Operator System Perspectives at Truncated Noncommutative Geometry

Institute for Mathematics,
Astrophysics and Particle Physics

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Chapter 1

Introduction

It is the principle of spectral geometry to infer information about a geometric object from spectral data of operators associated to it; one may think of a Riemannian (spin-)manifold and the Laplace or Dirac operator. This principle is foundational for the field of noncommutative geometry which may be considered as a geometry of quantum systems where there is no notion of phase space and points, but rather measurement outcomes that present themselves as spectral data. In this thesis, we focus on the *metric* aspect of noncommutative geometry; it emerges from Connes' observation that the metric space structure of a compact Riemannian spin-manifold (M, g) is recovered by the C*-algebra of continuous functions together with the spin-Dirac operator D_M [29]. More precisely, by the Gelfand–Naimark theorem, M is homeomorphic to the set of characters $\widehat{\mathrm{C}(M)}$ equipped with the weak*-topology, and *Connes' distance formula* holds:

$$d_g(p, q) = \sup\{|f(p) - f(q)| \mid f \in \mathrm{C}(M), \|[D_M, f]\| \leqslant 1\}, \tag{1.1}$$

for all $p, q \in M$, where the geodesic distance d_g is defined as the infimum of the lengths of all paths connecting p and q. The key observation establishing this formula is that the extended seminorm $\|[D_M, \cdot]\|$ applied to a smooth function on M coincides with the supremum over all points in M of the euclidean norm of the gradient, i.e. the Lipschitz constant; exploiting the duality of the distance between points of M and the Lipschitz constant of functions on M yields the equality (1.1).

Connes' distance formula gives rise to a distance function on the set of probability measures on M by viewing $f(x)$ and $f(y)$ as the evaluations of the Dirac measures δ_x and δ_y at f and allowing for general probability measures μ and ν in place of δ_x and δ_y. This distance function may be considered as a version of Kantorovich's dual formulation [79, 80] of Monge's optimal transport cost function [101], and it is instructive to interpret it as the optimal cost of transporting the measure μ to the measure ν. See [141] for background on optimal transport. More generally, the C*-algebra of continuous functions $\mathrm{C}(M)$ may be replaced by any (unital) C*-algebra A, faithfully represented on a Hilbert space H, and the key properties of the spin-Dirac operator D_M are captured by the general requirements that there is an essentially self-adjoint operator D on H, with $\|[D, a]\| \in \mathcal{B}(H)$, for all a in a dense

$*$-subalgebra of A, and $(D-i)^{-1}$ a compact operator. A triple (A, H, D) with these properties is called a *spectral triple*. This formalism generalizes many geometric objects including Riemannian spin-manifolds, finite metric spaces and discrete groups with length functions. See [31] for a conceptual discussion. The operator D has discrete spectrum which besides 0 consists of real eigenvalues only. The Hilbert space H and the operator D are, up to unitary equivalence, characterized by the list of eigenvalues of D with multiplicities. We refer to the distance function on the state space $\mathcal{S}(A)$ given by

$$d(\mu,\nu) := \sup\{|\mu(a)-\nu(a)| \mid a \in A, \|[D,a]\| \leqslant 1\}, \tag{1.2}$$

as the *Connes distance*.

When considering potential manifestations of the spectral triple formalism in physical practice, one is always confronted with constraints on the spectral data of D; after all, any measurement device – as exact as one might wish for – will only detect a finite part of the infinite list of eigenvalues of D. This issue was noticed by Connes–van Suijlekom [33] and it was suggested to consider *spectral truncations* as introduced by D'Andrea–Lizzi–Martinetti [36] instead, i.e. compressed versions (PAP, PH, PDP) of spectral triples, where P is a spectral projection for D. Note that PAP is not a C*-algebra in general, as P is generally not an element of A; instead a spectral truncation is an instance of an *operator system spectral triple* (X, H, D), with X an operator system, i.e. a $*$-closed unital subspace of $\mathcal{B}(H)$ (a "C*-algebra without multiplication"), but the remaining spectral triple axioms are the same. Operator systems come with a notion of positivity and together with the unit this allows to define *states*, i.e. positive unital functionals. Moreover, for an operator system spectral triple (X, H, D), the Connes distance on the state space of X, defined as in (1.2), still makes sense. Connes–van Suijlekom then asked how much geometric information of a spectral triple (A, H, D) is captured by a spectral truncation (PAP, PH, PDP), and it seems natural to expect that spectral truncations (PAP, PH, PDP) approximate a spectral triple (A, H, D), as more spectral data is taken into account, i.e. as $P \to \mathbf{I}^H$.

The question about convergence of spectral truncations requires more mathematical precision. A way to approach this is to try to compare the state spaces $\mathcal{S}(PAP)$ and $\mathcal{S}(A)$ with their respective Connes distances using tools from the classical theory of metric spaces. However, without any additional assumptions not much can be said about these metric spaces; the key requirement, put forward by Rieffel [115], is that the topology induced by the Connes distance should coincide with the weak*-topology on the state space, thus yielding a compact metric space. Rieffel subsequently developed a theory of *compact quantum metric spaces* more generally for archimedean order-unit spaces with a *Lip-norm* [116]. He also proposed a way to compare compact quantum metric spaces quantitatively in terms of *quantum Gromov–Hausdorff distance* [119]. With these tools he gave mathematically precise meaning to a folklore statement in physics that "matrix algebras converge to the sphere".

For a seminorm L on X, the distance function

$$d^L(\phi,\psi) := \sup\{|\phi(x)-\psi(x)| \mid L(x) \leqslant 1\}$$

on the state space $\mathcal{S}(X)$ is called the *Monge–Kantorovich distance.* If the seminorm L is densely defined, *-invariant, and its kernel contains the scalar multiples of the unit of X, with the Monge–Kantorovich distance metrizing the weak*-topology, the pair (X, L) is called a *compact quantum metric space.*

Various ways to compare compact quantum metric spaces (X, L_X) and (Y, L_Y) have been proposed. Probably the most accessible is to compare the state spaces $\mathcal{S}(X)$ and $\mathcal{S}(Y)$, which are compact metric spaces when equipped with their respective Monge–Kantorovich distances, in Gromov–Hausdorff distance [33]. However, as this does not take into account that the metrics on the state spaces arise from Lip-norms, Rieffel proposed the notion of *quantum Gromov–Hausdorff distance* [118]. Quantum Gromov–Hausdorff distance of the compact quantum metric spaces (X, L_X) and (Y, L_Y) is an upper bound for the classical Gromov–Hausdorff distance of the state spaces $\mathcal{S}(X)$ and $\mathcal{S}(Y)$. In fact, both are computed by the smallest Hausdorff distance of $\mathcal{S}(X)$ and $\mathcal{S}(Y)$ in the disjoint union $\mathcal{S}(X) \sqcup \mathcal{S}(Y)$, but for quantum Gromov–Hausdorff distance only metrics on $\mathcal{S}(X) \sqcup \mathcal{S}(Y)$ are considered which arise as the Monge–Kantorovich distance of an appropriate Lip-norm on the direct sum $X \oplus Y$. One may thus think of quantum Gromov–Hausdorff distance as being finer in the sense that it is harder to be similar as (the state spaces of) compact quantum metric spaces than as classical compact metric spaces. And indeed, it was shown in [76] that quantum and classical Gromov–Hausdorff distance are in fact not equivalent.

For the purposes of this thesis, given the above motivation, operator system versions of compact quantum metric spaces and quantum Gromov–Hausdorff distance seem most appropriate. An operator system $X \subseteq \mathcal{B}(H)$ comes with a matrix-order structure; i.e. an $n \times n$ matrix with entries in X is positive if it is positive as an operator on H^n. This matrix-order structure is compatible under conjugation by rectangular matrices and the identity operator on H^n is an archimedean order-unit for the cone of positive $n \times n$ matrices over X. By the Choi–Effros theorem [27], the matrix-order structure together with the archimedean matrix order-unit characterizes operator systems abstractly – analogous to the Gelfand–Naimark–Segal characterization of C*-algebras. Thus it is appropriate to consider morphisms in the operator system category to be linear maps which preserve the unit and positive matrices, i.e. unital completely positive (ucp) maps.

As we want to take an operator system point of view in this thesis, it seems appropriate to consider a Gromov–Hausdorff type distance which takes the whole matrix-order structure into account; throughout this thesis we will therefore work in the setting of Kerr and Li's *complete* and *operator Gromov–Hausdorff distance* [85, 86]. A criterion for controlling operator Gromov–Hausdorff distance of quantum metric spaces (X, L_X) and (Y, L_Y) is to find Lip-norm bounded ucp maps $\tau : X \to Y$ and $\sigma : Y \to X$ whose compositions are close to the respective identity maps on X and Y, uniformly on the Lip-norm unit-balls; we give a proof in Subsection 2.3.2 building on [77, 137], but the fundamental idea really goes back to Rieffel's *bridges* [118]. Recall that the Gromov–Hausdorff distance of two compact metric spaces M and N is the infimum over Hausdorff distances for all metrics on the disjoint union $M \sqcup N$ which restrict to the respective metrics on M and N. Thus any such metric on $M \sqcup N$ gives an upper bound for Gromov–Hausdorff distance. Dually, a bridge between compact quantum metric spaces (X, L_X) and (Y, L_Y) gives rise to a Lip-norm on $X \oplus Y$, which induces L_X and L_Y, and thereby yields an upper bound for quantum/operator Gromov–Hausdorff distance.

Within the framework of compact quantum metric spaces and operator Gromov–Hausdorff distance, the question about convergence of spectral truncations can now be rigorously posed. For this purpose, assume that (A, H, D) is a spectral triple such that $(A, \|[D, \cdot]\|)$ is a compact quantum metric space; such a spectral triple is called a *spectral metric space*, as coined in [13]. As a consequence of [115, Theorem 1.8], if P is a finite-rank spectral projection for D, the spectral truncation (PAP, PH, PDP) is automatically a spectral metric space as well. One may hence ask whether spectral truncations of spectral metric spaces converge in operator Gromov–Hausdorff distance, as $P \to \mathbf{I}^H$. This question was the initial motivation for this thesis and we now give a brief overview over the main contributions and related aspects we investigate.

Overview of results

Chapter 2 In a preliminary chapter we give some background and an overview of the basic facts about spectral triples, operator systems, and compact quantum metric spaces, which the subsequent chapters are based on.

Chapter 3 We begin by studying spectral truncations of the d-torus $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$, for the Dirac operator associated to the trivial spin-structure. Our general strategy to prove convergence of the compact quantum metric spaces

$$(P\mathrm{C}(\mathbb{T}^d)P, \|[PDP, \cdot]\|) \to (\mathrm{C}(\mathbb{T}^d), \|[D, \cdot]\|)$$

is to apply the above-mentioned criterion, i.e. to find Lip-norm bounded ucp maps $\tau : \mathrm{C}(\mathbb{T}^d) \to P\mathrm{C}(\mathbb{T}^d)P$ and $\sigma : P\mathrm{C}(\mathbb{T}^d)P \to \mathrm{C}(\mathbb{T}^d)$ whose compositions approximate the respective identity maps on $\mathrm{C}(\mathbb{T}^d)$ and $P\mathrm{C}(\mathbb{T}^d)P$ in norm controlled by Lip-norm. The canonical candidate for τ is the compression map $f \mapsto PfP$ which is Lip-norm contractive and ucp. It is generally less obvious what a good candidate for the map σ might be. Inspired by the use of Berezin quantization in [120, 128], we let σ be the formal adjoint of τ after equipping $\mathrm{C}(\mathbb{T}^d)$ with the L^2- and $P\mathrm{C}(\mathbb{T}^d)P$ with the Hilbert–Schmidt inner product, as was done in the case of the circle [136]. The compositions $\sigma \circ \tau$ and $\tau \circ \sigma$ turn out to be Fourier and Schur multiplication respectively with a symbol, which is the Fourier transform of the square of the spherical Dirichlet kernel; we dub this convolution kernel the *spectral Fejér kernel*. The fact that the spectral Fejér kernel is a square (and hence positive) allows us to establish it is a good kernel. Together with a transference result relating the norms of Fourier and Schur multiplication, this allows us to conclude the Lip-norm estimates which are sufficient to show that spectral truncations of the d-torus converge in operator Gromov–Hausdorff distance Theorem 3.3.9.

Chapter 4 In dealing with spectral truncations of tori, it is apparent that a major role is played by methods from harmonic analysis. These were conceptualized in

[56] investigating truncations of compact groups; there, no Dirac operator is considered, but rather the function algebra is compressed by a projection associated to the Peter–Weyl decomposition. The group is assumed to be equipped with a metric which is crucially required to be bi-invariant. The fact that (dual versions of) all of these notions are available in the setting of coamenable compact quantum groups (à la Woronowicz [145]), allows us to investigate their *Peter–Weyl truncations*; given such a quantum group $(\mathrm{C}(\mathbb{G}), \Delta)$, its GNS space admits a Peter–Weyl decomposition $\mathrm{L}^2(\mathbb{G}) \cong \bigoplus_{\pi\in\widehat{\mathbb{G}}} H_\pi \otimes H_\pi^*$ and as a "spectral" projection P we consider the projection onto the subspace $\bigoplus_{\pi\in\Lambda} H_\pi \otimes H_\pi^*$, for some family $\Lambda \subseteq \widehat{\mathbb{G}}$ of irreducible unitary corepresentations. The comultiplication Δ and the compression map $\mathrm{C}(\mathbb{G}) \ni a \mapsto PaP$ together induce right and left coactions on the operator system $P\mathrm{C}(\mathbb{G})P$, which are moreover ergodic. Invariant metrics for groups correspond to invariant Lip-norms for quantum groups which were introduced and studied in [97]. Ergodicity of the coactions on the operator system $P\mathrm{C}(\mathbb{G})P$ allows to induce a bi-invariant Lip-norm on it by slightly adapting one of the main results of *loc.cit.* to coactions on operator systems (rather than C*-algebras).

In order to apply the criterion for operator Gromov–Hausdorff convergence, the canonical candidate for the map $\tau : \mathrm{C}(\mathbb{G}) \to P\mathrm{C}(\mathbb{G})P$ is again the compression map. The reverse map $\sigma : P\mathrm{C}(\mathbb{G})P \to \mathrm{C}(\mathbb{G})$ will be a *slice map* $x \mapsto (\phi \otimes \mathbf{I}^{\mathrm{C}(\mathbb{G})})\alpha(x)$, for an appropriate state $\phi \in \mathcal{S}(\mathrm{C}(\mathbb{G}))$ and α the right coaction on $P\mathrm{C}(\mathbb{G})P$; it turns out that one can indeed find a state ϕ which allows to obtain the required Lip-norm estimates of the compositions $\sigma\circ\tau$ and $\tau\circ\sigma$, Theorem 4.3.13. This uses the well-behaved interplay of the weak*-topology and the Monge–Kantorovich distance on $\mathrm{C}(\mathbb{G})$, as well as of Lip-norm invariance and slice maps.

We also review the complementary result for convergence of Fourier truncations obtained by Rieffel in [123]; in this case, one deals with operator subsystems of $\mathrm{C}(\mathbb{G})$ rather than compressions, and the maps for applying the criterion for operator Gromov–Hausdorff convergence are a slice map $a \mapsto (\mathbf{I} \otimes \mu)\Delta(a)$ for τ, for a state $\mu \in \mathcal{S}(\mathrm{C}(\mathbb{G}))$ with an appropriate support condition, and the inclusion map for σ. For convergence of Peter–Weyl truncations, it is crucial that there is a state ϕ on the compression $P\mathrm{C}(\mathbb{G})P$ such that its pullback by the compression map approximates the counit ε on $\mathrm{C}(\mathbb{G})$, and similarly for convergence of Fourier truncations the state μ should satisfy an appropriate support condition and approximate the counit; this is the main place where the coamenability assumption is used.

Chapter 5 The close connection between Peter–Weyl and Fourier truncations as well as of the respective methods of their proofs of convergence require more clarification. It was shown in [33, 50] that the Toeplitz operator system $\mathrm{C}(\mathrm{S}^1)^{(\{0,\dots,N\})}$ and Fejér–Riesz operator system $\mathrm{C}(\mathrm{S}^1)_{(\{-N,\dots,N\})}$ (for notation see Chapter 5 below) arising respectively as spectral and Fourier truncations of the circle are dual operator systems. It might seem reasonable to expect that this is true for Peter–Weyl and Fourier truncations in general. However, the operator systems in question are in general not dual, as we explore in Chapter 5. In fact, the crucial point in the proof of the duality of the Toeplitz and Fejér–Riesz system in [33, 50] is the operator-valued Fejér–Riesz lemma; however, failure of an analogous result already in the case of the 2-torus has long been known [127]. Moreover, this was used in *loc.cit.* to show that positive semi-definite functions on (large enough) rectangles in $\mathbb{Z}^2$ cannot be extended to positive semi-definite functions on $\mathbb{Z}^2$. Such kinds of factorization and extension problems have been heavily researched with connections to many areas of mathematics, see e.g. [130, 131, 10]. In Chapter 5, we provide an operator system perspective at these problems and clarify their relation to the problem of duality of the operator systems arising from spectral/Peter–Weyl and Fourier truncations.

To this end, we work in the setting of a discrete group Γ (assumed amenable whenever necessary). We then analyze for which such discrete groups Γ and finite subsets $\Sigma \subseteq \Gamma$ it is true that the *Toeplitz system* $\mathrm{C}^*(\Gamma)^{(\Sigma)} \subseteq \mathrm{M}_\Sigma$ consisting of Toeplitz matrices $(T_{st^{-1}})_{s,t\in\Sigma}$ is dual to the *Fourier system*, i.e. operator subsystem

$$\mathrm{C}^*(\Gamma)_{(\Sigma\Sigma^{-1})} := \mathrm{span}\{\delta_{st^{-1}} \mid s,t \in \Sigma\} \subseteq \mathrm{C}^*(\Gamma).$$

Note that, for Γ amenable, this is a special case of the question about duality of the operator systems arising from Peter–Weyl and Fourier truncations in Chapter 4, since $\mathrm{C}^*(\Gamma)$ is a coamenable compact quantum group. In Theorem 5.3.18, we show that the following three statements are equivalent:

(1) the Toeplitz system $\mathrm{C}^*(\Gamma)^{(\Sigma)}$ and the Fourier system $\mathrm{C}^*(\Gamma)_{(\Sigma\Sigma^{-1})}$ are each other's operator system duals;

(2) for every positive semi-definite Toeplitz matrix $(T_{st^{-1}})_{s,t\in\Sigma} \in \mathrm{M}_\Sigma(\mathrm{M}_n)^+$, there is a positive semi-definite function $u : \Gamma \to \mathrm{M}_n$ such that $u(st^{-1}) = T_{st^{-1}}$, for all $s,t \in \Sigma$, $n \in \mathbb{N}$;

(3) every positive element $x \in \mathrm{M}_n(\mathrm{C}^*(\Gamma)_{(\Sigma\Sigma^{-1})})^+$ admits a sum-of-squares decomposition, i.e. there are $y_1, \ldots, y_r \in \mathrm{M}_n \otimes \mathrm{span}\{\delta_s \mid s \in \Sigma\}$ such that $x = \sum_{i=1}^r y_i y_i^*$.

The crucial observation which facilitates the proof is that the Toeplitz system is dual to the operator system spanned by sums of squares. The duality between sums of squares and Toeplitz matrices has long been known and exploited in the study of positive semi-definite extension problems. However, we contribute in rephrasing Fejér–Riesz type lemmas and their equivalence with extension properties in an operator system way; that is, a Fejér–Riesz lemma holds if and only if the canonical inclusion map from the sums-of-squares operator system to the group C^*-algebra is a complete order embedding. Equivalently the dual map, which is restriction of functions in the Fourier–Stieltjes algebra, is a quotient map (in the category of matrix-ordered vector spaces); this in turn can be rephrased as an extension property for positive semi-definite functions. As an immediate consequence, which one might now actually consider a corollary of [127], the minimal and maximal tensor products of the operator system of $n \times n$ Toeplitz matrices, $n \geqslant 3$, with itself are distinct, giving a more conceptual proof than the one in [50].

Our sums-of-squares operator systems serve as building blocks for operator systems related to the general problem of extending partially defined positive semi-definite functions on discrete groups. This allows us, for instance, to show that there must be matrix-valued positive semi-definite functions on the set $\{(-1,0),(0,-1),$

$(0,0),(0,1),(1,0)\}$ which do not admit positive semi-definite extensions to all of $\mathbb{Z}^2$; we achieve this by using the tools of amalgamated free products and minimal C*-covers of operator systems.

