\vee = \begin{pmatrix} \varpi^{-n_3} & 0 \\ 0 & \varpi^{n_3} \end{pmatrix}.$$

Moreover, $\eta(h_\alpha^\vee)$ is the image of $h_\alpha^\vee$ in $\mathrm{PGL}_2(F)$, so

$$\eta(h_\alpha^\vee) = \begin{pmatrix} \varpi^{-n_3} & 0 \\ 0 & \varpi^{n_3} \end{pmatrix} = \begin{pmatrix} \varpi^{-2n_3} & 0 \\ 0 & 1 \end{pmatrix}.$$

Our goal is to compute $1/N_\alpha = 2n_3/n_2$. We have

$$\begin{aligned} n_3/n_2 &= \frac{\mathrm{gcd}(n_1, n)/\mathrm{gcd}(n, 4)}{\mathrm{gcd}(n_1, n/\mathrm{gcd}(n, 4))} \\ &= 2^{\min(v_2(n_1), v_2(n)) - \min(v_2(n), 2) - \min(v_2(n_1), v_2(n) - \min(v_2(n), 2))} \\ &\in \{1/4, 1/2, 1\}, \end{aligned}$$

which confirms that $1/N_\alpha = 2n_3/n_2 \in \{1/2, 1, 2\}$. For $2 \nmid n$ we obtain $N_\alpha = 1/2$. If $n = 4$ and $n_1 = 2$, we obtain $N_\alpha = 1$. If $2 \nmid n_1$ and $4 \mid n$, we obtain $N_\alpha = 2$. However, recall that

$$n_1 = B_Q(e_1, e_1) = Q(2e_1) - 2Q(e_1) = 2Q(e_1),$$

since Q is a quadratic form. So we always have $2 \mid n_1$, hence the option $N_\alpha = 2$ is not possible. In this case, the only possibilities for N_α are $N_\alpha = 1$ and $N_\alpha = 1/2$.

5.3.2 Reduction to parameters for BD covers of $\mathrm{SL}_2(F)$

Recall that for a BD extension G , we want to determine the parameters of the affine Hecke algebra $\mathcal{H}(\mathcal{O}, G)$ in (5.3.1). It suffices to study the Hecke algebra $\mathcal{H}(\mathcal{O}, M_\alpha)$ in (5.3.4) for each root $\alpha \in \Sigma_{\mathcal{O}, \mu}$ individually. Fix a root α and write $M = M_\alpha$. By the considerations in the previous section, we may further reduce to the simply connected cover M_{sc} of the derived group of M . More precisely, M_{sc} is a BD extension of the simply connected cover M'_{sc} of the derived group of M' . The group M'_{der} is split, connected, semi-simple of rank one, hence isomorphic to $\mathrm{SL}_2(F)$ or $\mathrm{PGL}_2(F)$. The simply connected cover of either of these two groups is $\mathrm{SL}_2(F)$, so M_{sc} is a BD extension of $\mathrm{SL}_2(F)$. The parameters for M_{sc} and M are related as described in [Sol25, Proposition 2.4].

From now on, G denotes a BD extension of $G' = \mathrm{SL}_2(F)$ by μ_n . Let T be the maximal torus of G , which is a BD extension of $T' \cong F^\times$. The Weyl group of (G, T) is $W = N_G(T)/T = N_{G'}(T')/T' = \langle s_\alpha \rangle$. Let $\chi : Z(T) \rightarrow \mathbb{C}^\times$ be a genuine central character of T , and let σ be the unique irreducible genuine representation of T with central character χ , as described in §5.2.1. Let σ_1 be an irreducible subrepresentation of $\sigma|_{T_{\mathrm{cpt}}}$. Recall that we assumed that $\sigma = \sigma_0$, where σ_0 is as in (5.3.2). So σ is unitary, $\mu_\alpha(\sigma) = 0$ and $s_\alpha \sigma = \sigma$. We have that

$$\begin{aligned} \mathrm{End}_G(I_{TU_\alpha}^G(\mathrm{ind}_{T_{\mathrm{cpt}}}^T(\sigma_1))) &\cong \mathcal{H}(\mathcal{O}, G) \\ &:= \mathcal{H}(\{\pm h_\alpha^\vee\}, T_\sigma/T_{\mathrm{cpt}} \cong \mathbb{Z}, \{\pm \alpha^\sharp\}, (T_\sigma/T_{\mathrm{cpt}})^\vee \cong \mathbb{Z}, \lambda_\alpha, \lambda_\alpha^*, q_F) \end{aligned}$$

is an affine Hecke algebra of type $\widetilde{A}_1$. It remains to specify the poles q_α, q_{α^*} of μ_α . They are given by (5.3.2) and (5.3.3), but we do not know their explicit values yet. We will determine them by comparing $\mathcal{H}(\mathcal{O}, G)$ (and its module category) to the Hecke algebra $\mathcal{H}(G, \rho_\chi)$ coming from a type (J_χ, ρ_χ) for σ . Types for principal series representations of BD extensions were constructed in [Wan24], following the strategy in [Roc98]. There are conditions on the characteristic of the residue field, however by [Roc98, Remark 4.14] the results hold for (BD extensions of) $\mathrm{SL}_N(F)$ without restrictions on the residue characteristic.

The commutator on T is given by a Weyl group-invariant bilinear form B_Q on the cocharacter lattice $Y = \mathrm{Hom}(\mathbb{C}^\times, T) \cong \mathbb{Z}$, i.e.

$$[y(a), z(b)] = (a, b)_n^{B_Q(y, z)} \text{ for all } y, z \in Y, a, b \in F^\times,$$

where $(\ , \)_n$ is the n -th Hilbert symbol. Consider the sublattice of Y given by

$$Y_{Q, n} = \{y \in Y : B_Q(y, z) \in n\mathbb{Z} \text{ for all } z \in Y\} \cong \mathbb{Z},$$

which describes the center of T . Let $W_{\mathrm{ex}} := Y \rtimes W$ be the extended affine Weyl group and let $W_\chi := \{w \in W_{\mathrm{ex}} : w \cdot \chi = \chi\}$. We assumed that $s_\alpha \sigma = \sigma$

and by (5.2.1), σ is uniquely determined by its central character. So $s_\alpha \chi = \chi$ and $W_\chi = Y_{Q,n} \rtimes \langle s_\alpha \rangle$. By the results in [Wan24], there exists a type (J_χ, ρ_χ) for σ and the group W_χ indexes a $\mathbb{C}$ -basis for the Hecke algebra $\mathcal{H}(G, \rho_\chi) \cong \text{End}_G(\text{ind}_{J_\chi}^G(\rho_\chi))$. The multiplication relations for this algebra are also given; in general it is a semidirect product of an Iwahori-Hecke algebra of affine type with a group algebra. Since (J_χ, ρ_χ) is a cover of the type $(T_{\text{cpt}}, \sigma_1)$ for T , we have by (2.2.4) that

$$\text{ind}_{J_\chi}^G(\rho_\chi) \cong I_{TU_\alpha}^G(\text{ind}_{T_{\text{cpt}}}^T(\sigma_1)).$$

By (2.2.3), this tells us rather abstractly that $\mathcal{H}(G, \rho_\chi) \cong \mathcal{H}(\mathcal{O}, G)$. In particular, $\mathcal{H}(G, \rho_\chi)$ is itself an affine Hecke algebra of type $\widetilde{A}_1$. Moreover, it is possible to retrieve q_α, q_{α^*} from $\mathcal{H}(G, \rho_\chi)$. For this, we will compare the two algebras and their module categories more explicitly.