Literature

Spectral truncations, i.e. compressions of spectral triples (A, H, D) by a spectral projection P for D, were already studied in [36]. However, there the state space $\mathcal{S}(PAP)$ was equipped with the metric arising as the pullback by the compression map $A \ni a \mapsto PaP$ of the Monge–Kantorovich distance on the state space $\mathcal{S}(A)$. A similar approach was taken in [60]. One might argue that this approach is not intrinsic to the spectral truncation problem, as the quantum metric structure of the truncation requires knowledge of the original quantum metric structure. Instead, the paradigm of the spectral truncation formalism is to approximate noncommutative geometry when lacking full spectral (and hence metric) information; in fact when spectral truncations are treated as in [36, 60], one might think of them as quantum metric subspaces of spectral triples. However, the ucp compression map $A \to PAP$ may in general not be a quotient map of operator systems (which one might want to require for a compact quantum metric subspace).

Our results on spectral truncations of tori in Chapter 3 extend and build on previous work for the circle [136], an analysis for pure states of the truncated circle [65], and a first approach to truncations of tori in terms of products of circles (yet, not spectral for the spin-Dirac operator) [14]. We also mention the approach to spectral truncations using computer simulations [59]. Further results on convergence of spectral truncations can be found in [135, 17]. Besides convergence as compact quantum metric spaces other aspects of truncated noncommutative geometry have been studied, including K-theory [138, 139], noncommutative integration [66], and an approach in terms of tolerance relations [34, 35, 57].

For compact quantum metric spaces modeled on C*-algebras rather than operator systems, it seems appropriate to consider Lip-norms which take the additional algebraic structure into account in that they satisfy a *Leibniz condition.* Various adaptations of quantum Gromov–Hausdorff distance respecting this additional property (and extra structure) have been developed by Latrémolière; in particular, the *quantum Gromov–Hausdorff propinquity* [91] and the *spectral Gromov–Hausdorff propinquity* [92] shall be mentioned at this point. These define respectively metrics on the collection of all C*-algebras with Leibniz Lip-norms and on the collection of all spectral metric spaces, (i.e. spectral triples for which the Monge–Kantorovich distance on states induced by the Connes distance metrizes the weak*-topology). Remarkably it is shown in [91, 92] that if two C*-algebras with Leibniz Lip-norms are at quantum Gromov–Hausdorff propinquity 0 they must be *-isomorphic, and similarly if two spectral triples are at spectral Gromov–Hausdorff propinquity 0 they must be unitarily equivalent. These results are in line with the fact that if two operator systems with Lip-norms are at operator Gromov–Hausdorff distance 0, they are completely order isomorphic [86].

The question about convergence of additional structure on compact quantum metric spaces was considered by Rieffel. Enhancing his convergence result for matrix algebras to the sphere [120] he also showed convergence of cotangent bundles [121], vector bundles [122] and Dirac operators [124].

We point out that an arbitrary spectral triple is far from being automatically a spectral metric space, see Example 2.3.11; instead, it has taken significant effort to establish the quantum metric structure of some of the spectral triples which are particularly relevant in noncommutative geometry [1, 77, 100]. See also [74] for a characterization of compact quantum metric spaces in terms of finite-dimensional approximations.

Another approach to quantum metric spaces is due to Kuperberg–Weaver [89]; in their sense, a quantum metric space (or W*-*metric space*) is a von Neumann algebra together with an appropriate filtration by operator systems, mimicking the classical metric axioms. Spectral triples give rise to W*-metric spaces and conversely W*-metric space structures induce Leibniz Lipschitz seminorms in the sense of Rieffel. This approach appears to be suitable for certain applications in quantum information theory, in particular quantum graphs; however, due to its von Neumann algebraic flavor it seems less appropriate for dealing with *compact* quantum metric spaces and thus to approach questions about (quantum Gromov–Hausdorff distance type) convergence of spectral truncations. Moreover, as pointed out in [89], W*-metric spaces disregard spin-geometric aspects of spectral triples.

A candidate object for a compact quantum metric spaces is a discrete group Γ with a proper length function ℓ; we discuss in Example 2.1.6 how such a length function induces a Lipschitz seminorm on the reduced group C*-algebra of Γ and even a spectral triple. The main classes of discrete groups for which it is known to date that they are compact quantum metric spaces in the sense of Rieffel include $\mathbb{Z}^d$ [117], groups of rapid decay [2], groups with a Haagerup-type property [105], and nilpotent groups [28]. See also [7] for a discussion of this matter for length functions on discrete quantum groups. These are classes of examples to which Rieffel's results on Fourier truncations of compact quantum groups apply [123]. For our Peter–Weyl truncations studied in Chapter 4 we have to additionally require a bi-invariant Lip-norm; if such a Lip-norm is induced by a length function, the bi-invariance condition corresponds to invariance of the length function for the action by inner automorphisms.

All the operator systems we deal with in this thesis are unital; for the relevant theory we refer to the monographs [106, 111, 19, 48, 23]. Davidson–Kennedy built a noncommutative Choquet theory around them [40, 41, 37]. In another approach to noncommutative geometry aiming at incorporating constraints on spatial resolution, nonunital operator systems [142] play a central role [33, 34]; a noncommutative Choquet theory point of view at these was given in [84].

Outlook

Let us end this introduction with some remarks on potential extensions of the results obtained in this thesis. To date convergence of spectral truncations for the canonical spectral triples arising from spin-manifolds has only been proven for tori with trivial spin-structure, Chapter 3, and even this has taken significant effort [136, 14, 94]. Towards a generalization for a non-abelian Lie group G we point out that the Peter–

Weyl decomposition

$$\mathrm{L}^2(S) \cong \bigoplus_{\pi \in \widehat{G}} H_\pi \otimes H_\pi^* \otimes V,$$

of the Hilbert space of L^2-spinors on G is generally not a spectral decomposition for the Dirac operator. Rather $\mathrm{L}^2(S) \cong \mathrm{L}^2(G) \otimes V$ carries a tensor product of representations of G, where the one on V arises from the extension of the coadjoint action of G on $\mathfrak{g}^*$ to $\mathbb{C}\ell(\mathfrak{g}^*)$, with $\mathfrak{g}^*$ the dual of the Lie algebra of G, *cf.* [124]; to obtain the spectral decomposition of the Dirac operator on G one then has to decompose this tensor product of representations into its irreducible components. See [70, 11, 71] for some computations of the spectral decomposition of Dirac operators in the setting of homogeneous spaces. This obstructs the potential for relating spectral truncations (for a Dirac operator) of a compact Lie group to Peter–Weyl truncations as in [56] and Chapter 4. Instead, other methods and possibly case-by-case analyses taking into account the particular representation theory of the Lie group in question, appear to be required.

Similarly as for Fourier truncations [123] it would be desirable to treat Peter–Weyl truncations of coactions, rather than just of compact quantum groups as in Chapter 4; the obstruction to this is that the maps – compression and slice maps – which we use for the operator Gromov–Hausdorff distance estimate, necessitate both a left and a right coaction as well as a bi-invariant Lip-norm. Similar choices of maps have played an important role in the study of many problems of compact quantum metric spaces, but new ideas – potentially more geometric and depending less on (co-)actions – are called for.

The problems of extending partially defined positive semi-definite functions and matrices have appeared in many contexts and have long been studied from different points of view. Rather than providing a survey here, we refer to [131, 130, 10] as well as [8, 9] and references therein. The operator system point of view which we present in Chapter 5 gives a new perspective on arguments which have been similarly applied before; this facilitates the import of techniques related to the tensor product theory and C^*-covers of operator systems. We expect that generalizations to a setting of locally compact groupoids will be possible. At a technical level this would involve nonunital operator systems. It might allow to put extension problems such as those for groups and matrices on a common footing.

Chapter 2

Preliminaries

In this thesis all Hilbert spaces are complex and separable with inner products anti-linear in the first component. The sets of $m \times n$ and $n \times n$ matrices over the complex numbers will be denoted $\mathrm{M}_{m,n}$ and M_n respectively. We denote the unit in M_n by $\mathbf{1}_n$, and the unit in a unital C*-algebra A by $\mathbf{1}_A$. For a vector space V the identity map $V \to V$ will be denoted $\mathbf{I}^V$.

2.1 Spectral triples

The concept of a spectral triple was popularized by Connes [29, 30, 31] as a way to capture (noncommutative) geometric properties by means of functional analytic, i.e. *spectral*, data; below, we exemplify this by discussing spin^c-manifolds and discrete groups with length functions, preparing for later chapters. See [31] for motivation on spectral triples and the textbooks [61, 140] for more detail.

Definition 2.1.1. A *spectral triple* (A, H, D) consists of a unital *-algebra A, a Hilbert space H with $A \subseteq \mathcal{B}(H)$, and an essentially self-adjoint operator D on H, such that the following two properties hold:

(1) The commutators $[D, a]$ extend to bounded operators on H, for all $a \in A$.

(2) The resolvent $(D - i)^{-1}$ is a compact operator on H.

Equivalently, one may require the *-algebra A in the above definition to be a C*-algebra (by taking the closure) in which case condition (1) is only required to hold for all elements in a given dense *-subalgebra.

Condition (1) gives rise to a metric on the state space of A defined by the operator D, see Proposition 2.1.5. The *compact resolvent condition* (2) ensures that the operator D has discrete spectrum $\sigma(D)$ consisting of eigenvalues only; in fact, the resolvent $(D - i)^{-1}$ is a compact normal operator, so its non-zero spectrum consists of eigenvalues only and one checks that $0 \neq \lambda \in \sigma((D - i)^{-1})$ if and only if $\frac{1}{\lambda} + i \in \sigma(D)$. Moreover, as pointed out in [31], the Hilbert space H together with the operator D is characterized up to unitary equivalence by $\sigma(D)$ and the dimensions of the associated eigenspaces.

Example 2.1.2. We recall the example of a Riemannian spin^c-manifold. We mainly follow [61, 140], see also [68] and the standard references [73, 54, 16] for a principal bundle point of view.

Let (M, g) be an oriented closed (i.e. compact without boundary) d-dimensional Riemannian manifold. Using the musical isomorphism induced by g, the tangent and cotangent bundle are identified whenever convenient. For d even, consider the complex Clifford algebra bundle $\mathbb{C}\ell(\mathrm{T}M) \to M$ with fibers $\mathbb{C}\ell(\mathrm{T}M)_p \cong \mathbb{C}\ell(\mathrm{T}_pM, Q_g)$, $p \in M$, isomorphic to the complex Clifford algebra for the quadratic form $Q_g(v) := g(v, v)$. For d odd, consider instead the even part $\mathbb{C}\ell^0(\mathrm{T}M) \to M$ of the complex Clifford algebra bundle with fibers $\mathbb{C}\ell^0(\mathrm{T}M)_p \cong \mathbb{C}\ell^0(\mathrm{T}_pM, Q_g)$, $p \in M$, isomorphic to the even part of the complex Clifford algebra for the quadratic form Q_g. We use the notation $\mathbb{C}\ell^{(0)}(\mathrm{T}M)$ to refer to $\mathbb{C}\ell(\mathrm{T}M)$ and $\mathbb{C}\ell^0(\mathrm{T}M)$ respectively, depending on whether d is even or odd.

A *spin^c-structure* on M is a vector bundle $S \to M$ such that the (even part of the) complex Clifford algebra bundle $\mathbb{C}\ell^{(0)}(\mathrm{T}M)$ is isomorphic to the endomorphism bundle $\mathrm{End}(S)$. The vector bundle S is called the *spinor bundle* and sections of it are called *spinors*. The action of the algebra of sections $\Gamma(\mathbb{C}\ell^{(0)}(\mathrm{T}M))$ on spinors $\Gamma(S)$ is called *Clifford multiplication* and denoted $\mathfrak{c}$. The Clifford algebra $\mathbb{C}\ell(\mathrm{T}_pM, Q_g)$ is isomorphic to standard complex Clifford algebra $\mathbb{C}\ell_d$ which has exactly one irreducible representation on $\mathbb{C}^{2^{\frac{d}{2}}}$ if d is even, and exactly two irreducible representations on $\mathbb{C}^{2^{\frac{d-1}{2}}}$ if d is odd; hence $\dim(S_p) = 2^{\lfloor \frac{d}{2} \rfloor}$. The manifold M is called *spin^c* if it admits a spin^c-structure. There is a topological obstruction to being spin^c and generally spin^c-structures are not unique.

The Levi-Civita connection on the cotangent bundle of M lifts to a family of connections on the spinor bundle suitably compatible with Clifford multiplication and parametrized by purely imaginary one-forms; these connections are called *Clifford connections* and we fix one of them which we call the *spin^c-connection* and which we denote $\nabla^S : \Gamma^\infty(S) \to \Omega^1(M) \otimes \Gamma^\infty(S)$. If M is spin, i.e. it admits a spin^c-structure together with a *charge conjugation* operator $J : \Gamma^\infty(S) \to \Gamma^\infty(S)$, then there is a unique spin^c-connection commuting with J, called the *spin-connection*, and we work with this connection on the spinor bundle in this case.

Whenever one has a homomorphism $\mathfrak{c} : \Gamma(\mathbb{C}\ell(\mathrm{T}M)) \to \Gamma(\mathrm{End}(S))$ and a Clifford connection on S (i.e. a connection which is suitably compatible with the Clifford multiplication $\mathfrak{c}$) one can define a (generalized) *Dirac operator* by

$$D : \Gamma^\infty(S) \xrightarrow{\nabla^S} \Omega^1(M) \otimes \Gamma^\infty(S) \xrightarrow{-i\mathfrak{c}} \Gamma^\infty(S). \tag{2.1}$$

Here we view $\Omega^1(M) \subseteq \Gamma^\infty(\mathbb{C}\ell(\mathrm{T}M))$ using the musical isomorphism arising from the Riemannian metric. In our situation, where M is spin^c with $\mathfrak{c}$ an isomorphism and where a spin^c-connection ∇^S is fixed, we call D the *spin^c-Dirac operator* and denote it D_M.

One checks that a Dirac operator D as in (2.1) is a symmetric elliptic first-order differential operator. A symmetric first-order differential operator on a closed manifold is shown to be essentially self-adjoint by using Friedrich's mollifiers. A first-order differential operator has bounded commutators with smooth functions acting by pointwise multiplication on L^2-spinors. Moreover, applying Gårding's inequality and Rellich's lemma one obtains that a symmetric elliptic first-order differential operator has compact resolvent. See e.g. [68, Chapter 10] for details. Altogether, we obtain the *canonical spectral triple*

$$\big(\mathrm{C}(M), \mathrm{L}^2(S), D_M\big)$$

associated to a spin$^{\mathrm{c}}$-manifold M with spinor bundle $S \to M$.

Remark 2.1.3. A *spin-structure* on an oriented closed Riemannian manifold M is a spin$^{\mathrm{c}}$-structure $S \to M$ together with a certain anti-unitary operator J on $\Gamma^\infty(S)$ called *charge conjugation*. The manifold M is called *spin* if it admits a spin-structure. Again there is a topological obstruction to being spin and generally spin-structures are not unique. However, once a spin-structure is fixed there is a unique spin$^{\mathrm{c}}$-connection, called the *spin-connection*, on the spinor bundle which commutes with the charge conjugation operator. The associated Dirac operator is called the *spin-Dirac operator*. A spin-structure gives rise to a *chirality* operator γ on $\Gamma^\infty(S)$ and a *KO-dimension*.

Spin-manifolds are generalized in noncommutative geometry as *real* spectral triples, i.e. spectral triples with additional operators J and γ satisfying axioms analogous to the properties of charge conjugation and chirality. For our purposes, spin$^{\mathrm{c}}$-manifolds will be sufficient, so we only refer once again to [68, 61, 140] for details. It should be mentioned though that Connes showed how to reconstruct a spin-manifold from a real spectral triple $(A, H, D; J, \gamma)$, with the C*-algebra A commutative and under (five) appropriate extra assumptions [32].

Next, we treat the case of the *trivial* spin$^{\mathrm{c}}$-structure on the d-torus which is the spin$^{\mathrm{c}}$-structure we will consider in Chapter 3.

Example 2.1.4. Consider the flat d-dimensional torus $\mathbb{T}^d := \mathbb{R}^d/2\pi\mathbb{Z}^d$. By flatness we have $\mathbb{C}\ell^{(0)}(\mathrm{T}M) \cong \mathbb{T}^d \times \mathbb{C}\ell_d^{(0)}$ and a spinor bundle $S \to \mathbb{T}^d$ is given by the trivial bundle $S = \mathbb{T}^d \times \mathbb{C}^{2^{\lfloor \frac{d}{2} \rfloor}}$, where the fibers are irreducible representations of the (even part of the) standard complex Clifford algebra; this is the *trivial spin$^{\mathrm{c}}$-structure* on $\mathbb{T}^d$. As the spin$^{\mathrm{c}}$-connection we choose the flat spin-connection yielding the spin$^{(\mathrm{c})}$-Dirac operator

$$D_{\mathbb{T}^d} = -i\partial_\mu \otimes \gamma_d^\mu$$

on $\mathrm{L}^2(S) \cong \mathrm{L}^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^{\lfloor \frac{d}{2} \rfloor}}$ (with domain the smooth spinors). Here γ_d^μ are the generators of the standard complex Clifford algebra $\mathbb{C}\ell_d$. The Dirac operator $D_{\mathbb{T}^d}$ squares to $\Delta_{\mathbb{T}^d} \otimes \mathbf{1}_{2^{\lfloor \frac{d}{2} \rfloor}}$, where $\Delta_{\mathbb{T}^d}$ is the Laplace operator. With this one readily checks that the spectrum of $D_{\mathbb{T}^d}$ is given by the eigenvalues $\lambda_n^\pm := \pm\|n\|$, for $n \in \mathbb{Z}^d$, with $\|\cdot\|$ the euclidean norm on $\mathbb{Z}^d \subseteq \mathbb{R}^d$, so the eigenspinors of $D_{\mathbb{T}^d}$ are indexed by $\mathbb{Z}^d$. The non-zero eigenvalues $\lambda_n^\pm$ have multiplicity $\mathcal{N}(\|n\|)2^{\lfloor \frac{d}{2} \rfloor - 1}$, where $\mathcal{N}(\|n\|)$ is the number of lattice points $m \in \mathbb{Z}^d$ with $\|m\| = \|n\|$. The eigenvalue 0 has multiplicity $2^{\lfloor \frac{d}{2} \rfloor}$.

The spectrum of the spin-Dirac operator for the non-trivial spin-structures on flat tori is computed in [53]. See also [58] for an overview of explicit computations of some spectra of spin-Dirac operators.

In this thesis, we mainly deal with the *metric* aspect of noncommutative geometry. For this, the following observation due to Connes [29, 30] is foundational.

Proposition 2.1.5. *Let (A, H, D) be a spectral triple.*

(1) The following defines a metric on the state space $\mathcal{S}(A)$ of A:

$$d(\mu, \nu) := \sup\{|\mu(a) - \nu(a)| \mid a \in A, \|[D, a]\| \leqslant 1\}, \tag{2.2}$$

for all $\mu, \nu \in \mathcal{S}(A)$.

(2) Let $(A, H, D) = (\mathrm{C}(M), \mathrm{L}^2(S), D_M)$ be the canonical spectral triple of a spin$^{\mathrm{c}}$-manifold M with fixed spinor bundle $S \to M$, a choice of spin$^{\mathrm{c}}$-connection and the associated spin$^{\mathrm{c}}$-Dirac operator D_M as in Example 2.1.2. Then the metric d from (2.2) coincides with the Riemannian distance on $M \cong \mathcal{P}(\mathrm{C}(M))$, i.e.

$$d(\delta_p, \delta_q) = d_g(p, q), \tag{2.3}$$

for all $p, q \in M$. Here $\delta_p \in \mathcal{P}(\mathrm{C}(M))$ denotes the evaluation functional at $p \in M$.

Note that the homeomorphism $M \cong \mathcal{P}(\mathrm{C}(M))$ is to be understood in terms of the subspace topology on the pure state space of $\mathrm{C}(M)$ inherited from the weak*-topology on the state space. The metric (2.2) is called the *Connes distance* and (2.3) is known as the *Connes distance formula.* For a (noncommutative) spectral triple, the topology on $\mathcal{S}(A)$ induced by the metric (2.2) does in general not coincide with the weak*-topology. However, when it does one may speak of a *compact quantum metric space*, to which we come back in Section 2.3.

Proof (sketch). (1) Let $\mu, \nu \in \mathcal{S}(A)$ be states. Assume $d(\mu, \nu) = 0$. Then, by rescaling $\mu(a) = \nu(a)$, for all $a \in A$ with $\|[D, a]\| < \infty$. Since $\mathrm{Dom}(\|[D, \cdot]\|) := \{a \in A \mid \|[D, a]\| < \infty\}$ is a dense *-subalgebra of A, we have that $\mu = \nu$. Assuming now that $\mu \neq \nu$, it follows from norm-density of $\mathrm{Dom}(\|[D, \cdot]\|)$ in A that there is an element $a \in A$ with $\|[D, a]\| \leqslant 1$ and $\mu(a) \neq \nu(a)$, so $d(\mu, \nu) > 0$. Symmetry and the triangle inequality follow directly from the definition of the Connes distance.

(2) This is sketched in [29, Proposition 1] and more details are given in [140, Proposition 4.23]; we only outline the main steps. Using the local formula for D_M, one checks that $[D_M, f](\psi) = -i\mathfrak{c}(\mathrm{d}f \otimes \psi)$, for all $\psi \in \Gamma^\infty(S)$, which yields $\|[D_M, f]\| = \sup_{p \in M} \|\mathrm{grad}_p(f)\|$, for all $f \in \mathrm{C}^\infty(M)$. Now we have the identity $\sup_{p \in M} \|\mathrm{grad}_p(f)\| = \|f\|_{\mathrm{Lip}}$, with $\|f\|_{\mathrm{Lip}} := \sup_{p \neq q} \frac{|f(p) - f(q)|}{d_g(p,q)}$ the Lipschitz constant of f, and the resulting identity

$$\|[D, f]\| = \|f\|_{\mathrm{Lip}} \tag{2.4}$$

is in fact valid for all Lipschitz continuous functions $f \in \mathrm{C}(M)$. With this one obtains $d(\delta_p, \delta_q) \leqslant d_g(p,q)$, for all $p, q \in M$.

Conversely, the functions $f_p(q) := d_g(p,q)$, for all $p, q \in M$, are Lipschitz continuous with Lipschitz constant 1, so (2.4) implies $d(\delta_p, \delta_q) \geqslant |f_p(p) - f_p(q)| = d_g(p,q)$. □

Example 2.1.6 ([29]). Let Γ be a discrete group equipped with a length function $\ell : \Gamma \to [0, \infty)$, i.e. $\ell(e) = 0$, $\ell(t^{-1}) = \ell(t)$, and $\ell(st) \leqslant \ell(s) + \ell(t)$, for all $s, t \in \Gamma$. Define the operator $D_\ell : \mathrm{Dom}(D_\ell) \to \ell^2(\Gamma)$ with $\mathrm{Dom}(D_\ell) := \mathbb{C}[\Gamma] \subseteq \ell^2(\Gamma)$ as the operator given by pointwise multiplication with ℓ, i.e.

$$D_\ell((\psi_t)_{t\in\Gamma}) := (\ell(t)\psi_t)_{t\in\Gamma},$$

for all finitely supported functions $(t \mapsto \psi_t) \in \mathbb{C}[\Gamma]$. Then

$$\|[D_\ell, \lambda(a)]\| = \sum_{t\in\Gamma} a_t \ell(t), \tag{2.5}$$

for all $a = \sum_{t\in\Gamma} a_t \delta_t \in \mathbb{C}[\Gamma]$, where λ is the left regular representation $G \to \mathcal{B}(\ell^2(\Gamma))$. In fact, observe that

$$\begin{aligned} \|[D_\ell, \lambda(t)]\| &= \|\lambda(t) D_\ell \lambda(t)^{-1} - D_\ell\| \\ &= \|D_{\ell(t^{-1}\cdot)} - D_\ell\| \\ &= \sup_{s\in\Gamma} |\ell(t^{-1}s) - \ell(s)| \\ &= \ell(t), \end{aligned}$$

for all $t \in \Gamma$, which readily implies (2.5).

Moreover, if the length function ℓ is proper the triple $(\mathrm{C}^*_\mathrm{r}(\Gamma), \ell^2(\Gamma), D_\ell)$ is a spectral triple. Indeed, one checks that D_ℓ is a symmetric operator with dense range and $\mathbb{C}[\Gamma]$ is a core. Properness of ℓ implies the resolvent $(D_\ell - i)^{-1}$, given by $(D_\ell - i)^{-1}\psi(t) = \frac{1}{\ell(t) - i}\psi(t)$, is approximated by the finite rank operators

$$\ell^2(\Gamma) \ni \psi \mapsto \left(t \mapsto \begin{cases} \frac{1}{\ell(t)-i}\psi(t), & \text{if } \ell(t) \leqslant N \\ 0, & \text{otherwise} \end{cases} \right),$$

for $N \in \mathbb{N}$, so $(D_\ell - i)^{-1}$ is a compact operator on $\ell^2(\Gamma)$. The spectrum of D_ℓ is the image of the length function $\ell(\Gamma) \subseteq \mathbb{R}_0^+$ which is point spectrum only and accumulates at ∞ by properness of ℓ. The eigenspace V_λ associated to an eigenvalue $\lambda \in \mathbb{R}^+$ is spanned by δ_t with $t \in \ell^{-1}(\{\lambda\})$, and V_λ is finite-dimensional by properness of ℓ.

Length functions are of interest inducing candidates for Lip-norms. See Section 2.3 and Chapter 4 below, and recall the examples where it is known that they indeed induce Lip-norms from [117, 2, 105, 28] mentioned in the introduction. For a more spin-geometric version of length functions see the *Clifford length functions* in [64, 3].

As put forward in [33], the spectral triples axioms make sense if the C*-algebra is replaced by an *operator system*, i.e. a unital subspace $X \subseteq \mathcal{B}(H)$ which is closed under taking adjoints.

Definition 2.1.7. An *operator system spectral triple* (X, H, D) consists of an operator system X, a Hilbert space H with $X \subseteq \mathcal{B}(H)$, and an essentially self-adjoint operator D on H, such that the following two properties hold:

(1) The commutators $[D, x]$ extend to bounded operators on H, for all elements $x \in X$.

(2) The resolvent $(D - i)^{-1}$ is a compact operator on H.

The following example from [33] is the key example for an operator system spectral triple in this thesis.

Example 2.1.8. Let (A, H, D) be a spectral triple. Assume the operator D is self-adjoint (by passing to its unique self-adjoint extension). Let $\Omega \subseteq \mathbb{R}$ be a Borel set and let $P \in \mathcal{B}(H)$ be the associated spectral projection for D. Note that the operator PDP coincides with the operator $g(D)$ given by Borel functional calculus for the function g given by $g(\lambda) := \lambda\chi_\Omega(\lambda)$, for all $\lambda \in \mathbb{R}$, and with χ_Ω the indicator function for Ω. Hence PDP is self-adjoint by the spectral theorem [114, Theorem VIII.6]. If Ω is bounded, it is clear that the resolvent of PDP is compact. If Ω is unbounded, we have $(PDP - i)^{-1} = h(D)$ with $h \in \mathrm{C}_0(\mathbb{R})$ given by $h(\lambda) := (\lambda\chi_\Omega(\lambda) - i)^{-1}$. Since D has compact resolvent it follows by the spectral theorem that $h(D)$ is compact.

Hence the triple (PAP, PH, PDP) is an operator system spectral triple, called the *spectral truncation* of (A, H, D) for P. Note that the spectral projection P commutes with the operator D, so $PDP = DP$. We emphasize that generally $P \notin A$, so PAP is generally not a corner in A.

Definition 2.1.9. Let G be a compact group. We say that an (operator system) spectral triple (X, H, D) is an (operator system) G*-spectral triple* if there is a strongly continuous unitary representation U of G on H such that $U_g X U_g^* = X$ and $U_g D U_g^* = D$, for all $g \in G$.

Since $U_g D = DU_g$ implies that $U_g P = PU_g$, for all spectral projections P for D, we have the following fact.

Proposition 2.1.10. *Let G be a compact group, (A, H, D) a G-spectral triple and P a spectral projection for D. Then the spectral truncation (PAP, PH, PDP) is an operator system G-spectral triple.*

2.2 Operator systems

We collect some facts about operator systems. Much of this will not be needed until Chapter 5. We first focus exclusively on the positivity structure and will only discuss matrix-norms towards the end of this subsection.

2.2.1 Order-unit spaces

Let V be a complex $*$-vector space and denote by V_{h} its hermitian part. A convex *cone* in V is any subset $\mathcal{C} \subseteq V_{\mathrm{h}}$ satisfying $[0, \infty) \cdot \mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$. A convex cone $\mathcal{C}$ in V is called *spanning* if $\mathcal{C} - \mathcal{C} = V_{\mathrm{h}}$ and *proper* if $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$. A complex $*$-vector space V together with a proper spanning convex cone $\mathcal{C}$ is called an *ordered vector space*. We often write $V^+ := \mathcal{C}$ and call it the *positive cone* of (the ordered vector space) V. The positive cone V^+ induces a partial order on V by defining $v \leqslant w$ if and only if $w - v \in V^+$.

Given a complex $*$-vector space V with a proper convex cone $\mathcal{C}$, we say that an element $e \in \mathcal{C}$ is an *order-unit* if, for every hermitian element $v \in V_{\mathrm{h}}$, there is a positive real number $r > 0$ such that $re - v \in \mathcal{C}$. Note that if e is an order-unit then, for every $v \in V_{\mathrm{h}}$, there is a positive real number $r > 0$ such that $v = re - (re - v) \in \mathcal{C} - \mathcal{C}$, so the cone $\mathcal{C}$ is spanning. A $*$-vector space V together with a proper convex cone V^+ and an order-unit e_V is called an order-unit space.

For an order-unit space (V, V^+, e_V), the following defines a seminorm on V_{h}, called the *order seminorm*:

$$p_V(v) := \inf\{r > 0 \mid -re_V \leqslant v \leqslant re_V\}$$

There are many extensions of the order seminorm to V, but all of them induce the same topology on V [108], called the *order topology*.

An order-unit space (V, V^+, e_V) as well as its order-unit e_V are called *archimedean* if, for all $v \in V_{\mathrm{h}}$, we have that $re_V + v \geqslant 0$, for all $r > 0$, implies that $v \in V^+$. If V is an archimedean order-unit space, the order seminorm p_V is a norm and the positive cone V^+ is closed in V for the order topology [108].

A linear map $\phi : V \to W$ between ordered vector spaces V, W is called *self-adjoint* if $\phi(v^*) = \phi(v)^*$, for all $v \in V$. If the map ϕ furthermore satisfies $\phi(V^+) \subseteq W^+$, it is called *positive*. We denote the cone of all positive maps $V \to W$ by $\mathcal{P}(V, W)$. If V, W are order-unit spaces, ϕ is called *unital* if $\phi(e_V) = e_W$. In this case $\phi(V^+) \subseteq W^+$ implies that ϕ is self-adjoint.

For ordered vector spaces V, W, a positive map $\phi : V \to W$ is called an *order isomorphism* if it is bijective with $\phi^{-1} : W \to V$ positive. An injective positive map ϕ is an *order embedding* if ϕ is an order isomorphism onto its image.

For an ordered vector space V, denote by $(V^*)^+ := \mathcal{P}(V, \mathbb{C})$ the set of positive linear functionals $V \to \mathbb{C}$ and by V^* its complex linear span. Then $(V^*, (V^*)^+)$ is an ordered vector space. If V is an order-unit space, unital positive linear functionals on V are called *states*. Define the *state space* as the set $\mathcal{S}(V) := \{\phi \in (V^*)^+ \mid \phi(e_V) = 1\}$ of all states. The state space is equipped with the weak$*$-topology (induced by the order-topology on V), for which it is a compact convex set by the Banach–Alaoglu theorem.

2.2.2 Matrix-order unit spaces and operator systems

A *matrix-ordering* on a complex $*$-vector space V is a family of positive cones $\mathcal{C}_n \subseteq \mathrm{M}_n(V)_{\mathrm{h}}$ such that $(\mathrm{M}_n(V), \mathcal{C}_n)$ is an ordered vector space, for every $n \in \mathbb{N}$, and which is *compatible*, i.e. $A\mathcal{C}_m A^* \subseteq \mathcal{C}_n$, for all $A \in \mathrm{M}_{n,m}$, $m, n \in \mathbb{N}$. Given a matrix-ordered vector space $(V, (\mathrm{M}_n(V)^+)_{n\in\mathbb{N}})$, an element $e \in V$ is called a *matrix order-unit* if the element

$$e_n := \begin{pmatrix} e & & \\ & \ddots & \\ & & e \end{pmatrix} \in \mathrm{M}_n(V)$$

is an order-unit for the ordered vector space $(\mathrm{M}_n(V), \mathrm{M}_n(V)^+)$, for all $n \in \mathbb{N}$. A *matrix order-unit space* is a matrix-ordered vector space together with a matrix order-unit. A matrix order-unit e is called *archimedean* if e_n is archimedean, for all $n \in \mathbb{N}$. An *operator system* is a matrix-ordered vector space together with an archimedean matrix order-unit.

For a linear map $\phi : X \to Y$ between matrix-ordered vector spaces X, Y, we define its matrix amplification $\phi^{(n)} : \mathrm{M}_n(X) \to \mathrm{M}_n(Y)$ by $\phi^{(n)}((x_{ij})_{i,j}) := (\phi(x_{ij})_{i,j})$, for all $(x_{ij})_{i,j} \in \mathrm{M}_n(X)$. The map ϕ is called *completely positive* if $\phi^{(n)} : \mathrm{M}_n(X) \to \mathrm{M}_n(Y)$ is a positive map, for all $n \in \mathbb{N}$. We denote the cone of all completely positive maps $X \to Y$ by $\mathcal{CP}(X, Y)$. We sometimes use the abbreviations "cp" and "ucp" for "completely positive" and "unital and completely positive" respectively. We denote the categories of matrix-ordered vector spaces with cp maps by $\mathbf{MVS}$, of matrix order-unit spaces with ucp maps by $\mathbf{MOU}$ and of operator systems with ucp maps by $\mathbf{OSy}$.

If A is a unital C*-algebra and $X \subseteq A$ an operator subsystem, every positive linear functional $\phi : X \to \mathbb{C}$ can be extended to a positive linear functional on A by *Krein's theorem*. More generally, every cp map $\Phi : X \to \mathcal{B}(H)$ can be extended to a cp map $A \to \mathcal{B}(H)$ by *Arveson's extension theorem* [4].

For matrix-ordered vector spaces X, Y, a *complete order isomorphism* is a completely positive map $\phi \in \mathcal{CP}(X, Y)$ such that $\phi^{(n)} \in \mathcal{P}(\mathrm{M}_n(X), \mathrm{M}_n(Y))$ is an order isomorphism, for every $n \in \mathbb{N}$. We say that X and Y are completely order isomorphic if there is a complete order isomorphism $X \to Y$, and write $X \cong Y$. A completely positive map ϕ is a *complete order embedding* if ϕ is a complete order isomorphism onto its image.

Let H be a Hilbert space. Then every unital self-adjoint subspace $X \subseteq \mathcal{B}(H)$ is an operator system with positive matrix-cones $\mathrm{M}_n(X)^+ := \mathrm{M}_n(X) \cap \mathcal{B}(H^n)^+$, for all $n \in \mathbb{N}$, and archimedean matrix-order unit $e_X := \mathbf{1}_{\mathcal{B}(H)}$. Sometimes operator systems defined as archimedean matrix-order unit spaces are called *abstract* operator systems (*cf.* [106]), whereas self-adjoint unital spaces of operators acting on a Hilbert space are called *concrete* operator systems. [27, Theorem 4.4] states that these definitions coincide:

Theorem 2.2.1 (Choi–Effros theorem)**.** *Let X be an operator system. Then there is a Hilbert space H and a complete order embedding $\phi : X \to \mathcal{B}(H)$ such that $\phi(e_X) = \mathbf{1}_{\mathcal{B}(H)}$.*

When dealing with a concrete operator system $X \subseteq \mathcal{B}(H)$ we may write X_{sa} instead of X_{h} to denote the real subspace of self-adjoint operators in X.

In Chapter 4 we will make use of idempotent ucp maps onto an operator subsystem: given operator systems $Y \subseteq X$ we say that a map $E : X \to Y$ is a *ucp conditional expectation* if $E(y) = y$, for all $y \in Y$.

For convenience, we record the following well-known consequence of the *Kadison function representation.*

Lemma 2.2.2. *For every element $x \in X$ of an operator system X, the following holds:*

$$\sup_{\phi \in \mathcal{S}(X)} |\phi(x)| \leqslant \|x\| \leqslant 2 \sup_{\phi \in \mathcal{S}(X)} |\phi(x)|$$

Proof. The first inequality is immediate since states are positive functionals of norm 1. Indeed, we have $x = \mathrm{Re}(x) + i\mathrm{Im}(x)$, where $\mathrm{Re}(x) = \frac{x+x^*}{2}$ and $\mathrm{Im}(x) = \frac{i(x^*-x)}{2}$ are self-adjoint, so that $\|\mathrm{Re}(x)\| = \sup_{\phi \in \mathcal{S}(X)} |\phi(\mathrm{Re}(x))|$ and similarly for $\mathrm{Im}(x)$ [78, Theorem 4.3.9]. The claim then follows by triangle inequality. □

Let $X \subseteq \mathcal{B}(H)$ be an operator system and, for a directed set $\mathcal{L}$, let $(P_\Lambda)_{\Lambda \in \mathcal{L}}$ be a net of orthogonal projections in $\mathcal{B}(H)$. For every $\Lambda \in \mathcal{L}$, set $H_\Lambda := P_\Lambda H$. Assume that the net $(P_\Lambda)_{\Lambda \in \mathcal{L}}$ is a *join semilattice* for the relation of containment of ranges, i.e. for all $\Lambda_1, \Lambda_2 \in \mathcal{L}$, the orthogonal projection $P_{\Lambda_1 \vee \Lambda_2}$ onto the closed subspace $H_{\Lambda_1} + H_{\Lambda_2}$ is in the net. Assume furthermore that the net $(P_\Lambda)_{\Lambda \in \mathcal{L}}$ converges strongly to the identity $\mathbf{I}^H \in \mathbf{B}(H)$. Let $\tau_\Lambda : \mathcal{B}(H) \to \mathcal{B}(H_\Lambda)$ be the compression map, i.e. $\tau_\Lambda(T) := P_\Lambda T P_\Lambda$, for all bounded operators $T \in \mathcal{B}(H)$, and denote by $X_\Lambda := \tau_\Lambda(X)$ the operator subsystem of $\mathcal{B}(H_\Lambda)$ given by the image of the operator system X under τ_Λ. We set $\mathcal{S}_\mathcal{L} := \bigcup_{\Lambda \in \mathcal{L}} \tau_\Lambda^* \mathcal{S}(X_\Lambda) \subseteq \mathcal{S}(X)$, where $\tau_\Lambda^* : \mathcal{S}(X_\Lambda) \to \mathcal{S}(X)$ is the pullback of the map τ_Λ.

Lemma 2.2.3 ([56, Proposition 16])**.** *The set $\mathcal{S}_\mathcal{L}$ is dense in the state space $\mathcal{S}(X)$ for the weak*-topology.*

Proof. Observe that the set $\mathcal{S}_\mathcal{L}$ is convex. Indeed, every subset $\tau_\Lambda^* \mathcal{S}(X_\Lambda) \subseteq \mathcal{S}_\mathcal{L}$ is convex, since the pullback map $\tau_\Lambda^* : \mathcal{S}(X_\Lambda) \to \mathcal{S}(X)$ is affine. Now, observe that for closed subspaces $H_{\Lambda_1} \subseteq H_{\Lambda_2}$ we have that $P_{\Lambda_1} P_{\Lambda_2} = P_{\Lambda_2} P_{\Lambda_1} = P_{\Lambda_1}$, so that we can consider the restriction $\tau_{\Lambda_1}\big|_{X_{\Lambda_2}}(T) := P_{\Lambda_1} a P_{\Lambda_1}$, for all elements $T \in X_{\Lambda_2}$ and $a \in X$ with $\tau_{\Lambda_2}(a) = T$. In particular, since τ_Λ is onto, for all closed subspaces H_Λ, $\Lambda \in \mathcal{L}$, we have that $\tau_\Lambda^* : \mathcal{S}(X_\Lambda) \to \mathcal{S}(X)$ and $(\tau_{\Lambda_1}\big|_{X_{\Lambda_2}})^* : \mathcal{S}(X_{\Lambda_1}) \to \mathcal{S}(X_{\Lambda_2})$ are injections. Therefore, if $\phi \in \tau_{\Lambda_1}^* \mathcal{S}(X_{\Lambda_1}), \psi \in \tau_{\Lambda_2}^* \mathcal{S}(X_{\Lambda_2})$ are states, any convex combination $t\phi + (1-t)\psi$ (for $0 \leqslant t \leqslant 1$) is a state

$$t(\tau_{\Lambda_1}\big|_{X_{\Lambda_1 \vee \Lambda_2}})^* \phi + (1-t)(\tau_{\Lambda_2}\big|_{X_{\Lambda_1 \vee \Lambda_2}})^* \psi$$

in $\tau_{\Lambda_1 \vee \Lambda_2}^* \mathcal{S}(X_{\Lambda_1 \vee \Lambda_2})$, which establishes convexity of $\mathcal{S}_\mathcal{L}$.