Proposition 5.3.6. *Let G be a BD extension of $G' = \text{SL}_2(F)$ and let $\sigma = \sigma_0$ be as in (5.3.2). Suppose that μ_α has a zero on $\mathcal{O} = \sigma X_{\text{nr}}(T)$. Then in the formula of $\mu_\alpha(\sigma, X_\alpha)$, we have that $q_\alpha = q_F$, $q_{\alpha^*} = 1$, i.e.*

$$\mu_\alpha(\sigma \otimes \cdot) = \mu_\alpha(\sigma, X_\alpha) = c_\alpha \frac{(1 - X_\alpha)(1 - X_\alpha^{-1})}{(1 - q_F X_\alpha)(1 - q_F X_\alpha^{-1})}.$$

Proof. Since $\mathcal{H}(G, \rho_\chi) \cong \mathcal{H}(\mathcal{O}, G)$, we know that $\mathcal{H}(G, \rho_\chi)$ is an Iwahori-Hecke algebra of type $\widetilde{A}_1$. Suppose that χ has positive depth. The only Levi subgroups of G are G and T . Thus by [Wan24, Theorem 4.8], either $\mathcal{H}(G, \rho_\chi) \cong \mathbb{C}[\mathcal{O}]$ or there exists a character χ_1 of G such that $\chi\chi_1$ is a depth zero character of $T(\mathcal{O}_F)$ and $\mathcal{H}(G, \rho_\chi) \cong \mathcal{H}(G, \rho_{\chi\chi_1})$. Since we already know that $\mathcal{H}(G, \rho_\chi)$ is affine of type $\widetilde{A}_1$, the first case is not possible. Therefore, we may assume from now on that χ has depth zero. Note that we might have to twist by an unramified character of T again, to ensure that $s_\alpha \chi \cong \chi$. This is fine though, because it does not change the depth. Hence, we may assume that χ has depth zero and that $s_\alpha \chi \cong \chi$.

In [Wan24, Theorem 4.6], we are given the Iwahori-Matsumoto presentation of the Iwahori-Hecke algebra $\mathcal{H}(G, \rho_\chi)$, see also [Sol21, Theorem 1.8]. The affine Weyl group is $W_\chi = Y_{Q,n} \rtimes \langle s_\alpha \rangle \cong \mathbb{Z} \rtimes S_2$ and the based root datum is

$$\mathcal{R}_\chi = (Y_{Q,n} \cong \mathbb{Z}, R = \{\pm 1\}, Y_{Q,n}^\vee \cong \mathbb{Z}, R^\vee = \{\pm 2\}, \Delta_\chi = \{\alpha_\chi\} = \{1\}).$$

The affine reflections are $S_\chi = \{s_{\alpha_\chi}, s'_{\alpha_\chi} : y \mapsto 1 - y\}$. By [Wan24, Theorem 4.6], there exists a basis $\{\vartheta_w : w \in W_\chi\}$ of $\mathcal{H}(G, \rho_\chi)$ such that with respect to this basis, $\mathcal{H}(G, \rho_\chi)$ is the Iwahori-Hecke algebra $\mathcal{H}(W_\chi, q_F)$ with relations

$$\vartheta_s^2 = q_F \vartheta_1 + (q_F - 1) \vartheta_s \text{ for } s \in S_\chi \text{ and } \vartheta_v \vartheta_w = \vartheta_{vw} \text{ if } \ell(v) + \ell(w) = \ell(vw).$$

Recall from [Sol25, Corollary 1.3] that the parameters q_α, q_{α^*} of μ_α correspond to the non-unitary $\sigma \otimes \chi' \in \mathcal{O}$ for which $I_{TU_\alpha}^G(\sigma \otimes \chi')$ is reducible. More precisely, $I_{TU_\alpha}^G(\sigma \otimes \chi')$ is irreducible for all $\chi' \in \mathcal{O}$ with $\chi'(h_\alpha^\vee) \in \mathbb{C}^\times \setminus \{q_\alpha, q_\alpha^{-1}, -q_{\alpha^*}, -q_{\alpha^*}^{-1}\}$. We chose σ such that $1 \neq q_\alpha \geq q_{\alpha^*} \geq 1$. The equivalences of categories

$$\mathcal{H}(G, \rho_\chi) - \text{Mod} \cong \text{Rep}_\sigma(G) \cong \mathcal{H}(\mathcal{O}, G) - \text{Mod}$$

respect parabolic induction [Sol19, Condition 3.1, Lemma 4.1, Lemma 5.1]. That means that we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{C}[Y_{Q,n}] - \text{Mod} & \longleftarrow & \text{Rep}_\sigma(T) & \longrightarrow & \mathbb{C}[T_\sigma/T_{\text{cpt}}] - \text{Mod} \\ \downarrow \text{ind}_{\mathbb{C}[Y_{Q,n}]}^{\mathcal{H}(W_\chi, q_F)} & & \downarrow I_{TU_\alpha}^G & & \downarrow \text{ind}_{\mathbb{C}[T_\sigma/T_{\text{cpt}}]}^{\mathcal{H}(\mathcal{O}, G)} \\ \mathcal{H}(W_\chi, q_F) - \text{Mod} & \longleftarrow & \text{Rep}_\sigma(G) & \longrightarrow & \mathcal{H}(\mathcal{O}, G) - \text{Mod}, \end{array}$$

where the horizontal arrows are equivalences of categories.

For $t \in \text{Hom}(Y_{Q,n}, \mathbb{C}^\times) \cong \mathbb{C}^\times$, the induced representation $\text{ind}_{\mathbb{C}[Y_{Q,n}]}^{\mathcal{H}(W_\chi, q_F)}(\mathbb{C}_t)$ is irreducible for all $t \in \mathbb{C}^\times \setminus \{q_F, q_F^{-1}, -1\}$, see [Sol21, §2.2]. So on the $\mathcal{H}(\mathcal{O}, G)$ -side, for $t \in \text{Hom}(T_\sigma/T_{\text{cpt}}, \mathbb{C}^\times) \cong \mathbb{C}^\times$, the induced representation $\text{ind}_{\mathbb{C}[T_\sigma/T_{\text{cpt}}]}^{\mathcal{H}(\mathcal{O}, G)}(\mathbb{C}_t)$ is irreducible for all $t \in \mathbb{C}^\times \setminus \{q_F, q_F^{-1}, -1\}$. Recall that the based root datum for $\mathcal{H}(\mathcal{O}, G)$ is

$$\mathcal{R}_\mu = (\{\pm h_\alpha^\vee\}, T_\sigma/T_{\text{cpt}} \cong \mathbb{Z}, \{\pm \alpha^\sharp\}, (T_\sigma/T_{\text{cpt}})^\vee \cong \mathbb{Z}, \Delta_\mu = \{h_\alpha^\vee\}).$$