Now, since the subspace $\sum_{H_\Lambda \in \mathcal{L}} H_\Lambda$ is dense in H by strong convergence $P_\Lambda \to \mathbf{I}^H$, the set $\mathcal{S}_\mathcal{L}$ contains a dense subset $\mathcal{S}_{\mathcal{L},\mathrm{vec}}$ of the vector states on X, so that an element $x \in X$ is positive if the complex number $\rho(x)$ is positive, for all vector states $\rho \in \mathcal{S}_{\mathcal{L},\mathrm{vec}}$, and thus for all states $\rho \in \mathcal{S}_\mathcal{L}$. Therefore, by [78, Theorem 4.3.9], the set $\mathrm{co}(\mathcal{S}_\mathcal{L}) = \mathcal{S}_\mathcal{L}$ is weak*-dense in $\mathcal{S}(X)$ as claimed. □

For a matrix-ordered vector space X with ordered vector space dual X^*, set $\mathrm{M}_n(X^*)^+ := \mathcal{CP}(X, \mathrm{M}_n)$, for all $n \in \mathbb{N}$. If X is a matrix order-unit space define the *matrix state spaces* as $\mathcal{S}_n(X) := \{\phi \in \mathrm{M}_n(X^*)^+ \mid \phi(e_X) = \mathbf{1}_n\}$. If X is an operator system, we have the identification

$$\mathrm{M}_n(X^*)^+ = \mathcal{CP}(X, \mathrm{M}_n) \cong \mathcal{P}(\mathrm{M}_n(X), \mathbb{C}) = (\mathrm{M}_n(X)^*)^+$$

by [106, Theorem 6.1]. If X is a finite-dimensional operator system, by [27, Corollary 4.5], one can (non-canonically) choose an archimedean matrix order-unit for the matrix-ordered vector space X^*, giving it the structure of an operator system which we then denote by X^{d}. For (matrix-)ordered vector spaces V, W, every (completely) positive map $\phi : V \to W$ induces a (completely) positive map $\phi^* : W^* \to V^*$.

Archimedeanization

Matrix order-unit spaces can be turned into operator systems in the following way [107]. Let X be a matrix order-unit space. Set $N := \bigcap_{\phi \in \mathcal{S}(X)} \ker(\phi)$. One checks that identifying $\mathrm{M}_n(X/N) \cong \mathrm{M}_n(X)/\mathrm{M}_n(N)$ induces a matrix order-unit space structure on the quotient X/N. Define

$$\begin{aligned}\mathcal{C}_n^{\mathrm{arch}} := \{&(x_{ij})_{i,j} + \mathrm{M}_n(N) \in \mathrm{M}_n(X)/\mathrm{M}_n(N) \mid \\ &r(e_X)_n + (x_{ij})_{i,j} + \mathrm{M}_n(N) \in \mathrm{M}_n(X)^+ + \mathrm{M}_n(N), \text{ for all } r > 0\},\end{aligned}$$

for all $n \in \mathbb{N}$. We sometimes write $\mathrm{Arch}(\mathrm{M}_n(X)^+) := \mathcal{C}_n^{\mathrm{arch}}$ if explicit reference to X is required. Then the triple $(X/N, (\mathcal{C}_n^{\mathrm{arch}})_{n \in \mathbb{N}}, e_X + N)$ is an operator system, called the *archimedeanization* of the matrix order-unit space X, and we denote it by $\mathrm{Arch}(X)$. Ucp maps from the matrix order-unit space X to any operator system Y uniquely factor through the archimedeanization. More precisely, the following universal property holds: There is a unique surjective ucp map $q : X \to \mathrm{Arch}(X)$ such that, for every ucp map $\phi : X \to Y$ with Y an operator system, there is a unique ucp map $\tilde{\phi} : \mathrm{Arch}(X) \to Y$ such that $\tilde{\phi} \circ q = \phi$. If the subspace $N \subseteq X$

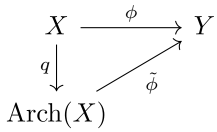

Figure 2.1: The universal property of the archimedeanization.

is $\{0\}$, the cone $\mathcal{C}_n^{\mathrm{arch}}$ is the closure of the cone $\mathrm{M}_n(X)^+$ in the order topology. The universal property of the archimedeanization implies functoriality:

Lemma 2.2.4. *The archimedeanization is a functor from* $\mathbf{MOU}$ *to* $\mathbf{OSy}$*, denoted* Arch.

Proof. Let $\phi : X \to Y$ be a ucp map in $\mathbf{MOU}$. Then there are canonical ucp maps $q_X : X \to \mathrm{Arch}(X)$, $q_Y : Y \to \mathrm{Arch}(Y)$. Since $q_Y \circ \phi$ is a ucp map from X to the operator system $\mathrm{Arch}(Y)$, by the universal property of the archimedeanization there is a unique map $\tilde{\phi} : \mathrm{Arch}(X) \to \mathrm{Arch}(Y)$ such that $\tilde{\phi} \circ q_X = q_Y \circ \phi$. □

Quotients

From [83, Section 3] we extract how to define quotients in the $\mathbf{MVS}$ category. To this end, let X be a matrix-ordered vector space and let $J \subseteq X$ be a *-invariant subspace. Assume moreover that J is a complete order-ideal, i.e. if $x \in \mathrm{M}_n(J)$ and $0 \leqslant y \leqslant x$ then $y \in \mathrm{M}_n(J)$. Set

$$\mathrm{M}_n(X/J)^+ := \{(x_{ij} + J)_{i,j} \mid \text{there exists } y \in \mathrm{M}_n(J) \text{ such that } x + y \in \mathrm{M}_n(X)^+\}.$$

One readily checks that $\mathrm{M}_n(X/J)^+$ is a convex spanning cone for the *-vector space $\mathrm{M}_n(X/J)$ and that this family of cones is compatible. To see note properness, note that as in the proof of [108, Proposition 2.41], any element of the intersection $\mathrm{M}_n(X/J)^+ \cap (-\mathrm{M}_n(X/J)^+)$ can be written as $x + \mathrm{M}_n(J) = -y + \mathrm{M}_n(J)$ with $x, y \in \mathrm{M}_n(X)^+$. Then there is an element $j \in \mathrm{M}_n(J)$ such that $j = x + y \in \mathrm{M}_n(X)^+$. Hence $0 \leqslant x \leqslant j$ which implies that $x \in \mathrm{M}_n(J)$, since J is a complete order-ideal by assumption. So $x + \mathrm{M}_n(J) = 0 + \mathrm{M}_n(J)$, i.e. the intersection $\mathrm{M}_n(X/J)^+ \cap (-\mathrm{M}_n(X/J)^+)$ is trivial, whence the cone $\mathrm{M}_n(X/J)^+$ is proper. This shows that $(X/J, (\mathrm{M}_n(X/J)^+)_{n\in\mathbb{N}})$ is a matrix-ordered vector space again, and we say it is a *quotient* of X. Note that the canonical quotient map $q : X \to X/J$ is cp.

If X and Y are matrix-ordered vector spaces and $\phi : X \to Y$ is a cp map, then $\ker(\phi)$ is clearly *-invariant and moreover it is a complete order-ideal, since $x \in \ker(\phi^{(n)})$ and $0 \leqslant y \leqslant x$ implies $0 \leqslant \phi^{(n)}(y) \leqslant \phi^{(n)}(x) = 0$. It follows that complete order-ideals in matrix-ordered vector spaces are precisely the kernels of cp maps to matrix-ordered vector spaces. We say that the cp map $\phi : X \to Y$ is an $\mathbf{MVS}$-*quotient map* if it is onto and the induced map $\tilde{\phi} : X/\ker(\phi) \to Y$ is a complete order isomorphism (i.e. inverse map $\tilde{\phi}^{-1} : Y \to X/\ker(\phi)$ is cp).

We now discuss quotients in the $\mathbf{OSy}$ category [83]. A quotient of an operator system by a complete order-ideal (even if it does not contain the matrix order-unit) is not necessarily an operator system again, but one rather needs to consider quotients by kernels of ucp maps to operator systems. To this end, let X be an operator system. A subspace $J \subseteq X$ is a *kernel* if there is an operator system Y and a ucp map $\phi : X \to Y$ such that $J = \ker(\phi)$. Equivalently, J is a kernel if and only if there is a family of states $\{\phi_\iota\}_{\iota\in I} \subseteq \mathcal{S}(X)$ such that $J = \bigcap_{\iota\in I} \ker(\phi_\iota)$. Setting

$$\mathrm{M}_n(X/J)^+ := \{(x_{ij} + J)_{i,j} \in \mathrm{M}_n(X/J) \mid (x_{ij})_{i,j} \in \mathrm{M}_n(X)^+\}$$

and $e_{X/J} := e_X + J$, the quotient X/J inherits the structure of a matrix order-unit space from X. Then $\mathrm{Arch}(X/J)$ is called an *operator system quotient* and it has the universal property of a quotient in the category $\mathbf{OSy}$ as one expects: i.e. every ucp map from X to an operator system factors uniquely through the quotient $\mathrm{Arch}(X/J)$. A kernel $J \subseteq X$ is called *completely proximinal* if $\mathrm{M}_n(X/J)^+ = \mathrm{Arch}(\mathrm{M}_n(X/J)^+)$, for all $n \in \mathbb{N}$, in which case $X/J = \mathrm{Arch}(X/J)$.

Tensor products

We recall some elements of the theory of tensor products for operator systems from [82]. Let X and Y be operator systems and denote by $X \otimes Y$ the algebraic tensor product. An *operator system structure* on $X \otimes Y$ is a family of matrix cones $\tau := (\mathcal{C}_n)_{n\in\mathbb{N}}$ over $X \otimes Y$ which satisfies the following properties:

(T1) The triple $(X \otimes Y, (\mathcal{C}_n)_{n\in\mathbb{N}}, e_X \otimes e_Y)$ is an operator system, denoted $X \otimes_\tau Y$;

(T2) We have $\mathrm{M}_k(X)^+ \otimes \mathrm{M}_l(Y)^+ \subseteq \mathcal{C}_{kl}$, for all $k, l \in \mathbb{N}$;

(T3) For all ucp maps $\phi : X \to \mathrm{M}_k$, $\psi : Y \to \mathrm{M}_l$, the map $\phi \otimes \psi : X \otimes_\tau Y \to \mathrm{M}_{kl}$ is ucp.

An operator system structure τ is *functorial* if it satisfies

(T4) For all operator systems S, T and ucp maps $\phi : X \to S$, $\psi : Y \to T$, the map $\phi \otimes \psi$ is a ucp map $X \otimes_\tau Y \to S \otimes_\tau T$.

There is a compatible family of positive cones over $X \otimes Y$, given by

$$\mathcal{D}_n^{\max}(X, Y) := \{A(x \otimes y)A^* \mid x \in \mathrm{M}_k(X)^+, y \in \mathrm{M}_l(Y)^+, A \in \mathrm{M}_{n,kl}\},$$

for all $n \in \mathbb{N}$. Its archimedeanization defines a functorial operator system structure on $X \otimes Y$, called the *maximal tensor product* and denoted by $\max$. It is maximal in the sense that, for every compatible family $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of positive cones over $X \otimes Y$, we have $\mathcal{D}_n^{\max}(X, Y) \subseteq \mathcal{C}_n$, for all $n \in \mathbb{N}$. In particular, if τ is any operator system structure on $X \otimes Y$ and $\phi : X \otimes_\tau Y \to Z$ is a ucp map with Z an operator system, then the induced map (by archimedeanization) $\tilde{\phi} : X \otimes_{\max} Y \to Z$ is ucp.

The *minimal tensor product* of two operator systems X and Y, which are by the Choi–Effros theorem unitally completely order embedded in C*-algebras A and B respectively, is the operator system structure inherited from the inclusion of $X \otimes Y$ in the spatial tensor product $A \otimes_{\min} B$. This tensor product is independent of the choice of inclusions $X \subseteq A$ and $Y \subseteq B$. Moreover it is a functorial tensor product and it is minimal in the sense that, for every compatible family $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of positive cones over $X \otimes Y$, we have that $\mathcal{C}_n \subseteq \mathrm{M}_n(X \otimes_{\min} Y)^+$, for all $n \in \mathbb{N}$. If X, Y are finite-dimensional operator systems, we have that $(X \otimes_{\max} Y)^{\mathrm{d}} \cong X^{\mathrm{d}} \otimes_{\min} Y^{\mathrm{d}}$ and $(X \otimes_{\min} Y)^{\mathrm{d}} \cong X^{\mathrm{d}} \otimes_{\max} Y^{\mathrm{d}}$.

For two operator systems X, Y and two functorial operator system structures σ, τ on $X \otimes Y$, we say that the pair (X, Y) is (σ, τ)*-nuclear* if the operator systems $X \otimes_\sigma Y$ and $X \otimes_\tau Y$ are canonically unitally completely order isomorphic. We say that an operator system X is (σ, τ)-nuclear if the pair (X, Y) is (σ, τ)-nuclear, for every operator system Y.

C*-covers

A C**-cover* of an operator system X is a unital C*-algebra A together with a unital complete order embedding $\iota : X \to A$ such that A is generated by the image $\iota(X)$. Every operator system X admits a maximal and a minimal C*-cover, denoted respectively by $(\mathrm{C}^*_{\max}(X), \iota_{\max})$ and $(\mathrm{C}^*_{\min}(X), \iota_{\min})$; they are characterized by the fact that, for any C*-cover (A, ι) there are unique *-homomorphisms $\mathrm{C}^*_{\max}(X) \to A \to \mathrm{C}^*_{\min}(X)$ such that the diagram in Figure 2.2 commutes. The maximal C*-cover is sometimes called the *universal* C**-cover*, denoted by $\mathrm{C}^*_{\mathrm{u}}(X)$, and its existence was shown in [87, Proposition 8]. The minimal C*-cover is often called the C**-envelope* and denoted $\mathrm{C}^*_{\mathrm{e}}(X)$; its existence was first shown by Hamana [63], see also [106]. Its construction in terms of boundary representations

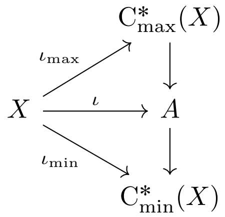

Figure 2.2: The universal properties of the maximal and minimal C*-cover.

was a 45-years long program initiated by Arveson [4], with important contributions by Muhly–Solel [102], Dritschel–McCullough [46], Arveson [5], and its final resolution by Davidson–Kennedy [39].

Propagation number

The *propagation number* $\mathrm{prop}(X)$ of an operator system X is the smallest positive integer $n \in \mathbb{N}$ (if it exists) such that the subspace $X^{\circ n} := \mathrm{span}\{x_1 \dots x_n \mid x_i \in X\} \subseteq C^*_{\min}(X)$ spanned by products of at most n elements of X inside the minimal C*-cover is equal to the minimal C*-cover; if no such $n \in \mathbb{N}$ exists, the propagation number is set to be ∞. The propagation number was introduced by Connes–van Suijlekom [33] and it is an invariant for stable equivalence, i.e. $\mathrm{prop}(X \otimes_{\min} \mathcal{K}) = \mathrm{prop}(X)$, where $\mathcal{K}$ is the C*-algebra of compact operators on a separable Hilbert space, (essentially since $C^*_{\min}(X \otimes_{\min} \mathcal{K}) \cong C^*_{\min}(X) \otimes \mathcal{K}$). Moreover, since stable equivalence is a characterization of (various notions of) Morita equivalence for operator systems [49] the propagation number is a Morita invariant.

Amalgamated direct sums

Let X and Y be operator systems and let V be an operator system together with unital complete order embeddings $X \xhookleftarrow{\iota_X} V \xhookrightarrow{\iota_Y} Y$. The *amalgamated direct sum (of X and Y over V)* [86] is the unique operator system $X \oplus_V Y$ which comes with ucp maps

$$X \xrightarrow{\phi_X} X \oplus_V Y \xleftarrow{\phi_Y} Y$$

and which satisfies the following universal property: for every operator system Z and ucp maps $X \xrightarrow{\psi_X} Z \xleftarrow{\psi_Y} Y$ with $\psi_X \circ \iota_X = \psi_Y \circ \iota_Y$, there is a unique ucp map $\psi : X \oplus_V Y \to Z$ such that $\psi \circ \phi_X = \psi_X$ and $\psi \circ \phi_Y = \psi_Y$. In other words, the amalgamated direct sum is the pushout of the diagram

$$X \xhookleftarrow{\iota_X} V \xhookrightarrow{\iota_Y} Y$$

in the category **OSy**, see Figure 2.3. The existence of the amalgamated direct sum follows from the existence of the amalgamated free product of C*-algebras and by realizing $X \oplus_V Y$ as an operator subsystem of $C^*_{\max}(X) *_{C^*_{\max}(V)} C^*_{\max}(Y)$. From the universal property of the amalgamated direct sum one obtains the fact that the ucp maps ϕ_X, ϕ_Y are complete order embeddings, and using the universal property of

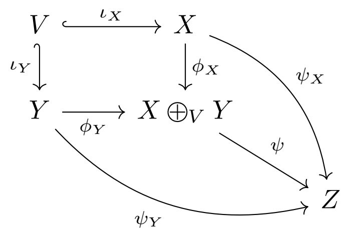

Figure 2.3: The universal property of the amalgamated direct sum.

the amalgamated free product of C*-algebras, one readily checks by diagram chasing that $\mathrm{C}^*_{\max}(X \oplus_V Y) \cong \mathrm{C}^*_{\max}(X) *_{\mathrm{C}^*_{\max}(V)} \mathrm{C}^*_{\max}(Y)$, *cf.* [86, Proposition 2.6], see also [99, Theorem 4.24].

More concretely, if V is finite-dimensional, we have $X \oplus_V Y = \frac{X \oplus Y}{J}$, where $J = \{(\iota_X(v), -\iota_Y(v)) \mid v \in V\}$. In fact, in this case J is the kernel of a ucp map onto an operator system and it is furthermore completely proximinal [81, Proposition 2.4], so that $\mathrm{Arch}(\frac{X \oplus Y}{J}) = \frac{X \oplus Y}{J}$. One readily checks the universal property of the amalgamated direct sum for $\frac{X \oplus Y}{J}$. If we do not specify the operator subsystem V, we usually assume $V \cong \mathbb{C}e_X \cong \mathbb{C}e_Y$ and just speak of the *amalgamated direct sum* or *coproduct* of X and Y, which we then denote by $X \oplus_{\mathbf{1}} Y$, see also [81, 55, 26]. An operator system X is *hyperrigid* inside its minimal C*-cover if for every non-degenerate representation π of $\mathrm{C}^*_{\min}(X)$ on a Hilbert space H, the only ucp extension of the map $\pi|_X : X \to \mathcal{B}(H)$ is π itself [6]. If two operator systems X and Y are hyperrigid in their respective minimal C*-covers, we have $\mathrm{C}^*_{\min}(X \oplus_{\mathbf{1}} Y) \cong \mathrm{C}^*_{\min}(X) *_{\mathbf{1}} \mathrm{C}^*_{\min}(Y)$ [26, Theorem 4.11]. In particular, this is the case if X and Y contain *enough unitaries* in their respective minimal C*-covers in the sense of [81]; in this case $\mathrm{C}^*_{\min}(X \oplus_{\mathbf{1}} Y) \cong \mathrm{C}^*_{\min}(X) *_{\mathbf{1}} \mathrm{C}^*_{\min}(Y)$ is already shown in [51].

2.2.3 Operator spaces

We now recall some basic operator space notions. See the standard references [106, 111, 19, 48] for details. An *operator space* is a complex vector space X together with a family of matrix-norms $\|\cdot\|_n$ on $\mathrm{M}_n(X)$, $n \in \mathbb{N}$, which is compatible in the sense that $\|AxB\|_n \leqslant \|A\|\|x\|_n\|B\|$, for all elements $x \in \mathrm{M}_n(X)$ and scalar matrices $A, B \in \mathrm{M}_n$, and such that $\|x \oplus y\|_{m+n} = \max\{\|x\|_n, \|y\|_m\}$, for all $x \in \mathrm{M}_n(X), y \in \mathrm{M}_m(X)$. A linear map $\phi : X \to Y$ between operator spaces is called completely bounded (cb) if there exists a positive real number $R > 0$ such that $\|\phi^{(n)}(x)\|_n \leqslant R$, for all $n \in \mathbb{N}$, and $x \in \mathrm{M}_n(X)$ with $\|x\|_n \leqslant 1$; in this case the number $\|\phi\|_{\mathrm{cb}} := \sup_{n \in \mathbb{N}} \sup_{\|x\|_n \leqslant 1} \|\phi^{(n)}(x)\|_n$ is called the cb-norm of ϕ. We denote the set of all cb maps $\phi : X \to Y$ by $\mathcal{CB}(X, Y)$. A cb map $\phi \in \mathcal{CB}(X, Y)$ is called completely contractive (cc) if $\|\phi\|_{\mathrm{cb}} \leqslant 1$ and completely isometric if $\|\phi^{(n)}(x)\|_n = \|x\|_n$, for all $n \in \mathbb{N}$, $x \in \mathrm{M}_n(X)$. We denote the category of operator spaces with cc maps by **OSp**.