Since $h_\alpha^\vee$ is the basis element, we obtain that for $\chi' \in \mathcal{O} \cong \text{Hom}(T_\sigma/T_{\text{cpt}}, \mathbb{C}^\times)$, the representation $I_{TU_\alpha}^G(\sigma \otimes \chi')$ is irreducible for $\chi'(h_\alpha^\vee) \in \mathbb{C}^\times \setminus \{q_F, q_F^{-1}, -1\}$. Hence the set $\{q_F, q_F^{-1}, -1\}$ agrees with the set $\{q_\alpha, q_\alpha^{-1}, -q_{\alpha^*}, -q_{\alpha^*}^{-1}\}$. Since we chose $q_\alpha \geq q_{\alpha^*} \geq 1$, we find that $q_\alpha = q_F$ and $q_{\alpha^*} = 1$. $\square$

Corollary 5.3.7. *The μ -function in (5.3.2) always looks like one of the following two options:*

$$\begin{aligned} \mu_\alpha(\sigma, X_\alpha) &= c_\alpha \frac{(1 - X_\alpha)(1 - X_\alpha^{-1})}{(1 - q_F X_\alpha)(1 - q_F X_\alpha^{-1})}, \text{ or} \\ \mu_\alpha(\sigma, X_\alpha) &= c_\alpha \frac{(1 - X_\alpha)(1 - X_\alpha^{-1})}{(1 - q_F^{1/2} X_\alpha)(1 - q_F^{1/2} X_\alpha^{-1})} \frac{(1 + X_\alpha)(1 + X_\alpha^{-1})}{(1 + q_F^{1/2} X_\alpha)(1 + q_F^{1/2} X_\alpha^{-1})}. \end{aligned}$$

Hence, the labels λ, λ^* in (5.3.3) of the Hecke algebra $\mathcal{H}(\mathcal{O}, G)$ are given by:

- $\lambda(h_\alpha^\vee) = 1, \lambda^*(h_\alpha^\vee) = 1$ in the first case;
- $\lambda(h_\alpha^\vee) = 1, \lambda^*(h_\alpha^\vee) = 0$ in the second case.

Proof. This follows from Proposition 5.3.6 and the fact that the parameters for $M = M_\alpha$ and the parameters for the BD extension M_{sc} of $\text{SL}_2(F)$ are related as described before Example 5.3.5. Note that the case $N_\alpha = 2$ is not possible, since $q_{\tilde{\alpha}} = q_F \neq 1 = q_{\tilde{\alpha}^*}$. $\square$

Remark 5.3.8. In principle, in this chapter we study principal series representations of BD covering groups. However, the result of Corollary 5.3.7 holds a bit more generally, namely for principal series representations of (not necessarily BD) covering groups of split reductive groups. This is because the reduction to one root α still works, and for covering groups of split groups we can always reduce to a BD cover of the simply connected cover by taking a fiber product, see Section 5.3.1, after Lemma 5.3.4. So we can reduce to a principal series representation of a BD extension of $\text{SL}_2(F)$ which has parameters as given in Proposition 5.3.6.

5.3.3 Some examples of subgroups of affine Weyl groups and cocycles

Recall that to $\mathfrak{s}_G = [\sigma, T]_G$ we can associate two Hecke algebras, namely the Hecke algebra $\mathcal{H}(G, \rho_\chi)$ arising from a type (J_χ, ρ_χ) [Wan24], and the algebra $\mathcal{H}(\mathcal{O}, G) \rtimes \mathbb{C}[R(\mathcal{O}), \mathfrak{h}]$ from (5.3.1). The type (J_χ, ρ_χ) for $\mathfrak{s}_G$ is a cover of the type $(T_{\text{cpt}}, \sigma_1)$ for $\mathfrak{s}_T$, so by (2.2.3) and (2.2.4), the two algebras must be isomorphic:

$$\mathcal{H}(G, \rho_\chi) \cong \mathcal{H}(\mathcal{O}, G) \rtimes \mathbb{C}[R(\mathcal{O}), \mathfrak{h}]. \quad (5.3.5)$$

The group $W_{\text{ex}} = Y \rtimes W$ is the extended affine Weyl group and

$$W_\chi = \{w \in W_{\text{ex}} : w \cdot \chi = \chi\}.$$

Then $\mathcal{H}(G, \rho_\chi)$ has a $\mathbb{C}$ -basis indexed by W_χ and its structure is given by generators and relations in [Wan24, Theorem 4.6]. The algebra $\mathcal{H}(\mathcal{O}, G) \rtimes \mathbb{C}[R(\mathcal{O}), \mathfrak{h}]$ has a $\mathbb{C}[\mathcal{O}]$ -basis indexed by

$$W(T, \mathcal{O}) = \{w \in W : w \cdot \sigma \in \mathcal{O}\} = W(\Sigma_{\mathcal{O}, \mu}) \rtimes R(\mathcal{O}).$$

Recall that $\mathbb{C}[\mathcal{O}] \cong \mathbb{C}[Y_{Q,n}]$ and note that $Y_{Q,n}$ lies in W_χ . So by studying how the lattice $Y_{Q,n}$ lies in W_χ , we can get information about the cocycle $\mathfrak{h} : W(T, \mathcal{O})^2 \rightarrow \mathbb{C}[\mathcal{O}]^\times$. Therefore, in this section we give some examples of the subgroups W_χ of W_{ex} . In particular, in Example 5.3.10 we show that W_χ can not always be written as a semi-direct product of $Y_{Q,n}$ with a finite group, hence the cocycle $\mathfrak{h}$ can be nontrivial.

If W_χ can be written as a semi-direct product of $Y_{Q,n}$ with a finite group of finite order elements, the cocycle $\mathfrak{h}$ will only take values in $\mathbb{C}^\times$, but it might still be nontrivial. This question remains open and could be answered by studying the multiplication relations in the corresponding Hecke algebras.

Example 5.3.9. Assume that $n \neq 2$ and suppose that F contains all n -th roots of unity. Consider a BD extension G of $G' = \text{SL}_2(F)$ by $\mu_n = \langle \zeta_n \rangle$. As a set, the torus T equals $\mu_n \times T'$ where $T' \cong F^\times$ is the maximal torus of G' . We have that $Y = \alpha^\vee \mathbb{Z}$ and the multiplication on T is determined by

$$(1, \alpha^\vee(t))(1, \alpha^\vee(t')) = (t, t')_n (1, \alpha^\vee(tt')) \quad (5.3.6)$$

for $t, t' \in F^\times$, where $(\ , \)_n$ is the n -th Hilbert symbol. Then $B_Q(\alpha^\vee, \alpha^\vee) = 2$, so $Y_{Q,n} = nY$ if $2 \nmid n$ and $Y_{Q,n} = \frac{n}{2}Y$ if $2 \mid n$. Write $s : F^\times \rightarrow T$ for the map $t \mapsto (1, \alpha^\vee(t))$, then the equation (5.3.6) becomes

$$s(t)s(t') = (t, t')_n s(tt').$$

For genuine characters $\chi : T_{\text{cpt}} = \mu_n \times T'_{\text{cpt}} \rightarrow \mathbb{C}^\times$ the reflection s_α acts via $s_\alpha \chi(s(t)) = \chi(s(t))^{-1}$ for $t \in \mathcal{O}_F^\times$. Note that this does not mean that $s_\alpha \chi = \chi^{-1}$, because $\zeta_n \in Z(T)$, hence $s_\alpha \chi(\zeta_n) = \chi(\zeta_n) \neq \chi(\zeta_n)^{-1}$. Suppose that $s_\alpha \chi \neq \chi$. Let ϖ be a uniformizer of F and consider the element $s(\varpi^{-1}) \in N(T)$, which corresponds to the element $\alpha^\vee$ in the extended Weyl group $W_{\text{ex}} = Y \rtimes \langle s_\alpha \rangle \cong N(T)/T_{\text{cpt}}$. In T , we have

$$s(\varpi^{-1})s(t)s(\varpi^{-1})^{-1} = (t, \varpi^2)_n s(t).$$

This means that $\alpha^\vee$ and $s_\alpha \alpha^\vee$ act on χ via

$$\alpha^\vee \chi(s(t)) = (t, \varpi^2)_n \chi(s(t)) \text{ and } s_\alpha \alpha^\vee \chi(s(t)) = (\varpi^2, t)_n \chi(s(t))^{-1},$$

respectively. Consider now the example where $\chi : T_{\text{cpt}} \rightarrow \mathbb{C}^\times$ is the genuine character determined by $\chi(s(t)) = (\varpi, t)_n$. Then $\alpha^\vee \chi \neq \chi$ and $s_\alpha \chi \neq \chi$ since $n \neq 2$. However,

$$\begin{aligned} (s_\alpha \alpha^\vee) \cdot \chi(s(t)) &= (\varpi^2, t)_n \chi(s(t))^{-1} \\ &= (\varpi^2, t)_n (\varpi^{-1}, t)_n \\ &= (\varpi, t)_n \\ &= \chi(s(t)), \end{aligned}$$

so $s_\alpha \alpha^\vee \chi = \chi$. It follows that $W_\chi = \{w \in W_{\text{ex}} : w\chi = \chi\}$ is generated by $Y_{Q,n}$ and $s_\alpha \alpha^\vee$. We have that $(s_\alpha \alpha^\vee)^2 = s_\alpha \alpha^\vee s_\alpha \alpha^\vee = (\alpha^\vee)^{-1} \alpha^\vee = 1$. Hence, $W_\chi = Y_{Q,n} \rtimes \langle s_\alpha \alpha^\vee \rangle$.

Now let $\chi : T_{\text{cpt}} \rightarrow \mathbb{C}^\times$ be an arbitrary genuine character. Then two situations may occur. If there is no integer m such that $s_\alpha (\alpha^\vee)^m$ fixes χ , then $W_\chi = Y_{Q,n}$. If there does exist an integer m such that $s_\alpha (\alpha^\vee)^m$ fixes χ , then $W_\chi = Y_{Q,n} \rtimes \langle s_\alpha (\alpha^\vee)^m \rangle$ (the element $s_\alpha (\alpha^\vee)^m$ has order 2). So the cocycle $\natural$ only takes values in $\mathbb{C}^\times$, i.e. $\natural : W(T, \mathcal{O})^2 \rightarrow \mathbb{C}^\times$.

Example 5.3.10. Suppose that F contains all n -th roots of unity. Consider a BD extension G of $G' = \text{GL}_2(F)$ by $\mu_n = \langle \zeta_n \rangle$. Then $Y = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and $\alpha^\vee = e_1 - e_2$. We have that $T(\mathcal{O}_F) = \mu_n \times T'(\mathcal{O}_F)$. The extended Weyl group is $W_{\text{ex}} = Y \rtimes \langle s_\alpha \rangle \cong N(T)/T(\mathcal{O}_F)$. There is a section $s : T' \rightarrow T$ such that for $y, z \in Y$, $t_1, t_2 \in F^\times$,

$$\begin{aligned} s(y(t_1))s(z(t_2)) &= (t_1, t_2)_n^{D(y,z)} s(y(t_1)z(t_2)) \\ [s(y(t_1)), s(z(t_2))] &= (t_1, t_2)_n^{B_Q(y,z)}. \end{aligned}$$

Consider a genuine character $\chi : T(\mathcal{O}_F) \rightarrow \mathbb{C}^\times$ and write it as $\chi = \varepsilon \times \chi_1 \times \chi_2$ where $\varepsilon : \mu_n \hookrightarrow \mathbb{C}^\times$, $\chi_1 = \chi \circ s \circ e_1 : \mathcal{O}_F^\times \rightarrow \mathbb{C}^\times$ and $\chi_2 = \chi \circ s \circ e_2 : \mathcal{O}_F^\times \rightarrow \mathbb{C}^\times$. The action of s_α flips e_1 and e_2 , so $s_\alpha \chi = \varepsilon \times \chi_2 \times \chi_1$. The elements $y \in Y \subset W_{\text{ex}}$ correspond to $s(y(\varpi^{-1}))$ in $N(T)$. We have $y\chi = \varepsilon \times y\chi_1 \times y\chi_2$ and we want to determine $y\chi_1, y\chi_2$.

Let $t \in \mathcal{O}_F^\times$ and $y, z \in Y$. Then

$$s(y(\varpi^{-1}))s(z(t))s(y(\varpi^{-1}))^{-1} = (t, \varpi)_n^{B_Q(y,z)} s(z(t)).$$

This means that for $y \in Y$,

$$\begin{aligned} y\chi_1 &= \varepsilon((- , \varpi)_n^{B_Q(y, e_1)})\chi_1, \\ y\chi_2 &= \varepsilon((- , \varpi)_n^{B_Q(y, e_2)})\chi_2. \end{aligned}$$

Hence, for $y \in Y$ we find that

$$\begin{aligned} y\chi &= y(\varepsilon \times \chi_1 \times \chi_2) \\ &= \varepsilon \times y\chi_1 \times y\chi_2 \\ &= \varepsilon \times \varepsilon((- , \varpi)_n^{B_Q(y, e_1)})\chi_1 \times \varepsilon((- , \varpi)_n^{B_Q(y, e_2)})\chi_2, \\ s_\alpha y\chi &= \varepsilon \times \varepsilon((- , \varpi)_n^{B_Q(y, e_2)})\chi_2 \times \varepsilon((- , \varpi)_n^{B_Q(y, e_1)})\chi_1. \end{aligned}$$

This describes how in general, the elements of W_{ex} act on χ . For y to fix χ , we need $y \in Y_{Q,n} = \{y' \in Y : B_Q(y', z) \in n\mathbb{Z} \forall z \in Y\}$. Suppose that s_α does not fix χ , so $\chi_1 \neq \chi_2$. For $s_\alpha y$ to fix χ , we need

$$\begin{aligned}\chi_1 &= \varepsilon((- , \varpi)_n^{B_Q(y, e_2)})\chi_2, \\ \chi_2 &= \varepsilon((- , \varpi)_n^{B_Q(y, e_1)})\chi_1.\end{aligned}$$

So if we pick χ_1 freely, then we need

$$B_Q(y, e_1 + e_2) \in n\mathbb{Z} \text{ and } \chi_2 = \varepsilon((- , \varpi)_n^{B_Q(y, e_1)})\chi_1.$$

If these conditions are satisfied, then the character $\chi = \varepsilon \times \chi_1 \times \chi_2$ is fixed by $s_\alpha y$.