Let $X \subseteq \mathcal{B}(H)$ be a closed subspace. Then X is an operator space by isometrically identifying the space of matrices $\mathrm{M}_n(X)$ as a subspace of $\mathcal{B}(H^n)$. Every cb map $\phi : X \to \mathcal{B}(K)$ extends to a cb map $\tilde{\phi} : \mathcal{B}(H) \to \mathcal{B}(K)$ with $\|\tilde{\phi}\|_{\mathrm{cb}} = \|\phi\|_{\mathrm{cb}}$ by *Wittstock's extension theorem* [144]. Sometimes operator spaces defined as matrix-normed vector spaces as above are called *abstract* operator spaces (*cf.* [106]), whereas spaces of operators acting on a Hilbert space are called *concrete* operator spaces. [126, Theorem 3.1] states that these definitions coincide:

Theorem 2.2.5 (Ruan's theorem)**.** *Let X be an operator space. Then there is a Hilbert space H and a complete isometry $\phi : X \to \mathcal{B}(H)$.*

Note that this implies that every operator system is (unitally completely order isomorphic) to an operator space, and in fact, for an operator system X one can recover the matrix-norms from the matrix-order structure as $\|x\|_n = \inf\{r > 0 \mid \left(\begin{smallmatrix} r(e_X)_n & x \\ x^* & r(e_X)_n \end{smallmatrix}\right) \geqslant 0\}$. For a cp map ϕ we have $\phi(\mathbf{1}) = \|\phi\|_{\mathrm{cb}}$. A unital linear map is cp if and only if it is cc, and it is a complete order embedding if and only if it is completely isometric. By Wittstock's decomposition theorem [143], see also [106, Theorem 8.5], every cb map on an operator system is a linear combination of four cp maps. Given an operator space X, its dual X^* is an operator space when equipped with the matrix norms inherited by declaring the identification $\mathrm{M}_n(X^*) \cong \mathcal{CB}(X, \mathrm{M}_n)$ completely isometric [19, 1.2.20]. Moreover, the matrix-ordered vector space and operator space structures on X^* are compatible in the sense that the proper convex cones $\mathcal{CP}(X, \mathrm{M}_n)$ span $\mathcal{CB}(X, \mathrm{M}_n)$ and are closed (for the cb-norm); this shows that X^* is a *(positively generated) matrix-ordered operator space.* See [72] for more on these.

Recall from [19, 1.2.14] that if X is an operator space and $Y \subseteq X$ a closed subspace, the quotient X/Y is an operator space, for the matrix-norms given by

$$\|(x_{ij} + Y)_{i,j}\|_n := \inf\{\|x + y\|_n \mid y \in \mathrm{M}_n(Y)\}.$$

For operator spaces X and Z, a cc map $\phi : X \to Z$ is called an $\mathbf{OSp}$-quotient map if it is onto and the induced map $\tilde{\phi} : X/\ker(\phi) \to Z$ is a complete isometry (i.e. the inverse map $\tilde{\phi}^{-1} : Z \to X/\ker(\phi)$ is cc).

A map $\phi : X \to Y$ between operator spaces is cb if and only if $\phi \otimes \mathbf{I}^Z : X \otimes Z \to Y \otimes Z$ is cb, for every operator space Z [42], see also [111, Proposition 2.1.1]. It follows that a map $\phi : X \to Y$ between operator systems is cp if and only if $\phi \otimes \mathbf{I}^Z : X \otimes Z \to Y \otimes Z$ is cp, for every operator system Z.

Below, for Hilbert spaces H, K and subspaces $X \subseteq \mathcal{B}(H), Y \subseteq \mathcal{B}(K)$, we denote by $X \otimes_{\min} Y$ the *spatial* tensor product, i.e. the completion of the algebraic tensor product of X, and Y in $\mathcal{B}(H \otimes K)$, where $H \otimes K$ is the Hilbert space tensor product. We will refer to the following result as the *Fubini theorem* for cb maps. In the special case that the maps ϕ_1, ϕ_2 below are linear functionals, we refer to it as the Fubini theorem for slice maps which was proven in [134].

Lemma 2.2.6. *Let $X_1 \subseteq \mathcal{B}(H_1)$, $X_2 \subseteq \mathcal{B}(H_2)$ be operator spaces and let $\phi_1 : X_1 \to \mathcal{B}(K_1)$, $\phi_2 : X_2 \to \mathcal{B}(K_2)$ be cb maps. Then the map $\phi_1 \otimes \phi_2 : X_1 \otimes X_2 \to \mathcal{B}(K_1 \otimes K_2)$ extends uniquely to a cb map $\phi_1 \otimes \phi_2 : X_1 \otimes_{\min} X_2 \to \mathcal{B}(K_1 \otimes K_2)$ on the spatial tensor product such that $\|\phi_1 \otimes \phi_2\|_{\mathrm{cb}} \leqslant \|\phi_1\|_{\mathrm{cb}}\|\phi_2\|_{\mathrm{cb}}$. In particular, the* *following commutativity property holds:*

$$(\phi_1 \otimes \mathbf{I}^{\mathcal{B}(H_2)})(\mathbf{I}^{X_1} \otimes \phi_2) = (\mathbf{I}^{\mathcal{B}(H_1)} \otimes \phi_2)(\phi_1 \otimes \mathbf{I}^{X_2}) = \phi_1 \otimes \phi_2 \qquad (2.6)$$

Proof. For the unique extension of the map $\phi_1 \otimes \phi_2$ to the spatial tensor product and the norm estimate see [106, Theorem 12.3]. The equation (2.6) holds for the algebraic tensor product and thus extends uniquely to the spatial tensor product. □

2.2.4 Complete order embeddings and quotient maps

The following proposition is central to our investigation of the extension problem for positive semi-definite functions, see Theorem 5.3.18 and Theorem 5.4.16 below. We give a full proof of the equivalence of (1) and (3) for which we could not find a reference in the infinite dimensional case. Note that this is in line with [52, Proposition 1.8] which implies the equivalence of (1) and (3) in the case that the operator systems X and Y are finite-dimensional.

Proposition 2.2.7. *Let X and Y be operator systems and let $\phi : X \to Y$ be a ucp map. We view the dual spaces X^* and Y^* both as operator spaces and as matrix-ordered vector spaces, for the matrix-norms inherited from declaring the identification $\mathrm{M}_n(X^*) \cong \mathcal{CB}(X, \mathrm{M}_n)$ completely isometric and for the positive matrix-cones given by $\mathrm{M}_n(X^*)^+ := \mathcal{CP}(X, \mathrm{M}_n)$, and analogously for Y. Then the following statements are equivalent:*

(1) The map $\phi : X \to Y$ is a complete order embedding.

(2) The map $\phi : X \to Y$ is a complete isometry.

(3) The map $\phi^* : Y^* \to X^*$ is an $\mathbf{MVS}$-quotient map.

(4) The map $\phi^* : Y^* \to X^*$ is an $\mathbf{OSp}$-quotient map.

Proof. The equivalence (1)⇔(2) was already mentioned above and is well-known, see e.g. [19, 1.3.3].

For the equivalence (2)⇔(4) see [19, 1.4.3]; note that the implication (2)⇒(4) requires (the matrix-valued version of) Wittstock's extension theorem to extend cb maps $X \to \mathrm{M}_n$ to cb maps $Y \to \mathrm{M}_n$ (if X is completely isometrically identified with its image $\phi(X) \subseteq Y$). The converse implication uses the completely isometric inclusion of an operator space in its bidual.

For the equivalence (1)⇔(3), assume first (1) that $\phi : X \to Y$ is a complete order embedding. In particular ϕ is one-to-one, so ϕ^* is cp onto. Denote by $q : Y^* \to Y^*/\ker(\phi^*)$ the canonical cp quotient map and by $\widetilde{\phi^*} : Y^*/\ker(\phi^*) \to X^*$ the induced cp map such that $\widetilde{\phi^*} \circ q = \phi^*$. In order to show (3), we need to show that the inverse $(\widetilde{\phi^*})^{-1}$ is cp. To this end, let $x^* \in \mathrm{M}_n(X^*)^+$. We view x^* as a cp map $x^* : X \to \mathrm{M}_n$. By assumption, the map $\phi^{-1} : \phi(X) \to X$ is well-defined and ucp, so we may push x^* forward by ϕ to a cp map on the image $\phi(X) \subseteq Y$, i.e. $\phi_* x^* := x^* \circ \phi^{-1} \in \mathcal{CP}(\phi(X), \mathrm{M}_n)$. By (the matrix-valued version of) Arveson's extension theorem (i.e. Krein's theorem together with the identification of cp maps

$Y \to \mathrm{M}_n$ with positive functionals on $\mathrm{M}_n(Y)$ [106, Theorem 6.2]), there is a cp map $y^* \in \mathcal{CP}(Y, \mathrm{M}_n)$ extending $\phi_* x^*$, i.e.

$$y^*(\phi(x)) = \phi_* x^*(\phi(x)) = x^* \circ \phi^{-1}(\phi(x)) = x^*(x) \tag{2.7}$$

for all $x \in X$. Now, viewing x^* and y^* as elements of $\mathrm{M}_n(X^*)^+$ and $\mathrm{M}_n(Y^*)^+$ respectively, the equation (2.7) yields

$$(\phi^*)^{(n)}(y^*) = x^*.$$

With this we obtain

$$\begin{aligned}((\widetilde{\phi^*})^{-1})^{(n)}(x^*) &= ((\widetilde{\phi^*})^{-1})^{(n)} \left((\phi^*)^{(n)}(y^*)\right) \\ &= ((\widetilde{\phi^*})^{-1})^{(n)} \circ (\widetilde{\phi^*})^{(n)} \circ q^{(n)}(y^*) \\ &= q^{(n)}(y^*) \in \mathrm{M}_n(Y^*/\ker(\phi^*))^+.\end{aligned}$$

This shows the map $(\widetilde{\phi^*})^{-1}$ is cp, as claimed.

Conversely, assume (3) that $\phi^* : Y^* \to X^*$ is an $\mathbf{MVS}$-quotient map. In particular ϕ^* is onto, so ϕ is one-to-one with inverse map $\phi^{-1} : \phi(X) \to X$. Identifying the dual $\phi(X)^*$ of the image $\phi(X) \subseteq Y$ with the quotient $Y^*/\phi(X)^\perp \cong Y^*/\ker(\phi^*)$ we may view the dual map $(\phi^{-1})^* : X^* \to \phi(X)^*$ of the inverse of ϕ as the inverse of the dual map of ϕ (modulo $\phi(X)^\perp$), i.e.

$$(\phi^{-1})^* = (\widetilde{\phi^*})^{-1} : X^* \to Y^*/\ker(\phi^*).$$

Note that the map $(\widetilde{\phi^*})^{-1}$ is cp by the assumption that ϕ^* is an $\mathbf{MVS}$-quotient map. Moreover, the dual map

$$((\widetilde{\phi^*})^{-1})^* : (Y^*/\ker(\phi^*))^* \cong \phi(X)^{**} \to X^{**}$$

is cp. Recall from [83, Proposition 6.2] that an operator system is canonically unitally completely order embedded into its bidual. So we may consider the restriction of $((\widetilde{\phi^*})^{-1})^*$ to the operator subsystem $\phi(X) \subseteq \phi(X)^{**}$ and want to show that $((\widetilde{\phi^*})^{-1})^*|_{\phi(X)} = \phi^{-1}$. Indeed, for $x \in X$, we may view $\phi(x)$ as an element of $\phi(X)^{**}$ and we have

$$((\widetilde{\phi^*})^{-1})^*((\phi(x)) = \phi(x) \circ (\widetilde{\phi^*})^{-1} = \phi(x) \circ (\phi^{-1})^* = \phi^{-1} \circ \phi(x) = x.$$

It follows that $\phi^{-1} = ((\widetilde{\phi^*})^{-1})^*|_{\phi(X)} : \phi(X) \to X$ is cp, and hence ϕ is a complete order embedding as claimed. □

Remark 2.2.8. We point out that the above statements are equivalent to ϕ^* being a quotient map in the category $\mathbf{MOS}$ of matrix-ordered operator spaces with cp cc maps [72]. In fact, this is implied by (3) and (4) together, and conversely, an $\mathbf{MOS}$-quotient map is in particular an $\mathbf{OSp}$-quotient map. In our applications in Chapter 5, we will mainly focus on the equivalence (1) and (3). The equivalence of (3) and (4) allows us to avoid discussing the category $\mathbf{MOS}$ in more detail.

2.3 Compact quantum metric spaces

This small survey on compact quantum metric spaces is an expansion of the one in [93, Section 5]. The notions of *Lip-norms* and *compact quantum metric spaces* are due to Rieffel [115, 116, 119] who developed them for order-unit spaces. We work exclusively in the setting of operator systems, which we usually be consider to be unitally completely order embedded in $\mathcal{B}(H)$, for some Hilbert space H. When modeled on operator systems, compact quantum metric spaces can be compared in terms of *complete* or equivalently *operator Gromov–Hausdorff distance* [85, 86].

2.3.1 Lip-normed operator systems

By a seminorm we mean an *extended* seminorm, i.e. it may take value ∞.

Definition 2.3.1. Let X be an operator system. A seminorm $L : X \to [0, \infty]$ is called a *Lipschitz seminorm* if it satisfies the following properties:

(1) Its *domain* $\mathrm{Dom}(L) := \{x \in X \mid L(x) < \infty\}$ is dense in X;

(2) it satisfies $L(x^*) = L(x)$, for all $x \in X$;

(3) we have $\mathbb{C}\mathbf{1}_X \subseteq \ker(L)$.

A Lipschitz seminorm L is called a *Lip-norm* if additionally the following property holds:

(4) The induced *Monge–Kantorovich* distance

$$d^L(\phi, \psi) := \sup\{|\phi(x) - \psi(x)| \mid L(x) \leqslant 1\}$$

metrizes the weak*-topology on the state space $\mathcal{S}(X)$.

Remark 2.3.2. This operator system version of compact quantum metric spaces is compatible with Rieffel's order-unit space formulation, as shown in [77, Proposition 2.1.8]. In fact, setting $\mathrm{Dom}(L_{\mathrm{sa}}) := \mathrm{Dom}(L) \cap X_{\mathrm{sa}}$ and $L_{\mathrm{sa}} := L|_{X_{\mathrm{sa}}}$ defines a Lip-norm in the sense of Rieffel [116] on the real archimedean order-unit space $\mathrm{Dom}(L_{\mathrm{sa}})$, and one checks that the Monge–Kantorovich distance $d^{L_{\mathrm{sa}}}$, defined on $\mathcal{S}(X_{\mathrm{sa}}) = \mathcal{S}(X)$ in the obvious way, is equal to d^L. Conversely, if L^0 is a Lip-norm on X_{sa} in the sense of [116] one readily shows that

$$L^0_{\mathrm{OSy}}(x) := \sup_{\theta \in [0, 2\pi]} L^0(\cos(\theta)\mathrm{Re}(x) + \sin(\theta)\mathrm{Im}(x))$$

defines a Lip-norm on X. Moreover, $(L^0_{\mathrm{OSy}})_{\mathrm{sa}} = L^0$.

Lemma 2.3.3 ([115, Proposition 1.4])**.** *Let X be an operator system and $L : X \to [0, \infty]$ be a Lipschitz seminorm. Then the topology on the state space $\mathcal{S}(X)$ induced by the Monge–Kantorovich distance d^L is finer than the weak*-topology.*

Proof. Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(X)$ and assume that $d^L(\phi_n, \phi) \to 0$, for some $\phi \in \mathcal{S}(X)$. We show that $\phi_n \to \phi$ weak*. Note that by assumption for every $\varepsilon > 0$ and large enough n we have $\sup_{x\in\mathrm{Dom}(L)} \frac{|\phi_n(x)-\phi(x)|}{L(x)} = d^L(\phi_n, \phi) < \varepsilon$. Hence $\hat{x}(\phi_n) := \phi_n(x) \to \phi(x) =: \hat{x}(\phi)$, for all $x \in \mathrm{Dom}(L)$. By the identification $X \ni x \mapsto \hat{x}$ we have $\mathrm{Dom}(L) \subseteq \mathrm{C}(\mathcal{S}(X))$ – the C*-algebra of continuous functions on $\mathcal{S}(X)$ with the weak*-topology – and since $\mathrm{Dom}(L)$ is a dense subspace of X and contains scalar multiples of the unit, we have that $\mathrm{Dom}(L) \subseteq \mathrm{C}(\mathcal{S}(X))$ is a strictly separating subspace containing the constant functions. This is enough to see that $\phi_n \to \phi$ in the weak*-topology on $\mathcal{S}(X)$. □

Remark 2.3.4. Let X be an operator system with connected state space $\mathcal{S}(X)$. Then the kernel of any Lip-norm L on X is actually equal to $\mathbb{C}\mathbf{1}_X$ [75, Lemma 2.2]. To this end note that any two states $\phi, \psi \in \mathcal{S}(X)$ must be at finite distance from each other, since d^L metrizes the weak*-topology, for which $\mathcal{S}(X)$ is compact (and thus has finite diameter as a metric space). Recall that states on X separate points (in the sense that if $\phi(x) = 0$, for all $\phi \in \mathcal{S}(X)$, it follows that $x = 0$ [78, Theorem 4.3.4(i)]). Hence, for every $x \in X \backslash \mathbb{C}\mathbf{1}_X$, there must be two states $\phi, \psi \in \mathcal{S}(X)$ such that $\phi(x) \neq \psi(x)$. If we now assume that $L(x) = 0$ we have that $d^L(\phi, \psi) \geqslant |\phi(\lambda x) - \psi(\lambda x)|$, for all $\lambda \in \mathbb{C}$, by definition of the Monge–Kantorovich distance. Hence we have $d^L(\phi, \psi) = \infty$ contradicting the assumption that d^L metrizes the compact set $\mathcal{S}(X)$.

Definition 2.3.5. A *compact quantum metric space* is an operator system together with a Lip-norm.

Definition 2.3.6. Let (X, H, D) be an (operator system) spectral triple. If the tuple $(X, \|[D, \cdot]\|)$ is a compact quantum metric space, we say that (X, H, D) is an (operator system) *spectral metric space.*

The notion of spectral metric space was coined in [13].

Definition 2.3.7. A *morphism* between two compact quantum metric spaces (X, L_X) and (Y, L_Y) is a ucp map $\Phi : X \to Y$, for which there is a constant $C \geqslant 0$ such that $L_Y(\Phi(x)) \leqslant CL_X(x)$, for all $x \in X$. A morphism Φ is called *Lip-norm contractive* if the constant C can be chosen to be 1.

Definition 2.3.8. Let X be an operator system and let $L : X \to [0, \infty]$ be a Lipschitz seminorm. Denote by $\|\cdot\|_{X/\mathbb{C}}$ and $L_{X/\mathbb{C}}$ the induced norm and seminorm on the quotient $X/\mathbb{C}\mathbf{1}_X$ respectively. The *radius* of X is the number

$$r_X := \inf\{r \in [0, \infty] \mid \|\cdot\|_{X/\mathbb{C}} \leqslant rL_{X/\mathbb{C}}\}.$$

Notation 2.3.9. For a vector space V, seminorms p and q on V and any positive real number $r > 0$, we set

$$\begin{aligned}
\mathrm{B}_r^p &:= \{v \in V \mid p(v) < r\}, \\
\overline{\mathrm{B}}_r^p &:= \{v \in V \mid p(v) \leqslant r\}, \\
\mathrm{B}_r^{p,q} &:= \mathrm{B}_r^p \cap \mathrm{B}_r^q, \text{ and} \\
\overline{\mathrm{B}}_r^{p,q} &:= \overline{\mathrm{B}}_r^p \cap \overline{\mathrm{B}}_r^q.
\end{aligned}$$

The following characterization of Lip-norms appears in [115, Theorem 1.8], see also [109, Theorem 6.3] for an even earlier version in a C*-algebraic context. Recall that a subset S of a normed space $(V, \|\cdot\|)$ is *totally bounded* if, for every $\varepsilon > 0$, there exist $v_1, \ldots, v_n \in S$ such that $S \subseteq \bigcup_{i=1}^{n} \mathrm{B}_{\varepsilon}^{\|\cdot\|}$.

Proposition 2.3.10. *Let X be an operator system and $L : X \to [0, \infty]$ a Lipschitz seminorm. Then L is a Lip-norm if and only if X has finite radius and the set $\overline{\mathrm{B}}_1^{\|\cdot\|, L}$ is totally bounded.*

We only show that L is a Lip-norm if X has finite radius and $\overline{\mathrm{B}}_1^{\|\cdot\|, L}$ is totally bounded, as this is how we usually use this proposition. We follow the argument from [115] where also the converse can be found.