For example, in [Wan24, Example 5.5] these conditions are satisfied, with $n = 4$, $B_Q(e_1, e_1) = B_Q(e_2, e_2) = 2$, $B_Q(e_1, e_2) = -2$, $y \in e_1 Y_{Q,n}$ and $\chi_1 = \varepsilon((- , \varpi)_4)$, $\chi_2 = \chi_1^{-1}$. In this case, $Y_{Q,n} = \mathbb{Z}(e_1 - e_2) \oplus \mathbb{Z}(e_1 + e_2)$ and W_χ is generated by $Y_{Q,n}$ and $s_\alpha e_1$. Then $(s_\alpha e_1)^2 = e_2 + e_1 \in Y_{Q,n}$, so W_χ has no finite order elements except the identity and can not be written as a semidirect product of $Y_{Q,n}$ with a group of order 2. Moreover, in this case $\mathcal{H}(G, \rho_\chi) = \mathbb{C}[W_\chi]$ has no finite order elements except the identity. It follows that $\mathcal{H}(G, \rho_\chi)$ can not be written as a semidirect product of $\mathbb{C}[Y_{Q,n}] = \mathbb{C}[\mathcal{O}]$ with a group of order 2, so the cocycle $\natural : W(T, \mathcal{O})^2 \rightarrow \mathbb{C}[\mathcal{O}]^\times$ is nontrivial and does not only take values in $\mathbb{C}^\times$.

So far, we have seen two examples of the cocycle $\natural : W(T, \mathcal{O})^2 \rightarrow \mathbb{C}[\mathcal{O}]^\times$. In the case where $G' = \text{SL}_2(F)$, the cocycle $\natural$ always takes values in $\mathbb{C}^\times$, but it might still be nontrivial. In the case where $G' = \text{GL}_2(F)$, we saw an example where $\natural$ is nontrivial and does not only take values in $\mathbb{C}^\times$. This seems to be due to the fact that the cocharacter lattice of $\text{GL}_2(F)$ is not generated by its simple coroots, i.e. $\text{GL}_2(F)$ is not simply connected. In contrast, the nicer result for a cover of $\text{SL}_2(F)$ may be due to the fact that $\text{SL}_2(F)$ is simply connected. However, the fact that it only has one simple coroot might also play a role. Therefore, let us see what happens for a simply connected group with two simple coroots.

Example 5.3.11. Let G be a BD extension of $G' = \text{SL}_3(F)$. Let σ_1 be an irreducible subrepresentation of $\sigma|_{T_{\text{cpt}}}$, where σ is a genuine irreducible representation of T . The torus

$$T' \cong F^{\times 2} \cong \left\{ \begin{pmatrix} t_1 & & \\ & t_1^{-1}t_2 & \\ & & t_2^{-1} \end{pmatrix} \right\}$$

is generated by the images of the simple coroots $\alpha^\vee := e_1 - e_2$ and $\beta^\vee := e_2 - e_3$. Then $Y = \mathbb{Z}\alpha^\vee \oplus \mathbb{Z}\beta^\vee$. There is a section $s : T' \rightarrow T$ such that for $y, z \in Y$, $t_1, t_2 \in F^\times$,

$$[s(y(t_1)), s(z(t_2))] = (t_1, t_2)_n^{B_Q(y, z)}.$$

Let $\chi : T(\mathcal{O}_F) \cong \mu \times \mathcal{O}_F^\times \times \mathcal{O}_F^\times \rightarrow \mathbb{C}^\times$ be a genuine character and write it as $\chi = \varepsilon \times \chi_\alpha \times \chi_\beta$ where $\varepsilon : \mu \hookrightarrow \mathbb{C}^\times$, $\chi_\alpha = \chi \circ s \circ \alpha^\vee$ and $\chi_\beta = \chi \circ s \circ \beta^\vee$. The

reflection s_α flips e_1 and e_2 , so we have

$$\begin{aligned} s_\alpha \cdot \alpha^\vee &= s_\alpha \cdot (e_1 - e_2) = e_2 - e_1 = -\alpha^\vee; \\ s_\alpha \cdot \beta^\vee &= s_\alpha \cdot (e_2 - e_3) = e_1 - e_3 = (e_1 - e_2) + (e_2 - e_3) = \alpha^\vee + \beta^\vee. \end{aligned}$$

Similarly, the reflection s_β flips e_2 and e_3 , so $s_\beta \cdot \beta^\vee = -\beta^\vee$ and $s_\beta \cdot \alpha^\vee = \alpha^\vee + \beta^\vee$. Then

$$s_\alpha \cdot \chi = \varepsilon \times \chi_\alpha^{-1} \times \chi_\alpha \chi_\beta \text{ and } s_\beta \cdot \chi = \varepsilon \times \chi_\alpha \chi_\beta \times \chi_\beta^{-1}.$$

Moreover, for $y \in Y$ we have (see Example 5.3.10),

$$y \cdot \chi = \varepsilon \times \varepsilon((- , \varpi)_n^{B_Q(y, \alpha^\vee)}) \chi_\alpha \times \varepsilon((- , \varpi)_n^{B_Q(y, \beta^\vee)}) \chi_\beta.$$

Any element in $Y \rtimes \langle s_\alpha, s_\beta \rangle$ is of the form yw , where

$$w \in \langle s_\alpha, s_\beta \rangle = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta\}$$

and $y \in Y$. In Lemma 5.3.12, we show that if yw fixes χ , then it can always be written as $y_2 y_1 w$, where $y_2 \in Y_{Q,n}$ and $y_1 w$ is a finite order element that fixes χ . So W_χ is a semidirect product of $Y_{Q,n}$ with finite order element(s). For every $w \in \langle s_\alpha, s_\beta \rangle$, one generator of the form yw is enough, because suppose there is another generator $y'w$, then it can always be written as a product of yw with an element in $Y_{Q,n}$. Moreover, we only need one generator of the form $ys_\alpha s_\beta$, $y's_\beta s_\alpha$, because $(ys_\alpha s_\beta)^2 = y's_\beta s_\alpha$ for some y' . So W_χ always looks like (a subgroup of):

$$W_\chi = Y_{Q,n} \rtimes \langle (n_1 \alpha^\vee) s_\alpha, (n_2 \beta^\vee) s_\beta, (n_3 (\alpha^\vee + \beta^\vee)) s_\alpha s_\beta s_\alpha, ys_\alpha s_\beta \rangle$$

where the first three generators have order two and the last generator has order three. It follows from the proof of Lemma 5.3.12 that the generators take the form as how they are given above. So W_χ can always be written as a semidirect product of $Y_{Q,n}$ with a finite group. So, like in the case of a cover of $\mathrm{SL}_2(F)$, the cocycle $\natural$ only has values in $\mathbb{C}^\times$.