Proof (that the condition is sufficient for L to be a Lip-norm). Assume X has finite radius and $\overline{\mathrm{B}}_1^{\|\cdot\|, L}$ is totally bounded. Denote by $q : X \to X/\mathbb{C}\mathbf{1}_X$ the canonical quotient map. We first show that total boundedness of $\overline{\mathrm{B}}_1^{\|\cdot\|, L}$ implies total boundedness of $\overline{\mathrm{B}}_1^{L_{X/\mathbb{C}}}$ for the quotient norm $\|\cdot\|_{X/\mathbb{C}}$. By the assumption that X has finite radius, there is a real number $r > 0$ such that, for all $x \in X$ with $L(x) \leqslant 1$, we have

$$\|q(x)\|_{X/\mathbb{C}} \leqslant r L_{X/\mathbb{C}}(q(x)) \leqslant r L(x) \leqslant r.$$

In other words we have $q(\overline{\mathrm{B}}_1^{L}) \subseteq q(\overline{\mathrm{B}}_r^{\|\cdot\|} \cap \overline{\mathrm{B}}_1^{L})$ where the latter set is contained in $q(\overline{\mathrm{B}}_r^{\|\cdot\|, L})$. Since $\overline{\mathrm{B}}_r^{\|\cdot\|, L}$ is totally bounded by assumption, this implies that $\overline{\mathrm{B}}_1^{L_{X/\mathbb{C}}} = q(\overline{\mathrm{B}}_1^{L})$ is totally bounded for the norm $\|\cdot\|_{X/\mathbb{C}}$.

We now show that total boundedness of $\overline{\mathrm{B}}_1^{L_{X/\mathbb{C}}}$ implies that the d^L-topology on the state space $\mathcal{S}(X)$ is coarser than the weak*-topology. Fix $\varepsilon > 0$. We need to show that for every state $\phi \in \mathcal{S}(X)$ the ε-d^L-ball $\mathrm{B}_{\varepsilon}^{d^L}(\phi)$ around ϕ contains a weak*-neighborhood of ϕ. To this end fix $\phi \in \mathcal{S}(X)$ and set

$$\mathcal{U}(\phi) := \{\psi \in \mathcal{S}(X) \mid |\phi(x) - \psi(x)| < \frac{\varepsilon}{3}, \text{ for all } x \in X\}.$$

Note that $\mathcal{U}(\phi)$ is a weak*-open neighborhood of ϕ in $\mathcal{S}(X)$. We show that, for every $\psi \in \mathcal{U}(\phi)$, we have $d^L(\phi, \psi) < \varepsilon$. By total boundedness of $\overline{\mathrm{B}}_1^{L_{X/\mathbb{C}}}$ there are $x_1, \ldots, x_n \in \overline{\mathrm{B}}_1^{L}$ such that $\overline{\mathrm{B}}_1^{L_{X/\mathbb{C}}} \subseteq \bigcup_{i=1}^{n} \mathrm{B}_{\frac{\varepsilon}{3}}^{L_{X/\mathbb{C}}}(q(x_i))$. Let $x \in X$ with $L(x) \leqslant 1$. It follows that there is an index $i = 1, \ldots, n$ such that $\|q(x) - q(x_i)\|_{X/\mathbb{C}} < \frac{\varepsilon}{3}$. Hence, there is an element $y \in \mathbb{C}\mathbf{1}_X$ such that $\|x - x_i - y\| < \frac{\varepsilon}{3}$. It follows that

$$\begin{aligned}
&|\phi(x) - \psi(x)| \\
&= |\phi(x) - \phi(x_i + y)| + |\phi(x_i + y) - \psi(x_i + y)| + |\phi(x_i + y) - \psi(x)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Hence $d^L(\phi, \psi) < \varepsilon$, so $\mathrm{B}_{\varepsilon}^{d^L}(\phi) \supseteq \mathcal{U}(\phi)$, i.e. the d^L-topology on $\mathcal{S}(X)$ is coarser than the weak*-topology. Since the d^L-topology is always finer than the weak*-topology by Lemma 2.3.3 they must coincide, so L is a Lip-norm as claimed. $\square$

Note that if X is a finite-dimensional operator system, every Lipschitz seminorm L on X with $\ker(L) = \mathbb{C}\mathbf{1}_X$ is a Lip-norm. Indeed, the condition on the kernel of L guarantees that X has finite radius and by compactness the set $\overline{\mathrm{B}}_1^{\|\cdot\|,L}$ is totally bounded.

Example 2.3.11. An example for a Lipschitz seminorm which is not a Lip-norm is given in [115], of which we give an operator system version: Let X be an infinite-dimensional operator system, denote by $q : X \to X/\mathbb{C}\mathbf{1}_X$ the canonical (normed space) quotient map and set $L(x) := \|q(x)\|_{X/\mathbb{C}}$, for all $x \in X$. Then L is a Lipschitz seminorm, but the set $\overline{\mathrm{B}}_1^{\|\cdot\|,L}$ is dense in the norm-unit ball $\overline{\mathrm{B}}_1^{\|\cdot\|}$ which is non-compact; thus $\overline{\mathrm{B}}_1^{\|\cdot\|,L}$ cannot be totally bounded. By Proposition 2.3.10, the Lipschitz seminorm L is not a Lip-norm.

In our operator system setting the following notion of Monge–Kantorovich distance on matrix state spaces is natural.

Definition 2.3.12. Let X be an operator system and $n \in \mathbb{N}$ a positive integer. Recall the matrix state space $\mathcal{S}_n(X) := \mathrm{UCP}(X, \mathrm{M}_n)$ of X. If L is a Lipschitz seminorm on X, we set

$$d^{L,n}(\phi, \psi) := \sup_{x \in X \setminus \ker(L_X)} \frac{\|\phi(x) - \psi(x)\|}{L(x)}$$

and call it the induced *Monge–Kantorovich* distance on $\mathcal{S}_n(X)$.

If L is a Lip-norm the Monge–Kantorovich distances metrize the point-norm topologies on all matrix-state spaces. Indeed, [85, Proposition 2.12] shows that this is the case for $d^{L_{\mathrm{sa}},n}$ and a matrix-state version of the argument sketched in Remark 2.3.2 shows $d^{L,n} = d^{L_{\mathrm{sa}},n}$, for all $n \in \mathbb{N}$, to arrive at our claim.

A Lipschitz seminorm L on an operator system X is called *lower semicontinuous* if, for every sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\|x_n - x\| \to 0$, for some $x \in X$, we have $L(x) \leqslant \liminf_n L(x_n)$. Equivalently, the Lipschitz seminorm L is lower semicontinuous if the Lipschitz seminorm unit-ball $\overline{\mathrm{B}}_1^{L}$ is norm-closed in X. Denote by L_{d^L} the Lipschitz seminorm on X given by the Lipschitz constant induced by the metric d^L on $\mathcal{S}(X)$ when viewing $X \subseteq \mathrm{C}(\mathcal{S}(X))$. Then L_{d^L} is a lower semicontinuous Lipschitz seminorm on X and we have $L_{d^L} \leqslant L$. Moreover, equality $L_{d^L} = L$ holds if L was lower semicontinuous to begin with. The Monge–Kantorovich distances induced by L and L_{d^L} on $\mathcal{S}(X)$ coincide, and the Lipschitz seminorm L_{d^L} is the largest lower semicontinuous seminorm smaller than L with the property that $d^{L_{d^L}} = d^L$. See Sections 3 and 4 in [116] for these results in the order-unit space setting. They pass to our operator system setting by Remark 2.3.2. Note that this discussion can be adapted to matrix-state spaces as well. In particular, when referring to matrix-state spaces we may always assume our Lipschitz seminorms to be lower semicontinuous without loss of generality.

A Lipschitz seminorm L on an operator system X is called *closed* if the Lipschitz seminorm unit-ball $\overline{\mathrm{B}}_1^{L}$ is closed in the completion $\overline{X}$ of X. The Minkowski functional $\overline{L} : \overline{X} \to [0, \infty]$ is defined by

$$\overline{L}(x) := \inf\{r \geqslant 0 \mid x \in r \cdot \mathrm{cl}(\overline{\mathrm{B}}_1^L)\},$$

where $\mathrm{cl}(\overline{\mathrm{B}}_1^L)$ denotes the closure of $\overline{\mathrm{B}}_1^L$ in $\overline{X}$. If L is lower semicontinuous, then $\overline{L}|_X = L$ and $d^{\overline{L},n} = d^{L,n}$, for all $n \in \mathbb{N}$.

Definition 2.3.13. For a Lipschitz seminorm L on an operator system X we define its *closure* by $L^{\mathrm{c}} := \overline{L_{d^L}}$.

Note that passing to the closure of a Lipschitz seminorm does not change the Monge–Kantorovich distances on the matrix-state spaces, i.e. $d^{L^{\mathrm{c}},n} = d^{L_{d^L},n} = d^{L,n}$, for all $n \in \mathbb{N}$. See [116, Section 4] for a discussion on closed Lip-norms. For the above definition of the closure as the "closure of the lower semicontinuous version of L" we followed [85], silently applying Remark 2.3.2.

In the next subsection we will discuss Gromov–Hausdorff type distances of compact quantum metric spaces. It will follow from the above two remarks that for these distances we may replace Lip-norms by their closures. In fact, this technicality is already taken care of automatically when passing to isometry classes of compact quantum metric spaces.

Definition 2.3.14 ([86])**.** Let (X, L_X) and (Y, L_Y) be compact quantum metric spaces. An *isometry* is a unital complete order isomorphism $\Phi : \overline{X} \to \overline{Y}$ such that $L_Y^{\mathrm{c}} \circ \Phi = L_X^{\mathrm{c}}$ on $\overline{X}_{\mathrm{sa}}$.

2.3.2 Complete Gromov–Hausdorff distance

Let (X, L_X) and (Y, L_Y) be compact quantum metric spaces. Different generalizations of Gromov–Hausdorff distance for compact metric spaces to the quantum setting have been proposed. The notion of classical Gromov–Hausdorff distance between the state spaces of compact quantum metric spaces was used e.g. in [36, 33, 136].

Definition 2.3.15. The *Gromov–Hausdorff distance* of (X, L_X) and (Y, L_Y) is the classical Gromov–Hausdorff distance of their state spaces, i.e.

$$\begin{aligned} \mathrm{dist}_{\mathrm{GH}}((X, L_X), (Y, L_Y)) &:= \mathrm{dist}_{\mathrm{GH}}((\mathcal{S}(X), d^{L_X}), (\mathcal{S}(Y), d^{L_Y})) \\ &= \inf \mathrm{dist}_{\mathrm{H}}^{\rho}(\mathcal{S}(X), \mathcal{S}(Y)), \end{aligned} \tag{2.8}$$

where the infimum runs over all metrics ρ on the disjoint union $\mathcal{S}(X) \sqcup \mathcal{S}(Y)$ which restrict to the respective Monge–Kantorovich distances on the summands. Here $\mathrm{dist}_{\mathrm{H}}^{\rho}$ is the usual Hausdorff distance [24, Definition 7.3.1] on the set of compact subsets of the space $\mathcal{S}(X) \sqcup \mathcal{S}(Y)$ equipped with such a metric ρ.

The following notion of quantum Gromov–Hausdorff distance goes back to [118] where it is phrased in terms of the order-unit space version of compact quantum metric spaces. We follow here the treatment in [77, Section 2.2]. For a Lipschitz seminorm L on an operator system X we denote by L_{sa} the restriction to the self-adjoint part X_{sa}.

Definition 2.3.16. A Lip-norm L on the direct sum $X \oplus Y$ is called *admissible* if $\mathrm{Dom}(L) = \mathrm{Dom}(L_X) \oplus \mathrm{Dom}(L_Y)$ and if the induced seminorms on X_{sa} and Y_{sa} under the respective coordinate projections $p_X : \mathrm{Dom}(L)_{\mathrm{sa}} \to \mathrm{Dom}(L_X)_{\mathrm{sa}}$ and $p_Y : \mathrm{Dom}(L)_{\mathrm{sa}} \to \mathrm{Dom}(L_Y)_{\mathrm{sa}}$ coincide with $(L_X)_{\mathrm{sa}}$ and $(L_Y)_{\mathrm{sa}}$ respectively.

Definition 2.3.17. The *quantum Gromov–Hausdorff distance* of (X, L_X) and (Y, L_Y) is given by

$$\mathrm{dist}^{\mathrm{q}}_{\mathrm{GH}}((X, L_X), (Y, L_Y)) := \inf \mathrm{dist}^{d^L}_{\mathrm{H}}(\mathcal{S}(X), \mathcal{S}(Y)),$$

where the infimum runs over all admissible Lip-norms L on $X \oplus Y$ with d^L the induced Monge–Kantorovich distance on $\mathcal{S}(X \oplus Y)$ and where $\mathrm{dist}^{d^L}_{\mathrm{H}}$ is the Hausdorff distance on the set of compact subsets of $\mathcal{S}(X \oplus Y)$.

Note that if L is an admissible Lip-norm on $X \oplus Y$, the metric d^L metrizes the weak*-topology on $\mathcal{S}(X \oplus Y) \cong \mathcal{S}(X) \sqcup \mathcal{S}(Y)$. Thus d^L is an element of the set of metrics ρ over which the infimum in the definition (2.8) of $\mathrm{dist}_{\mathrm{GH}}$ is taken, so that we always have

$$\mathrm{dist}_{\mathrm{GH}} \leqslant \mathrm{dist}^{\mathrm{q}}_{\mathrm{GH}}. \tag{2.9}$$

However, in general Gromov–Hausdorff distance $\mathrm{dist}_{\mathrm{GH}}$ and quantum Gromov–Hausdorff distance $\mathrm{dist}^{\mathrm{q}}_{\mathrm{GH}}$ are inequivalent metrics on the set of compact quantum metric spaces [76].

Gromov–Hausdorff distance can equivalently be computed by taking the infimum of the Hausdorff distances of the state spaces over over all isometric embeddings $\iota_{\mathcal{S}(X)} : \mathcal{S}(X) \hookrightarrow \mathcal{T}$ and $\iota_{\mathcal{S}(Y)} : \mathcal{S}(Y) \hookrightarrow \mathcal{T}$ into compact metric spaces $(\mathcal{T}, \rho)$ [24, Definition 7.3.10, Remark 7.3.12]. Similarly, [77, Lemma 2.2.2] and [96, Proposition 4.7] imply the following:

$$\begin{aligned} &\mathrm{dist}^{\mathrm{q}}_{\mathrm{GH}}((X, L_X), (Y, L_Y)) \\ &= \mathrm{dist}^{\mathrm{q}}_{\mathrm{GH}}((\mathrm{Dom}(L_X)_{\mathrm{sa}}, (L_X)_{\mathrm{sa}}), (\mathrm{Dom}(L_Y)_{\mathrm{sa}}, (L_Y)_{\mathrm{sa}})) \\ &= \inf \mathrm{dist}^{V}_{\mathrm{H}}\left(\overline{\mathrm{B}}^{(L_X)_{\mathrm{sa}}}_1, \overline{\mathrm{B}}^{(L_Y)_{\mathrm{sa}}}_1\right), \end{aligned} \tag{2.10}$$

where the infimum is taken over all archimedean order-unit spaces V which contain $\mathrm{Dom}(L_X)_{\mathrm{sa}}$ and $\mathrm{Dom}(L_Y)_{\mathrm{sa}}$ as order-unit subspaces. Equivalently, one can take the infimum over all normed spaces V which contain $\mathrm{Dom}(L_X)_{\mathrm{sa}}$ and $\mathrm{Dom}(L_Y)_{\mathrm{sa}}$ as subspaces such that their order-units coincide.

It is apparent that quantum Gromov–Hausdorff distance does not capture the complete order structure of the involved operator systems. To overcome this, the notions of *complete* [85] and *operator Gromov–Hausdorff distance* [86] were introduced by Kerr and Li, building also on [95, 96].

Definition 2.3.18. The *complete Gromov–Hausdorff distance* is given by

$$\mathrm{dist}^{\mathrm{s}}((X, L_X), (Y, L_Y)) := \inf \sup_{n \in \mathbb{N}} \mathrm{dist}^{d^{L,n}}_{\mathrm{H}}(\mathcal{S}_n(X), \mathcal{S}_n(Y)),$$

where the infimum is taken over all admissible Lip-norms L on $X \oplus Y$.

Remark 2.3.19. For the definition of quantum and complete Gromov–Hausdorff distance it is equivalent to consider admissible Lip-norms on $X \oplus Y$ (in the operator system sense of Definition 2.3.16) and admissible Lip-norms on $(X \oplus Y)_{\mathrm{sa}}$ (in Rieffel's order-unit sense [118]), as explained in [77, Lemma 2.2.2].

In [85, Definition 3.2], the n-distance is defined as

$$\mathrm{dist}^{\mathrm{s}}_n((X, L_X), (Y, L_Y)) := \inf \mathrm{dist}_{\mathrm{H}}^{d^{L,n}}(\mathcal{S}_n(X), \mathcal{S}_n(Y)),$$

where, again, the infimum is taken over all admissible Lip-norms L on $X \oplus Y$. Moreover we have

$$\mathrm{dist}^{\mathrm{s}}_m \leqslant \mathrm{dist}^{\mathrm{s}}_n \leqslant \mathrm{dist}^{\mathrm{s}},$$

for all $m, n \in \mathbb{N}$ with $m \leqslant n$. In particular it follows that

$$\mathrm{dist}^{\mathrm{q}}_{\mathrm{GH}} = \mathrm{dist}^{\mathrm{s}}_1 \leqslant \mathrm{dist}^{\mathrm{s}}. \tag{2.11}$$

The following example from [85] shows that quantum and complete Gromov–Hausdorff distance are inequivalent. Let A be a separable unital C*-algebra. By [117] we can choose a Lip-norm L on A. Then the canonical map from (A, L) to its opposite algebra (A^{op}, L) is a Lip-isometric isomorphism of archimedean order-unit spaces implying that $\mathrm{dist}^{\mathrm{q}}_{\mathrm{GH}}((A, L), (A^{\mathrm{op}}, L)) = 0$ by [118]. However, if A and A^{op} are not *-isomorphic, they are not isometric as compact quantum metric spaces by [85, Corollary 4.11], whence $\mathrm{dist}^{\mathrm{s}}((A, L), (A^{\mathrm{op}}, L)) > 0$.

We summarize some basic properties of complete Gromov–Hausdorff distance. Given any two compact quantum metric spaces (X, L_X) and (Y, L_Y) their complete Gromov–Hausdorff distance is finite [85, Proposition 3.7]. Moreover, complete Gromov–Hausdorff distance satisfies the triangle inequality [85, Proposition 3.4]. We give a sketch of the proof which is analogous to [118, Theorem 4.3]. Given compact quantum metric spaces (X, L_X), (Y, L_Y), and (Z, L_Z), and a real number $\varepsilon > 0$, let L_{XY} and L_{YZ} be admissible Lip-norms for (X, L_X), (Y, L_Y), and (Y, L_Y), (Z, L_Z) respectively, such that

$$\sup_{n \in \mathbb{N}} \mathrm{dist}_{\mathrm{H}}^{d^{L_{XY},n}}(\mathcal{S}_n(X), \mathcal{S}_n(Y)) \leqslant \mathrm{dist}^{\mathrm{s}}(X, Y) + \varepsilon,$$

and similarly for Y and Z. Set $L_{XYZ}(x, y, z) := \max\{L_{XY}(x, y), L_{YZ}(y, z)\}$. One checks that the quotient seminorm $L_{XZ}(x, z) := \inf_{y \in Y} L_{XYZ}(x, y, z)$ is an admissible Lip-norm for (X, L_X), (Z, L_Z). Upon verifying that

$$\begin{aligned} &\mathrm{dist}_{\mathrm{H}}^{d^{L_{XZ},n}}(\mathcal{S}_n(X), \mathcal{S}_n(Y)) \\ &\leqslant \mathrm{dist}_{\mathrm{H}}^{d^{L_{XY},n}}(\mathcal{S}_n(X), \mathcal{S}_n(Y)) + \mathrm{dist}_{\mathrm{H}}^{d^{L_{YZ},n}}(\mathcal{S}_n(Y), \mathcal{S}_n(Z)), \end{aligned}$$

the triangle inequality readily follows.

It turns out that the complete Gromov–Hausdorff distance of two compact quantum metric spaces (X, L_X) and (Y, L_Y) is 0 if and only if they are isometric. Indeed, the fact that isometric compact quantum metric spaces are at complete Gromov–Hausdorff distance 0 follows from our criterion below Proposition 2.3.21. The converse is shown in [85, Theorem 4.10]. It follows that the set of isometry classes of compact quantum metric spaces together with complete Gromov–Hausdorff distance is a metric space. Many desirable properties of this metric space were established by Kerr, including a quantum analogue of Gromov's compactness theorem [85, Theorem 6.3].

We will need a criterion to control complete Gromov–Hausdorff distance. Before proceeding to it, we record the following estimate of Hausdorff distance of compact subsets of a metric space.