Lemma 5.3.12. *In the setting of Example 5.3.11, if $yw \in Y \rtimes W$ fixes χ , then it can always be written as $yw = y_2 y_1 w$, where $y_2 \in Y_{Q,n}$ and $y_1 w$ is a finite order element that fixes χ .*

Proof. We first prove some things about the bilinear form B_Q . It is a Weyl group-invariant symmetric bilinear form on Y (associated to a Weyl group-invariant quadratic form Q on Y), so

$$B_Q(\alpha^\vee, \alpha^\vee) = B_Q(s_\alpha s_\beta s_\alpha \cdot \alpha^\vee, s_\alpha s_\beta s_\alpha \cdot \alpha^\vee) = B_Q(-\beta^\vee, -\beta^\vee) = B_Q(\beta^\vee, \beta^\vee).$$

Moreover,

$$\begin{aligned} B_Q(\alpha^\vee, \alpha^\vee) &= B_Q(s_\beta \cdot \alpha^\vee, s_\beta \cdot \alpha^\vee) \\ &= B_Q(\alpha^\vee + \beta^\vee, \alpha^\vee + \beta^\vee) \\ &= B_Q(\alpha^\vee, \alpha^\vee) + 2B_Q(\alpha^\vee, \beta^\vee) + B_Q(\beta^\vee, \beta^\vee) \\ &= 2B_Q(\alpha^\vee, \alpha^\vee) + 2B_Q(\alpha^\vee, \beta^\vee), \end{aligned}$$

hence

$$B_Q(\alpha^\vee, \alpha^\vee) = -2B_Q(\alpha^\vee, \beta^\vee),$$

so B_Q is determined by the value

$$B_Q(\alpha^\vee, \alpha^\vee) = Q(\alpha^\vee + \alpha^\vee) - Q(\alpha^\vee) - Q(\alpha^\vee) = 2Q(\alpha^\vee).$$

We have that $B_Q(\alpha^\vee, \beta^\vee) = -Q(\alpha^\vee)$.

By symmetry, it suffices to show the statement of the Lemma for the elements yw of the form:

$$ys_\alpha, ys_\alpha s_\beta, ys_\alpha s_\beta s_\alpha.$$

Any $ys_\alpha s_\beta$ already has order 3, so in that case there is nothing left to show.

Suppose that $\chi = \varepsilon \times \chi_\alpha \times \chi_\beta$ is fixed by ys_α . We have that

$$ys_\alpha \chi = \varepsilon \times \varepsilon((- , \varpi)_n^{B_Q(y, \alpha^\vee)}) \chi_\alpha^{-1} \times \varepsilon((- , \varpi)_n^{B_Q(y, \beta^\vee)}) \chi_\alpha \chi_\beta.$$

So $ys_\alpha \chi = \chi$ if and only if

$$\begin{aligned} \chi_\alpha &= \varepsilon((- , \varpi)_n^{B_Q(y, \alpha^\vee)/2}); \\ B_Q(y, \beta^\vee) + B_Q(y, \alpha^\vee)/2 &\in n\mathbb{Z}. \end{aligned}$$

Write $\chi_\alpha = \varepsilon((- , \varpi)_n^{N/2})$, then ys_α fixes χ if and only if

$$B_Q(y, \alpha^\vee) \in N + 2n\mathbb{Z} \text{ and } B_Q(y, \alpha^\vee + 2\beta^\vee) \in 2n\mathbb{Z}. \quad (5.3.7)$$

Write $y = n_1\alpha^\vee + n_2\beta^\vee$. Then

$$B_Q(y, \alpha^\vee) = (2n_1 - n_2)Q(\alpha^\vee) \text{ and } B_Q(y, \alpha^\vee + 2\beta^\vee) = 3n_2Q(\alpha^\vee).$$

So if $ys_\alpha \chi = \chi$, then

$$(2n_1 - n_2)Q(\alpha^\vee) \in N + 2n\mathbb{Z} \text{ and } 3n_2Q(\alpha^\vee) \in 2n\mathbb{Z}. \quad (5.3.8)$$

Adding the two equations, we find

$$2(n_1 + n_2)Q(\alpha^\vee) \in N + 2n\mathbb{Z}.$$

This implies that the element $y_1 := (n_1 + n_2)\alpha^\vee$ also satisfies (5.3.7), hence $y_1 s_\alpha \chi = \chi$. Moreover,

$$ys_\alpha = y_2 y_1 s_\alpha$$

where $y_2 = -n_2\alpha^\vee + n_2\beta^\vee$. By (5.3.8), we have that

$$B_Q(y_2, \alpha^\vee) = -3n_2Q(\alpha^\vee) \in 2n\mathbb{Z} \text{ and } B_Q(y_2, \beta^\vee) = 3n_2Q(\alpha^\vee) \in 2n\mathbb{Z}.$$

So $y_2 \in Y_{Q,n}$. The element $y_1 s_\alpha$ has order 2.

Now suppose that $ys_\alpha s_\beta s_\alpha \chi = \chi$. We have that

$$ys_\alpha s_\beta s_\alpha \chi = \varepsilon \times \varepsilon((- , \varpi)_n^{B_Q(y, \alpha^\vee)}) \chi_\beta^{-1} \times \varepsilon((- , \varpi)_n^{B_Q(y, \beta^\vee)}) \chi_\alpha^{-1}.$$

We deduce that $ys_\alpha s_\beta s_\alpha \chi = \chi$ if and only if

$$B_Q(y, \alpha^\vee - \beta^\vee) \in n\mathbb{Z};$$

$$\chi_\alpha = \varepsilon((- , \varpi)_n^{B_Q(y, \alpha^\vee)}) \chi_\beta^{-1}.$$

Write $\chi_\alpha = \varepsilon((- , \varpi)_n^N) \chi_\beta^{-1}$. Then $ys_\alpha s_\beta s_\alpha \chi = \chi$ if and only if

$$B_Q(y, \alpha^\vee) \in N + n\mathbb{Z} \text{ and } B_Q(y, \alpha^\vee - \beta^\vee) \in n\mathbb{Z}; \quad (5.3.9)$$

Write $y = n_1 \alpha^\vee + n_2 \beta^\vee$, then

$$B_Q(y, \alpha^\vee) = (2n_1 - n_2)Q(\alpha^\vee) \text{ and } B_Q(y, \alpha^\vee - \beta^\vee) = 3(n_1 - n_2)Q(\alpha^\vee).$$

So if $ys_\alpha s_\beta s_\alpha \chi = \chi$, then

$$(2n_1 - n_2)Q(\alpha^\vee) \in N + n\mathbb{Z} \text{ and } 3(n_1 - n_2)Q(\alpha^\vee) \in n\mathbb{Z}. \quad (5.3.10)$$

Let $y_1 := (2n_1 - n_2)\alpha^\vee + (2n_1 - n_2)\beta^\vee$. Then y_1 also satisfies (5.3.9), so $y_1 s_\alpha s_\beta s_\alpha$ fixes χ and has order two. Moreover, we have that

$$ys_\alpha s_\beta s_\alpha = y_2 y_1 s_\alpha s_\beta s_\alpha,$$

where $y_2 := (n_2 - n_1)\alpha^\vee + 2(n_2 - n_1)\beta^\vee$. Then $y_2 \in Y_{Q,n}$, because by (5.3.10),

$$B_Q(y_2, \alpha^\vee) = 0 \text{ and } B_Q(y_2, \beta^\vee) = 3(n_2 - n_1)Q(\alpha^\vee) \in n\mathbb{Z}. \quad \square$$

5.3.4 A brief discussion on the existence of a Shimura correspondence

A question of interest in the research area of representations of covers of reductive p -adic groups is the existence of a *Shimura correspondence* [Sav88; Sav04]. In a general sense, this is about finding an isomorphism between a Hecke algebra for a covering group and a Hecke algebra for a reductive group.