Lemma 2.3.20. *Let M, N be compact subsets of a metric space (Z, d) and let $f : M \to N$, $g : N \to M$ be set maps. Then the following holds:*

$$\mathrm{dist}_{\mathrm{H}}^d(M,N) \leqslant \max\left\{\sup_{m\in M} d(m, f(m)), \sup_{n\in N} d(g(n), n)\right\}$$

Proof. This follows from the definition of Hausdorff distance [24, Definition 7.3.1]. Indeed, for a subset $S \subseteq Z$ and a positive real number $r > 0$, set $U_r^d(S) := \{z \in Z \mid \inf_{s\in S} d(z,s) < r\}$. Then we have:

$$\begin{aligned}
&\mathrm{dist}_{\mathrm{H}}^d(M,N)\\
&= \inf\{r > 0 \mid M \subseteq U_r^d(N) \text{ and } N \subseteq U_r^d(M)\}\\
&= \inf\left\{r > 0 \mid \left(\inf_{n\in N} d(m,n) < r, \forall m \in M\right) \text{ and } \left(\inf_{m\in M} d(m,n) < r, \forall n \in N\right)\right\}\\
&\leqslant \inf\left\{r > 0 \mid (d(m, f(m)) < r, \forall m \in M) \text{ and } (d(g(n), n) < r, \forall n \in N)\right\}\\
&= \max\left\{\sup_{m\in M} d(m, f(m)), \sup_{n\in N} d(g(n), n)\right\}
\end{aligned}$$

□

The following sufficient condition for estimating complete Gromov–Hausdorff distance is analogous to the criteria [137, Proposition 4] and [77, Proposition 2.2.4]. They rely on the methods already used in [115, 116] and formalized in terms of the notion of *bridge* in Section 5 of [118]. Since we will not use bridges explicitly, we refer to *loc.cit.* for details and further intuition. For a compact quantum metric space (X, L_X), denote by $\mathrm{diam}(X) := \mathrm{diam}(\mathcal{S}(X), d^{L_X})$ the diameter of the state space of X.

Proposition 2.3.21. *Let (X, L_X) and (Y, L_Y) be compact quantum metric spaces and let $\varepsilon_X, \varepsilon_Y, C_\Phi, C_\Psi > 0$ be positive real numbers. Suppose that there are morphisms $\Phi : X \to Y$, $\Psi : Y \to X$ of compact quantum metric spaces with $L_Y(\Phi(x)) \leqslant C_\Phi L_X(x)$ and $L_X(\Psi(y)) \leqslant C_\Psi L_Y(y)$, for all $x \in X$, $y \in Y$. Assume furthermore that*

$$\|\Psi\Phi(x) - x\| \leqslant \varepsilon_X L_X(x) \quad \textit{and} \quad \|\Phi\Psi(y) - y\| \leqslant \varepsilon_Y L_Y(y).$$

Then the following estimate holds:

$$\operatorname{dist}^{\mathrm{s}}(X,Y) \leqslant \max\left\{\operatorname{diam}(X)\left|1-\frac{1}{C_\Phi}\right| + \frac{\varepsilon_X}{C_\Phi}, \operatorname{diam}(Y)\left|1-\frac{1}{C_\Psi}\right| + \frac{\varepsilon_Y}{C_\Psi}\right\}$$

Proof. We set $r := \max\left\{\operatorname{diam}(X)\left|1-\frac{1}{C_\Phi}\right| + \frac{\varepsilon_X}{C_\Phi}, \operatorname{diam}(Y)\left|1-\frac{1}{C_\Psi}\right| + \frac{\varepsilon_Y}{C_\Psi}\right\}$ and define a seminorm L on $X \oplus Y$ by

$$L(x,y) := \max\left\{L_X(x), L_Y(y), \frac{1}{r}\|y-\Phi(x)\|, \frac{1}{r}\|x-\Psi(y)\|\right\}.$$

It is shown in the proof of [77, Proposition 2.2.4] that L is an admissible Lip-norm.

Denote by $\iota_{\mathcal{S}_n(X)} : \mathcal{S}_n(X) \to \mathcal{S}_n(X) \sqcup \mathcal{S}_n(Y)$, $\iota_{\mathcal{S}_n(Y)} : \mathcal{S}_n(Y) \to \mathcal{S}_n(X) \sqcup \mathcal{S}_n(Y)$ the respective inclusion maps of matrix state spaces into the disjoint union of matrix state spaces. Observe that, for every positive integer $n \in \mathbb{N}$ and matrix state $\phi \in \mathcal{S}_n(X)$, we have

$$\begin{aligned}
&d^{L,n}(\phi, \Psi^*\phi) \\
&= \sup_{(x,y)\in X\oplus Y\setminus\mathbb{C}\mathbf{1}_{X\oplus Y}} \frac{\|\iota_{\mathcal{S}_n(X)}(\phi)(x,y) - \iota_{\mathcal{S}_n(Y)}(\Psi^*\phi)(x,y)\|}{L(x,y)} \\
&= \sup_{(x,y)\in X\oplus Y\setminus\mathbb{C}\mathbf{1}_{X\oplus Y}} \frac{\|\phi(x) - \phi(\Psi(y))\|}{\max\left\{L_X(x), L_Y(y), \frac{1}{r}\|y-\Phi(x)\|, \frac{1}{r}\|x-\Psi(y)\|\right\}} \\
&\leqslant r.
\end{aligned}$$

Similarly, for every positive integer $n \in \mathbb{N}$ and matrix state $\psi \in \mathcal{S}_n(Y)$, we have $d^{L,n}(\Phi^*\psi, \psi) \leqslant r$.

Now, using the Lipschitz maps $\Psi^* : \mathcal{S}_n(X) \to \mathcal{S}_n(Y)$, $\Phi^* : \mathcal{S}_n(Y) \to \mathcal{S}_n(X)$ on the subsets $\mathcal{S}_n(X), \mathcal{S}_n(Y)$ of the metric space $(\mathcal{S}_n(X) \sqcup \mathcal{S}_n(Y), d^{L,n})$, we obtain from Lemma 2.3.20 that

$$\operatorname{dist}_{\mathrm{H}}^{d^{L,n}}(\mathcal{S}_n(X), \mathcal{S}_n(Y)) \leqslant \max\left\{\sup_{\phi\in\mathcal{S}_n(X)} d^{L,n}(\phi, \Psi^*\phi), \sup_{\psi\in\mathcal{S}_n(Y)} d^{L,n}(\Phi^*\psi, \psi)\right\}.$$

Together with the previous observation that each of these two suprema is bounded by r, the claim follows:

$$\operatorname{dist}^{\mathrm{s}}(X,Y) \leqslant \sup_{n\in\mathbb{N}} \operatorname{dist}_{\mathrm{H}}^{d^{L,n}}(\mathcal{S}_n(X), \mathcal{S}_n(Y)) \leqslant r$$

□

The following distance is an operator system analogue of the order-unit space version given in [96]. Other than complete Gromov–Hausdorff distance it is defined directly at the operator system level of compact quantum metric spaces without passing to matrix state spaces. This is analogous to the equality (2.10) for quantum Gromov–Hausdorff distance.

Definition 2.3.22 ([86]). Let (X, L_X), (Y, L_Y) be compact quantum metric spaces. The *operator Gromov–Hausdorff distance* is given by

$$\mathrm{dist}^{\mathrm{op}}((X, L_X), (Y, L_Y)) := \inf \mathrm{dist}_{\mathrm{H}}(\iota_X(\overline{\mathrm{B}}_1^{L_X}), \iota_Y(\overline{\mathrm{B}}_1^{L_Y})),$$

where the infimum is taken over all unital complete order embeddings $\iota_X : X \hookrightarrow Z$, $\iota_Y : Y \hookrightarrow Z$ into an operator system Z.

Let (X, L_X), (Y, L_Y), and (Z, L_Z) be compact quantum metric spaces and assume that V and W are operator systems such that there are complete order embeddings $X \hookrightarrow V \hookleftarrow Y \hookrightarrow W \hookleftarrow Z$. Then it follows that X and Z are completely order embedded in the amalgamated direct sum $V \oplus_Y W$. The triangle inequality for $\mathrm{dist}^{\mathrm{op}}$ follows [86, Lemma 3.2]. The fact that two compact quantum metric spaces have operator Gromov–Hausdorff distance 0 if and only if they are isometric is established in [95, Theorem 3.15] and [86, Theorem 3.15], showing that the set of isometry classes of compact quantum metric spaces with operator Gromov–Hausdorff distance is a metric space.

Theorem 2.3.23 ([86, Theorem 3.7]). *Complete and operator Gromov–Hausdorff distance coincide. I.e. for all compact quantum metric spaces (X, L_X) and (Y, L_Y), we have*

$$\mathrm{dist}^{\mathrm{s}}((X, L_X), (Y, L_Y)) = \mathrm{dist}^{\mathrm{op}}((X, L_X), (Y, L_Y)).$$

The equality of complete and operator Gromov–Hausdorff distance allows to establish important properties of both of these distances. For instance, the set of isometry classes of compact quantum metric spaces with operator (equivalently complete) Gromov–Hausdorff distance is a complete metric space [86, Theorem 4.1], which furthermore contains a homeomorphic image of the set of isometry classes of classical compact metric spaces with Gromov–Hausdorff distance as a closed subset [86, Theorem 3.10].

From (2.9), (2.11) and Theorem 2.3.23 we obtain the following corollary:

Corollary 2.3.24. *Under the hypothesis of Proposition 2.3.21, the distances* $\mathrm{dist}_{\mathrm{GH}}$, $\mathrm{dist}^{\mathrm{q}}_{\mathrm{GH}}$, $\mathrm{dist}^{\mathrm{s}}_n$, *for all* $n \in \mathbb{N}$, $\mathrm{dist}^{\mathrm{s}}$ *and* $\mathrm{dist}^{\mathrm{op}}$ *are all bounded above by the number* r *from (the proof of) Proposition 2.3.21.*

Chapter 3

Spectral truncations of tori

This chapter is based on the published article [94]. The results were obtained in collaboration with Walter D. van Suijlekom.

3.1 Introduction

In this chapter, we discuss convergence of spectral truncations of the d-torus $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ with its canonical Riemannian metric and trivial spin-structure. To this end, consider the associated canonical spectral triple $(\mathrm{C}(\mathbb{T}^d), \mathrm{L}^2(S_{\mathbb{T}^d}), D_{\mathbb{T}^d})$ from Example 2.1.4, where we identify the Hilbert space $\mathrm{L}^2(S_{\mathbb{T}^d})$ of L^2-spinors with the tensor product $\mathrm{L}^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^{\lfloor \frac{d}{2} \rfloor}}$. We abbreviate $S := S_{\mathbb{T}^d}$ and $D := D_{\mathbb{T}^d}$. Recall from Example 2.1.4 that the spectrum of the spin-Dirac operator is given by $\sigma(D) = \{\pm(n_1^2 + \cdots + n_d^2)^{\frac{1}{2}} \mid n_i \in \mathbb{Z}\}$. Recall furthermore from Proposition 2.1.5 that the spin-Dirac operator gives rise to the Monge–Kantorovich distance $d^{\|[D,\cdot]\|}$ on the state space $\mathcal{S}(\mathrm{C}(\mathbb{T}^d))$, i.e. the set of probability measures on $\mathbb{T}^d$. The Connes distance $d^{\|[D,\cdot]\|}$ metrizes the weak*-topology and recovers the Riemannian distance when restricted to pure states, i.e. the Dirac measures on $\mathbb{T}^d$; we denote the Riemannian distance on $\mathbb{T}^d$ as well as the Monge–Kantorovich distance on $\mathcal{S}(\mathrm{C}(\mathbb{T}^d))$ by $d_{\mathbb{T}^d}$.

For any real number $\Lambda \geqslant 0$, let P_Λ be the spectral projection for D given by the orthogonal projection onto the subspace of $\mathrm{L}^2(S)$ which is spanned by the eigenspinors e_λ of the eigenvalues λ with $|\lambda| \leqslant \Lambda$. More concretely, we have

$$P_\Lambda \mathrm{L}^2(S_{\mathbb{T}^d}) = \mathrm{span}\{e_n \mid n \in \mathbb{Z}^d, (n_1^2 + \cdots + n_d^2)^{\frac{1}{2}} \leqslant \Lambda\} \otimes \mathbb{C}^{2^{\lfloor \frac{d}{2} \rfloor}},$$

with $e_n(x) := e^{in\cdot x}$, for all $x \in \mathbb{T}^d$. The projection P_Λ gives rise to the following *operator system spectral triple*:

$$\left(\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}, \mathrm{L}^2(S)_\Lambda, D_\Lambda\right), \tag{3.1}$$

where we use the notations $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)} := P_\Lambda \mathrm{C}(\mathbb{T}^d) P_\Lambda$, $\mathrm{L}^2(S)_\Lambda := P_\Lambda \mathrm{L}^2(S)$ and $D_\Lambda := P_\Lambda D P_\Lambda$. We write $d_\Lambda := d^{\|[D_\Lambda,\cdot]\|}$ for the Monge–Kantorovich distance induced by the Lipschitz seminorm $\|[D_\Lambda, \cdot]\|$ on the state space of the operator system

$\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$. Since the operator system $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ is finite-dimensional, the set

$$\overline{\mathrm{B}}_1^{\|\cdot\|,\|[D_\Lambda,\cdot]\|} := \{T \in \mathrm{C}(\mathbb{T}^d)^{(\Lambda)} \mid \|T\| \leqslant 1, \|[D_\Lambda, T]\| \leqslant 1\} \subseteq \overline{\mathrm{B}}_1^{\|\cdot\|}$$

is totally bounded. Finite-dimensionality also ensures that $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ has finite radius in the sense of Definition 2.3.8, and moreover it is clear that $\ker(\|[D_\Lambda, \cdot]\|) = \mathbb{C}\mathbf{1}_{\mathcal{B}(\mathrm{L}^2(S)_\Lambda)}$. It follows from Proposition 2.3.10 that the Monge–Kantorovich distance d_Λ metrizes the weak*-topology on the state space $\mathcal{S}(\mathrm{C}(\mathbb{T}^d)^{(\Lambda)})$ and hence the operator system spectral triple in (3.1) is an operator system spectral metric space.

The elements T of the operator system $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ can be represented as matrices of the form $T = (t_{k-l})_{k,l\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}}$, where $\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d} := \overline{\mathrm{B}}_\Lambda \cap \mathbb{Z}^d \subseteq \mathbb{R}^d$ is the set of $\mathbb{Z}^d$-lattice points in the closed euclidean ball of radius Λ, and where $t_{k-l} = \langle e_k, Te_l\rangle$. In particular, $\mathrm{C}(\mathbb{T}^1)^{(\Lambda)}$ is the operator system of $(2\lfloor\Lambda\rfloor + 1) \times (2\lfloor\Lambda\rfloor + 1)$-Toeplitz matrices which was investigated at length in [33] and [50].

Our goal is to show convergence in complete Gromov–Hausdorff distance of the compact quantum metric spaces $(\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}, \|[D_\Lambda, \cdot]\|)$ to $(\mathrm{C}(\mathbb{T}^d), \|[D, \cdot]\|)$, as $\Lambda \to \infty$, by employing the criterion in Proposition 2.3.21; this requires two morphisms of compact quantum metric spaces

$$\tau_\Lambda : \mathrm{C}(\mathbb{T}^d) \to \mathrm{C}(\mathbb{T}^d)^{(\Lambda)} \text{ and}$$
$$\sigma_\Lambda : \mathrm{C}(\mathbb{T}^d)^{(\Lambda)} \to \mathrm{C}(\mathbb{T}^d),$$

the compositions of which approximate the respective identity maps on $\mathrm{C}(\mathbb{T}^d)$ and $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ in Lip-norm. Following [136] we sometimes call such a sequence of pairs of maps $(\tau_\Lambda, \sigma_\Lambda)_{\Lambda\geqslant 0}$ a C^1*-approximate complete order isomorphism.* The canonical candidate for the map $\tau_\Lambda : \mathrm{C}(\mathbb{T}^d) \to \mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ is the compression map given by $\tau_\Lambda(f) := P_\Lambda f P_\Lambda$, for all $f \in \mathrm{C}(\mathbb{T}^d)$. It is easy to see that this map is ucp and Lip-norm contractive (Lemma 3.3.1). Inspired by the choice of the map given in [136] in the case of the circle, we propose the following map $\sigma_\Lambda : \mathrm{C}(\mathbb{T}^d)^{(\Lambda)} \to \mathrm{C}^\infty(\mathbb{T}^d)$ as a candidate for the map in the converse direction:

$$\sigma_\Lambda(T) := \frac{1}{\mathcal{N}_\mathrm{B}(\Lambda)} \mathrm{Tr}\left(\psi^\mathrm{t}\psi\alpha(T)\right),$$

for all $T \in \mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$. Here, the symbol α denotes the $\mathbb{T}^d$-action (3.4) on the operator system $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$, the vector $\psi \in \ell^2(\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d})$ is given by $\psi := \sum_{n\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} e_n$, and $\mathcal{N}_\mathrm{B}(\Lambda) := \#\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$ is the number of $\mathbb{Z}^d$-lattice points in the closed euclidean ball of radius Λ. Note that the image of an element $T = (t_{k-l})_{k,l\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}}$ of the operator system $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ under this map is a function on $\mathbb{T}^d$ as follows:

$$\begin{aligned}\sigma_\Lambda(T)(x) &= \frac{1}{\mathcal{N}_\mathrm{B}(\Lambda)} \mathrm{Tr}\left((1)_{k,l\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} \left(t_{k-l}e^{i(k-l)\cdot x}\right)_{k,l\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}}\right) \\ &= \frac{1}{\mathcal{N}_\mathrm{B}(\Lambda)} \mathrm{Tr}\left(\left(\sum_{m\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} t_{m-l}e^{i(m-l)\cdot x}\right)_{k,l\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}}\right)\end{aligned}$$

$$= \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)} \sum_{m,n \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} t_{m-n} e^{i(m-n)\cdot x},$$

for all $x \in \mathbb{T}^d$. Berezin quantization as in [120, Section 2] is another instance of inspiration for our choice of map σ_Λ by realizing that the map σ_Λ is the formal adjoint of the map τ_Λ when the C*-algebra $\mathrm{C}(\mathbb{T}^d)$ is equipped with the L^2-inner product and the operator system $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ is equipped with the Hilbert–Schmidt inner product. Similarly as for τ_Λ, it is easy to see that the map σ_Λ is ucp and Lip-norm contractive (Lemma 3.3.2).

In order to show that our choice of maps τ_Λ and σ_Λ gives rise to a C^1-approximate order isomorphism it remains to show that their compositions approximate the respective identities on $\mathrm{C}(\mathbb{T}^d)$ and $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ in Lip-norm. We show by direct computations (Lemma 3.3.3) that the maps $\sigma_\Lambda \circ \tau_\Lambda$ and $\tau_\Lambda \circ \sigma_\Lambda$ act on $\mathrm{C}(\mathbb{T}^d)$ and $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ respectively as follows:

$$\sigma_\Lambda \circ \tau_\Lambda(f) = \left(\mathfrak{m}_\Lambda \widehat{f}\right)^{\vee} =: \mathcal{F}_{\mathfrak{m}_\Lambda}(f)$$
$$\tau_\Lambda \circ \sigma_\Lambda(T) = (\mathfrak{m}_\Lambda(k-l) t_{k-l})_{k,l \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} =: \mathcal{S}_{\mathfrak{m}_\Lambda}(T)$$

The map $\mathcal{F}_{\mathfrak{m}_\Lambda}$ is known as *Fourier multiplication* and the map $\mathcal{S}_{\mathfrak{m}_\Lambda}$ as *Schur multiplication*, respectively with the *symbol*

$$\mathfrak{m}_\Lambda(n) := \frac{\mathcal{N}_{\mathrm{L}}(\Lambda, n)}{\mathcal{N}_{\mathrm{B}}(\Lambda)},$$

where $\mathcal{N}_{\mathrm{L}}(\Lambda, n) := \#\overline{\mathrm{L}}_\Lambda^{\mathbb{Z}^d}(n) := \#(\overline{\mathrm{B}}_\Lambda \cap \overline{\mathrm{B}}_\Lambda(n) \cap \mathbb{Z}^d)$ is the number of $\mathbb{Z}^d$-lattice points in the intersection of the closed ball of radius Λ with a copy of itself translated by n (we call this intersection a *lense*). Here $f \mapsto \widehat{f}$ denotes the Fourier transform on $\mathbb{T}^d$ and $a \mapsto \check{a}$ the inverse Fourier transform.