In our context, the question could be phrased as follows. Given an irreducible representation σ of a maximal BD torus T in a BD covering group G of a split group G' , does there exist a split reductive group G_r and an irreducible representation σ_r of a maximal torus T_r of G_r , such that the algebras

$$\text{End}_G(I_P^G(\text{ind}_{T_{\text{cpt}}}^T(\sigma_1))) \cong \mathcal{H}(\mathcal{O}, G) \rtimes \mathbb{C}[R(\mathcal{O}), \mathfrak{h}] \text{ and}$$

$$\text{End}_{G_r}(I_{P_r}^{G_r}(\text{ind}_{T_{r,\text{cpt}}}^{T_r}(\sigma_{r,1}))) \cong \mathcal{H}(\mathcal{O}_r, G_r) \rtimes \mathbb{C}[R(\mathcal{O}_r), \mathfrak{h}_r]$$

are isomorphic? For principal series representations of split reductive groups G_r , the cocycle $\mathfrak{h}_r$ is always trivial, which follows from (5.3.5) and the results in [Roc98]. So Example 5.3.10 shows that in general, the answer to the question is negative, and the place to look for counterexamples is in covers of reductive groups that are not simply connected. In fact, already in Section 5.2.2 we saw a counterexample, namely a cover $\tilde{T}$ of a split torus that is not a BD extension. There, we showed that the progenerator $\text{End}_{\tilde{T}}(\text{ind}_{\tilde{T}_{\text{cpt}}}^{\tilde{T}}(\sigma|_{\tilde{T}_{\text{cpt}}}))$ of

$\mathrm{Rep}_{\mathfrak{s}_{\tilde{T}}}(\tilde{T})$ has a description involving a nontrivial 2-cocycle. For an irreducible representation σ_r of T_r , it is always possible to work with the smaller progenerator $\mathrm{End}_{T_r}(\mathrm{ind}_{T_{r,\mathrm{cpt}}}^{T_r}(\sigma_{r,1}))$ of $\mathrm{Rep}_{\mathfrak{s}_{T_r}}(T_r)$ whose description does not come with a nontrivial cocycle. So there can not be a nice correspondence between the progenerators in this case.

However, Examples 5.3.9 and 5.3.11 do not disprove the existence of a Shimura correspondence for covers of simply connected groups, in the sense of the question asked above. To find a confirmatory answer (or to find a counterexample), one could try to generalize Lemma 5.3.12 to covers of simply connected groups and moreover show that the cocycle $\natural$ is trivial (or nontrivial) in these cases. If the outcome of this turns out to be hopeful for a positive answer, it still remains to find the actual correspondence, i.e. to find the group G_r and representation σ_r of T_r . Possibly, for BD covers of simply connected groups, there is an isomorphism between the Hecke algebras $\mathcal{H}(G, \rho_\chi)$ and $\mathcal{H}(G_{Q,n}, \rho_{\chi_{Q,n}})$ of [Wan24, §5]. The group $G_{Q,n}$ is a reductive group with maximal torus $T_{Q,n}$ that has $Y_{Q,n}$ as its cocharacter lattice.

Research Data Management

The research in this thesis has been carried out under the research data management policy of the Institute of Mathematics, Astrophysics and Particle Physics, Radboud University. No data has been produced or analyzed in this project.

Samenvatting

Dit proefschrift gaat over de representatietheorie van overdekkingsgroepen van p -adische groepen. In deze samenvatting zullen deze begrippen kort worden uitgelegd.

p -adische getallen

Laten we beginnen met de p -adische getallen. Deze getallen zijn anders dan de getallen die je waarschijnlijk gewend bent. De getallen die je misschien wel kent zijn de *gehele getallen*

$$\dots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 5, \dots$$

en de *rationale getallen* (breuken) $\frac{r}{s}$ waarbij r en s gehele getallen zijn, met s ongelijk aan 0. Een gebruikelijke manier om breuken te schrijven is met hun decimale uitbreiding, bijvoorbeeld

$$\frac{4}{3} = 1,333\dots = 1 \times 10^0 + 3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{-3} + \dots \quad (1)$$

Een andere manier om breuken te schrijven is met hun zogeheten p -adische uitbreiding. Hier staat de “ p ” voor een vast gekozen priemgetal. Dat is een geheel getal groter dan 1 dat alleen maar door zichzelf en door 1 deelbaar is. Voorbeelden hiervan zijn 2, 3, 5, 7, $\dots$. Er zijn er oneindig veel. Laten we als voorbeeld $p = 5$ nemen, dan ziet een 5-adische uitbreiding er zo uit:

$$a_\ell \times 5^\ell + a_{\ell+1} \times 5^{\ell+1} + a_{\ell+2} \times 5^{\ell+2} + \dots \quad (2)$$

De ℓ staat voor een geheel getal dat negatief mag zijn. De coëfficiënten $a_\ell, a_{\ell+1}, a_{\ell+2}, \dots$ kunnen elk gekozen worden uit 0, 1, 2, 3 en 4. Bovendien eisen we dat a_ℓ ongelijk is aan 0, zo begint de reeks echt bij de term $a_\ell \times 5^\ell$. De reeks mag oneindig ver doorgaan. Merk op dat de decimale schrijfwijze zoals in (1) oneindig veel termen met negatieve machten van het grondgetal 10 toelaat, terwijl de 5-adische schrijfwijze juist oneindig veel termen met positieve machten van het grondgetal 5 toelaat. Dit is omdat in de 5-adische setting, hoge machten van 5 juist kleine getallen zijn. Hoe groter de ℓ in de schrijfwijze (2), hoe kleiner het getal is.

Elke breuk heeft een 5-adische schrijfwijze zoals in (2), bijvoorbeeld

$$\frac{4}{3} = 3 \times 5^0 + 3 \times 5^1 + 1 \times 5^2 + 3 \times 5^3 + 1 \times 5^4 + 3 \times 5^5 + \dots \quad (3)$$

Laten we deze uitdrukking ter illustratie vermenigvuldigen met 3, om te laten zien dat we dan het getal 4 terugkrijgen:

$$\begin{aligned}
 & 3 \times (3 \times 5^0 + 3 \times 5^1 + 1 \times 5^2 + 3 \times 5^3 + 1 \times 5^4 + 3 \times 5^5 + \dots) \\
 &= 9 \times 5^0 + 9 \times 5^1 + 3 \times 5^2 + 9 \times 5^3 + 3 \times 5^4 + 9 \times 5^5 + \dots \\
 &= 4 \times 5^0 + 10 \times 5^1 + 3 \times 5^2 + 9 \times 5^3 + 3 \times 5^4 + 9 \times 5^5 + \dots \\
 &= 4 \times 5^0 + 0 \times 5^1 + 5 \times 5^2 + 9 \times 5^3 + 3 \times 5^4 + 9 \times 5^5 + \dots \\
 &= 4 \times 5^0 + 0 \times 5^1 + 0 \times 5^2 + 10 \times 5^3 + 3 \times 5^4 + 9 \times 5^5 + \dots \\
 &\vdots \\
 &= 4 \times 5^0 + 0 \times 5^1 + 0 \times 5^2 + 0 \times 5^3 + 0 \times 5^4 + 0 \times 5^5 + \dots
 \end{aligned}$$

Alle termen na de eerste term moeten we naar rechts blijven opschuiven om de uitdrukking in de vorm (2) te krijgen (want de coëfficiënten mogen alleen 0, 1, 2, 3 of 4 zijn). De termen worden steeds kleiner hoe meer we ze naar rechts opschuiven (want hoge machten van 5 zijn kleine getallen in de 5-adische setting). Hierdoor verdwijnen al deze termen en alleen de eerste term $4 \times 5^0 = 4$ blijft over. De 5-adische uitbreiding in (3) is dus inderdaad geldig. Algemener zijn alle breuken in zo'n 5-adische vorm te schrijven. Als we alle mogelijke 5-adische uitbreidingen van de vorm (2) beschouwen, dan krijgen we naast alle breuken ook nog extra getallen die geen breuken zijn. Al deze getallen samen noemen we de 5-adische getallen.

Neem nu een willekeurig priemgetal p . Op dezelfde manier als het voorbeeld van de 5-adische getallen, zijn de p -adische getallen alle getallen van de vorm

$$a_\ell \times p^\ell + a_{\ell+1} \times p^{\ell+1} + a_{\ell+2} \times p^{\ell+2} + \dots \quad (4)$$

De ℓ is een geheel getal dat negatief mag zijn. De coëfficiënten $a_\ell, a_{\ell+1}, \dots$ zijn gehele getallen tussen 0 en $p-1$, met bovendien a_ℓ ongelijk aan 0. De p -adische getallen bevatten naast alle breuken ook nog extra getallen die geen breuken zijn. Ze hebben een *algebraïsche structuur*, want we kunnen ze optellen en vermenigvuldigen. Maar ze hebben ook een *meetkundige structuur*, want er is een begrip van afstand tussen getallen. Namelijk, als we twee p -adische getallen van elkaar aftrekken, dan krijgen we weer een p -adisch getal. Als we dit getal schrijven op de manier in (4), dan geeft de ℓ de afstand tussen de oorspronkelijke twee getallen aan: hoe groter de ℓ , hoe dichter de getallen bij elkaar liggen.

Bijvoorbeeld, in de 5-adische getallen is de afstand tussen 25 en 0 kleiner dan de afstand tussen 5 en 0. Dit komt omdat de 5-adische uitbreiding van $25-0 = 1 \times 5^2$ bij $\ell = 2$ begint, terwijl de 5-adische uitbreiding van $5-0 = 1 \times 5^1$ bij $\ell = 1$ begint. Maar in de 3-adische getallen is voor beide getallen de afstand tot 0 gelijk, want de 3-adische uitbreidingen van beide getallen beginnen bij de term $\ell = 0$:

$$5 = 2 \times 3^0 + 1 \times 3^1 \text{ en } 25 = 1 \times 3^0 + 2 \times 3^1 + 2 \times 3^2.$$

Verschillende waarden voor p geven dus verschillende p -adische getallen met een verschillende meetkundige structuur. We bekijken de p -adische getallen maar voor één vaste p tegelijk.

p -adische groepen en overdekkingsgroepen

Een *groep* is een verzameling samen met een groepsbewerking die aan bepaalde eisen voldoet. De groepsbewerking geeft ons voor elke twee elementen in de verzameling een derde element in de verzameling. De p -adische getallen geven ons gelijk twee voorbeelden van groepen:

- De p -adische getallen met optelling als groepsbewerking is een groep en wordt genoteerd als $\mathbb{Q}_p^+$.
- De p -adische getallen met 0 weggelaten en met de vermenigvuldiging als groepsbewerking is een groep. Deze groep wordt genoteerd als $\mathbb{Q}_p^\times$.

De groepen $\mathbb{Q}_p^+$ en $\mathbb{Q}_p^\times$ zijn in zekere zin de bouwstenen van de zogeheten *p -adische groepen* die door wiskundigen worden bestudeerd. Deze groepen hebben naast hun groepsstructuur ook een meetkundige structuur die komt van de meetkundige structuur van de p -adische getallen.

Het onderzoek in dit proefschrift gaat over een iets algemener soort meetkundige groepen, namelijk *overdekkingsgroepen* van p -adische groepen. Bijvoorbeeld, een overdekkingsgroep van $\mathbb{Q}_p^\times$ bestaat als verzameling uit eindig veel kopieën van $\mathbb{Q}_p^\times$. Om de verzameling tot een meetkundige groep te maken, moeten we die nog een groepsbewerking geven. Hiervoor zijn verschillende mogelijkheden en sommige mogelijkheden geven ons weer een p -adische groep. Maar er zijn ook mogelijkheden die ons een groep geven die niet meer te zien is als p -adische groep. Hierdoor zijn de overdekkingsgroepen algemener en ze hebben een ingewikkeldere structuur. In dit proefschrift wordt onderzocht of bepaalde resultaten die bekend zijn voor p -adische groepen ook gelden voor de algemenere overdekkingsgroepen. Deze resultaten gaan over de *representatietheorie* van de betreffende groepen.

Representatietheorie

Een *representatie* van een abstracte groep (zoals een overdekking van een p -adische groep) is een manier om die groep op een andere manier te bekijken, namelijk door de groep te vertalen naar een groep van lineaire transformaties van een vectorruimte. Een voorbeeld van zo'n groep van transformaties is de symmetriegroep van een regelmatige veelhoek. Maar een groep kent veel verschillende mogelijke representaties met veel soorten eigenschappen. Deze eigenschappen kunnen ook te maken hebben met de meetkundige structuur van de oorspronkelijke groep. Wat je kunt proberen is om zo veel mogelijk van een bepaald soort representaties (van één vaste groep) allemaal tegelijk te beschrijven. Dan is het ook interessant om zo veel mogelijk te weten te komen over de precieze structuur van die beschrijving. Dat is in zekere zin wat er in dit proefschrift wordt onderzocht. Van bepaalde resultaten die hierover bekend zijn voor p -adische groepen wordt aangetoond dat ze ook gelden voor overdekkingsgroepen. Maar er wordt ook aangetoond dat bepaalde aspecten voor overdekkingsgroepen wel degelijk anders uitpakken dan voor p -adische groepen. Dat is kort gezegd waarom de overdekkingsgroepen (in ieder geval voor de auteur) zo interessant zijn.

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Curriculum Vitae

Janet Flikkema was born on July 4, 1998 in Leeuwarden, the Netherlands. She grew up in the province of Fryslân. Between the ages of five to sixteen, she was an avid figure skater and competed in multiple editions of the Dutch Championships. In 2014, at the age of sixteen, she moved to Nijmegen to live by herself and study Physics and Astronomy at Radboud University. Gradually during her Bachelor's studies, she decided to study Mathematics instead. She obtained her Bachelor's degree in Mathematics in January 2019. After that, she pursued a Master's degree in Mathematics at Radboud University, which she obtained in January 2022. During her student years, she spent part of her free time as a figure skating coach at Triavium in Nijmegen.

In March 2022, Janet started her PhD research at Radboud University, of which this PhD thesis is the result. During her PhD, she was a member of the Young Scientist Council of her research institute IMAPP. She also completed the *Nijmeegse Vierdaagse* three times during her PhD (and five times during her life so far). What Janet will do after her PhD, is still unknown.

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