We apply an "antiderivative trick" (Lemma 3.3.4) to see that for obtaining estimates of the maps $\mathbf{I}^{\mathrm{C}^\infty(\mathbb{T}^d)} - \mathcal{F}_{\mathfrak{m}_\Lambda}$ and $\mathbf{I}^{\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}} - \mathcal{S}_{\mathfrak{m}_\Lambda}$ in Lip-norm, one needs to estimate the following two maps:

$$\mathcal{F}_{\mathfrak{n}_\Lambda} := \frac{i}{2} \sum_{\mu=1}^{d} \mathcal{F}_{\mathfrak{n}_\Lambda^\mu} \otimes \{\gamma^\mu, \cdot\} : [D, \mathrm{C}^\infty(\mathbb{T}^d)] \to \mathrm{C}^\infty(\mathbb{T}^d);$$
$$\mathcal{S}_{\mathfrak{n}_\Lambda} := \frac{i}{2} \sum_{\mu=1}^{d} \mathcal{S}_{\mathfrak{n}_\Lambda^\mu} \otimes \{\gamma^\mu, \cdot\} : [D, \mathrm{C}(\mathbb{T}^d)^{(\Lambda)}] \to \mathrm{C}(\mathbb{T}^d)^{(\Lambda)},$$

where $\mathcal{F}_{\mathfrak{n}_\Lambda^\mu}$ and $\mathcal{S}_{\mathfrak{n}_\Lambda^\mu}$ are now respectively Fourier and Schur multiplication with the symbol

$$\mathfrak{n}_\Lambda^\mu(n) = \begin{cases} 0, \text{ if } n = 0 \\ (1 - \mathfrak{m}_\Lambda(n)) \frac{n_\mu}{\|n\|^2}, \text{ if } n \neq 0. \end{cases} \tag{3.2}$$

The bracket $\{\gamma^\mu, \cdot\}$ denotes the anti-commutator, i.e. $\{\gamma^\mu, A\} = \gamma^\mu A + A\gamma^\mu$, for all $A \in \mathrm{M}_{2^{\lfloor \frac{d}{2} \rfloor}}$. A variation of the classical Bożejko-Fendler *transference theorem* for Fourier and Schur multipliers (Lemma 3.3.5) then shows that the cb-norm of $\mathcal{S}_{\mathfrak{n}_\Lambda}$ is bounded by the cb-norm of $\mathcal{F}_{\mathfrak{n}_\Lambda}$. Since the latter map takes values in a commutative C*-algebra its cb-norm coincides with its norm, so this is what is left to estimate.

It is not hard to see that $\mathcal{F}_{\mathfrak{m}_\Lambda}$ is an approximate identity which is controlled in Lip-norm, if the convolution kernel $K_{\mathfrak{m}_\Lambda} = \check{\mathfrak{m}}_\Lambda$ is a *good kernel* (Lemma 3.2.2). We show that this is indeed the case by exploiting the fact that $K_{\mathfrak{m}_\Lambda}$ is the square of the *spherical Dirichlet kernel*, whence positive. Since this is analogous to the 1-dimensional case, we call $K_{\mathfrak{m}_\Lambda}$ the *spectral Fejér kernel*. As outlined above, our main result that $(\tau_\Lambda, \sigma_\Lambda)$ is a C^1-approximate complete order isomorphism and hence that spectral truncations of the d-torus converge for all $d \geqslant 1$ is now simply a corollary of the fact that the *spectral Fejér kernel* is *good*.

In the last section, we give a computation of the propagation number of the operator system $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$.

3.2 Preliminaries

3.2.1 Actions and commutators

Recall the usual action of $\mathrm{C}(\mathbb{T}^d)$ on $\mathrm{L}^2(S)$:

$$f(g \otimes v) = (fg) \otimes v = \sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} \widehat{f}(n-m)\widehat{g}(m) e_n \otimes v \tag{3.3}$$

This induces an action of $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ on $\mathrm{L}^2(S)_\Lambda$:

$$T\left(\sum_{k \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} a_k e_k \otimes v\right) = \sum_{k \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} \sum_{l \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} t_{k-l} a_l e_k \otimes v$$

Furthermore, we have the standard $\mathbb{T}^d$-action on $\mathrm{C}^\infty(\mathbb{T}^d)$ (as a subalgebra of $\mathcal{B}(\mathrm{L}^2(S))$):

$$\alpha_\theta(f) = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n) e_n e^{in\cdot\theta}$$

This induces an action of $\mathbb{T}^d$ on $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ (as an operator subsystem of $\mathcal{B}(\mathrm{L}^2(S)_\Lambda)$):

$$\alpha_\theta(T) = \left(t_{k-l} e^{i(k-l)\cdot\theta}\right)_{k,l \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} \tag{3.4}$$

One readily computes the commutator of the Dirac operator with smooth functions acting on $\mathrm{L}^2(S)$ as in (3.3):

$$[D, f] = \sum_{\mu=1}^{d} \sum_{n \in \mathbb{Z}^d} n_\mu \widehat{f}(n) e_n \otimes \gamma^\mu$$

Similarly, we have:

$$[D_\Lambda, T] = \sum_{\mu=1}^{d} \left((k_\mu - l_\mu)t_{k-l}\right)_{k,l\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} \otimes \gamma^\mu \tag{3.5}$$

3.2.2 Good kernels

We follow the convention in [132] and call an approximate identity in the Banach $*$-algebra $\mathrm{L}^1(\mathbb{T}^d)$ (with convolution) a *good kernel*:

Definition 3.2.1. For all $\Lambda > 0$, let $K_\Lambda \in \mathrm{L}^1(\mathbb{T}^d)$. The family $\{K_\Lambda\}_{\Lambda>0}$ is called a *good kernel*, if the following holds:

(1) We have $\int_{\mathbb{T}^d} K_\Lambda(x)dx = 1$, and

(2) for all $\delta > 0$, we have that $\int_{\mathbb{T}^d\setminus \mathrm{B}_\delta(0)} K_\Lambda(x)dx \to 0$, as $\Lambda \to \infty$.

Good kernels provide a way to approximate the identity on $\mathrm{C}^\infty(\mathbb{T}^d)$ not only in L^1- and sup-norm, but also in Lip-norm:

Lemma 3.2.2. *If K_Λ is a* good kernel, *then, for all $f \in \mathrm{C}^\infty(\mathbb{T}^d)$, the following holds:*

$$\|f - K_\Lambda * f\| \leqslant \gamma_\Lambda \|[D, f]\|,$$

where $\gamma_\Lambda \to 0$ as $\Lambda \to \infty$.

Proof. The proof is as in [14, Lemma 5.13]. For all $x \in \mathbb{T}^d$, we have:

$$\begin{aligned} |K_\Lambda * f(x) - f(x)| &\leqslant \int_{\mathbb{T}^d} |K_\Lambda(y)(f(x-y) - f(x))|dy \\ &\leqslant \int_{\mathbb{T}^d} |K_\Lambda(y)|\|y\|\|f\|_{\mathrm{Lip}} dy \\ &= \int_{\mathbb{T}^d} |K_\Lambda(y)|\|y\|dy\|[D, f]\|, \end{aligned}$$

where the fact that $\|f\|_{\mathrm{Lip}} = \|[D, f]\|$ is shown in [29, Proposition 1], see also [30, 140]. We set $\gamma_\Lambda := \int_{\mathbb{T}^d} |K_\Lambda(y)|\|y\|dy$. Let $\varepsilon > 0$. Let Λ_0 be large enough such that, for all $\Lambda \geqslant \Lambda_0$, we have $\int_{\|y\|\geqslant\varepsilon} |K_\Lambda(y)|dy < \varepsilon$. Then, for $\Lambda \geqslant \Lambda_0$, we obtain:

$$\begin{aligned} \gamma_\Lambda &= \int_{\|y\|\geqslant\varepsilon} |K_\Lambda(y)|\|y\|dy + \int_{\|y\|<\varepsilon} |K_\Lambda(y)|\|y\|dy \\ &\leqslant d\left(\int_{\|y\|\geqslant\varepsilon} |K_\Lambda(y)|dy + \varepsilon\int_{\|y\|<\varepsilon} |K_\Lambda(y)|dy\right) \\ &\leqslant d\left(\varepsilon + \varepsilon C\right), \end{aligned}$$

since $\|y\| \leqslant d$, for $y \in [0,1]^d$ and where $C = \sup_{\Lambda>0} \int_{\mathbb{T}^d} |K_\Lambda(y)|dy < \infty$. □

3.2.3 Fourier and Schur multipliers

Let Γ be a discrete group and let $\lambda : \Gamma \to \mathcal{B}(\ell^2(\Gamma))$ be its left-regular representation given by $\lambda_s f(t) = f(st)$. We denote by $\mathrm{C}^*_\mathrm{r}(\Gamma)$ the reduced group C^*-algebra, i.e. the completion of the group ring $\mathbb{C}[\Gamma]$ in $\mathcal{B}(\ell^2(\Gamma))$ with respect to the norm $\|x\|_\mathrm{r} := \|\lambda(x)\|_{\mathcal{B}(\ell^2(\Gamma))}$. A function $u : \Gamma \to \mathbb{C}$ gives rise to a *multiplier* on the group ring as follows:

$$\begin{aligned} \mathbb{C}[\Gamma] &\to \mathbb{C}[\Gamma] \\ \sum_{t\in\Gamma} a_t \delta_t &\mapsto \sum_{t\in\Gamma} u(t) a_t \delta_t \end{aligned}$$

If this map extends to a bounded linear map $\mathcal{M}_u : \mathrm{C}^*_\mathrm{r}(\Gamma) \to \mathrm{C}^*_\mathrm{r}(\Gamma)$ we call this extension the *multiplier on* $\mathrm{C}^*_\mathrm{r}(\Gamma)$ *with symbol* u. We record the obvious fact that if u is finitely supported it always induces a multiplier on $\mathrm{C}^*_\mathrm{r}(\Gamma)$.

Recall that if Γ is abelian, then $\mathrm{C}^*_\mathrm{r}(\Gamma) = \mathrm{C}^*(\Gamma) \cong \mathrm{C}(\widehat{\Gamma})$, where $\widehat{\Gamma}$ is the Pontryagin dual of Γ. In this case, we call the multiplier on $\mathrm{C}(\widehat{\Gamma})$ the *Fourier multiplier with symbol* u and denote it by $\mathcal{F}_u$. The Fourier multiplier takes on the following form:

$$\mathcal{F}_u(f) = \Big(t \mapsto u(t)\widehat{f}(t)\Big)^{\vee} = \check{u} * f \tag{3.6}$$

See e.g. [67, Chapter 6] for the relevant Fourier theory of locally compact abelian groups.

Let $k : \Gamma \times \Gamma \to \mathbb{C}$ be a function, also called a *kernel*. A kernel k gives rise to a linear map with domain $\mathcal{B}(\ell^2(\Gamma))$ given by $\mathcal{S}_k : (T_{s,t})_{s,t\in\Gamma} \mapsto (k(s,t)T_{s,t})_{s,t\in\Gamma}$, where $T_{s,t} = \langle \delta_s, T\delta_t \rangle$, for $T \in \mathcal{B}(\ell^2(\Gamma))$. If this map is bounded with range in $\mathcal{B}(\ell^2(\Gamma))$, we call it a *Schur multiplier*. See e.g. [133] for a survey and [110, Chapter 5] as a standard reference which includes a discussion of the connection with Grothendieck's theorem. We collect some well-known facts about Schur multipliers.

Proposition 3.2.3. *Let $k : \Gamma \times \Gamma \to \mathbb{C}$ be a kernel. Then the following are equivalent:*

(i) $\mathcal{S}_k$ is a Schur multiplier of norm $\|\mathcal{S}_k\| \leqslant 1$.

(ii) $\mathcal{S}_k$ is a completely bounded Schur multiplier of cb*-norm $\|\mathcal{S}_k\|_{\mathrm{cb}} \leqslant 1$.*

(iii) There exists a Hilbert space H and families of vectors $\{\xi_s\}_{s\in\Gamma}, \{\eta_t\}_{t\in\Gamma} \subset H$ with $\|\xi_s\|, \|\eta_t\| \leqslant 1$ such that $k(s,t) = \langle \xi_s, \eta_t \rangle$, for all $s, t \in \Gamma$.

For an elementary proof of the equivalence of (i) and (ii), we refer to [106, Theorem 8.7 and Corollary 8.8]. A proof of the equivalence of (ii) and (iii) can be found e.g. in [23, Theorem D.4], which we now sketch: Assuming that $\|\mathcal{S}_k\|_{\mathrm{cb}} \leqslant 1$, Wittstock's factorization theorem gives a factorization of $\mathcal{S}_k$ through $\mathcal{B}(H)$, for some Hilbert space H, which allows to construct appropriate ξ_s and η_t. For the converse implication, the map $\mathcal{S}_k$ is factorized through $\mathcal{B}(\ell^2(\Gamma) \otimes H)$ as $\mathcal{S}_k(T) = V^*(T \otimes \mathbf{I}^H)W$, for the contractions $V\delta_s := \delta_s \otimes \xi_s$ and $W\delta_t := \delta_t \otimes \eta_t$. The same works when tensoring with $\mathbf{1}_n$, for arbitrary $n \in \mathbb{N}$, which shows complete contractivity of $\mathcal{S}_k$.

We are interested in Schur multipliers $\mathcal{S}_k$ induced by functions $u : \Gamma \to \mathbb{C}$, i.e. $k(s,t) := u(st^{-1})$. We call such a Schur multiplier a *Schur multiplier with symbol* u and slightly abuse notation to denote it by $\mathcal{S}_u$. It is easy to see that $\mathcal{S}_u\big|_{\mathrm{C}^*_\mathrm{r}(\Gamma)} = \mathcal{M}_u$. Indeed, let $f \in \mathrm{C}^*_\mathrm{r}(\Gamma)$ and $\{\delta_s\}_{s\in\Gamma}$ be an orthonormal basis for $\ell^2(\Gamma)$. Then the matrix associated to f (viewed as an element of $\mathcal{B}(\ell^2(\Gamma))$) is a Toeplitz matrix in the following sense:

$$\langle \delta_s, f\delta_t\rangle = \langle \delta_s, \sum_{r\in\Gamma} f_r\lambda_r(\delta_t)\rangle = \sum_{r\in\Gamma}\langle \delta_s, f_r\delta_{rt}\rangle = f_{st^{-1}}$$

It follows that the matrix associated to $\mathcal{M}_u(f)$ is the following Toeplitz matrix:

$$\langle \delta_s, \mathcal{M}_u(f)\delta_t\rangle = u(st^{-1})f_{st^{-1}},$$

which shows that $\mathcal{S}_u((f_{st^{-1}})_{s,t\in\Gamma}) = ((\mathcal{M}_u(f))_{s,t})_{s,t\in\Gamma}$.

3.2.4 Some convex geometry

In order to compute the propagation number of the operator system $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$, some facts from convex geometry are required. Since the natural setting for these is locally convex spaces we formulate all the required results in this abstract language even though we only make use of them in the finite-dimensional case. See e.g. [125, Section 11] and [21, Chapter II] for much of the standard terminology. Note that this connects to the basic notions about ordered vector spaces in Section 2.2. However, here we focus on real vector spaces and deal with more explicit geometric properties of convex sets therein.

Throughout this section, let X be a real locally convex vector space with continuous dual X^*. Every linear functional l on X and every real number $\alpha \in \mathbb{R}$ give rise to a hyperplane in X given by $H_{l=\alpha} = \{x \in X \mid l(x) = \alpha\}$. Clearly, $H_{l=\alpha} = \ker(l-\alpha)$ and the hyperplane $H_{l=\alpha}$ is closed if and only if l is continuous which is the only case we consider. Every hyperplane $H_{l=\alpha}$ gives rise to an open positive and an open negative half-space denoted by $H_{l>\alpha}$ and $H_{l<\alpha}$ respectively and defined by $H_{l>\alpha} := \{x \in X \mid l(x) > \alpha\}$ and similarly for $H_{l<\alpha}$. Their respective closures are called the closed positive and the closed negative half-space associated to $H_{l=\alpha}$ and denoted by $H_{l\geqslant\alpha}$ and $H_{l\leqslant\alpha}$ respectively. Of course $H_{l\geqslant\alpha} = \{x \in X \mid l(x) \geqslant \alpha\}$ and similarly for $H_{l\leqslant\alpha}$.

If $K, L \subseteq X$ are two convex sets, they are said to be *separated* by the hyperplane $H_{l=\alpha}$ if $K \subseteq H_{l\geqslant\alpha}$ and $L \subseteq H_{l\leqslant\alpha}$. The sets K and L are called *properly separated* if additionally $K \nsubseteq H_{l=\alpha}$ or $L \nsubseteq H_{l=\alpha}$. The sets K and L are called *strictly separated* by $H_{l=\alpha}$ if $K \subseteq H_{l>\alpha}$ and $L \subseteq H_{l<\alpha}$. A hyperplane $H_{l=\alpha}$ is called a *supporting hyperplane* for a non-empty convex set $K \subset X$, if $K \subseteq H_{l\geqslant\alpha}$ and $x \in H_{l=\alpha}$, for at least one $x \in K$. A supporting hyperplane $H_{l=\alpha}$ for K is called *non-trivial* if $K \nsubseteq H_{l=\alpha}$.

Recall that a cone $C \subseteq X$ is called *pointed* if $0 \in C$, *salient* if it does not contain any 1-dimensional subspaces of X and *proper* if $C \cap (-C) = \{0\}$. We only consider convex cones. A convex pointed cone is salient if and only if it is proper. For $x \in X$, we call a set $C + x$ a *cone with vertex* x if C is a pointed cone. A cone with vertex x is called *proper* if $C - x$ is a proper cone.

Let $K \subset X$ be convex set with $0 \notin K$. Set $C := [0, \infty) \cdot K$. Clearly, C is a convex pointed cone. Moreover, C is the smallest convex pointed cone which which contains K in the sense that every convex pointed cone which contains K must contain C.

If C is a convex pointed cone, a subset $B \subset C$ is called a *base* (or *sole* in [21, II, §8.3]) if there exists a closed hyperplane $H \not\ni 0$ such that $B = H \cap C$ and such that C is the smallest convex pointed cone which contains B. It is well-known that a convex subset B of a convex pointed cone C is a base if and only if, for every $x \in C \backslash \{0\}$, there exists a unique pair $(\lambda, y) \in (0, \infty) \times B$ such that $x = \lambda y$.

The statement of the following lemma can be found in [21, II, §7.2, Exercise 21a].

Lemma 3.2.4. *Let $C \subset X$ be a locally compact, closed, convex, proper cone with vertex x. Then there exists a closed supporting hyperplane H of C such that $H \cap C = \{x\}$.*

Proof. To simplify notation we assume, without loss of generality, that $x = 0$. Let $U \subset X$ be a convex open neighborhood of 0 such that $K := \overline{U} \cap C$ is compact. We claim that $C = [0, \infty) \cdot K$, i.e. C is the smallest convex pointed cone which contains K. In fact, the inclusion $C \supseteq [0, \infty) \cdot K$ is clear from the cone property. To see that $C \subseteq [0, \infty) \cdot K$, let $y \in C$. Since every 0-neighborhood in a locally convex space is absorbent, there exists a positive scalar $\lambda > 0$ such that $y \in \lambda U$. Hence, $\frac{1}{\lambda} y \in U \cap \frac{1}{\lambda} C = U \cap C \subset K$ and therefore $y = \lambda \frac{1}{\lambda} y \in \lambda K \subset [0, \infty) \cdot K$.

By [21, II, §7.1, Proposition 2], there is an open half-space $H_{l<\alpha} \subset X$ such that $0 \in H_{l<\alpha} \cap K \subset U \cap C$. We may assume that the scalar α is positive, otherwise pass to the functional $-l$ instead. The boundary $\partial H_{l<\alpha} = H_{l=\alpha}$ of this half-space is a closed hyperplane of X which does not contain 0.

We claim that the cone C is the smallest closed convex pointed cone which contains $H_{l=\alpha} \cap C$, i.e. $C = [0, \infty) \cdot (H_{l=\alpha} \cap C)$. To see this, observe that the following inclusion holds:

$$[0, \infty) \cdot (H_{l=\alpha} \cap C) = [0, \infty) \cdot (H_{l \leqslant \alpha} \cap C) \subseteq [0, \infty) \cdot K = C$$

The converse inclusion $[0, \infty) \cdot (H_{l \leqslant \alpha} \cap C) \supseteq [0, \infty) \cdot K$ follows from the continuity and hence boundedness on the compact set K of the functional l by observing that there exists a positive real number $\lambda > 0$ such that $\lambda H_{l \leqslant \alpha} = H_{l \leqslant \lambda \alpha} \supseteq K$.

Now, set $H := H_{l=0}$. Then we have:

$$H \cap C = H \cap [0, \infty) \cdot (H_{l=\alpha} \cap C) = (H \cap [0, \infty) \cdot H_{l=\alpha}) \cap C = \{0\}$$

So H is the desired closed supporting hyperplane for C. □

Lemma 3.2.5. *Let $K \subset X$ be a non-empty compact convex subset. Let $l \in X^*$ be a continuous linear functional and $H_{l=0}$ be the associated closed hyperplane through 0. Then there exists an extreme point $x \in \operatorname{ex}(K)$ such that $K - x \subset H_{l \geqslant 0}$.*

Proof. By continuity of l, the image $l(K) \subset \mathbb{R}$ is bounded. Set $\alpha := \inf(l(K))$. In other words, α is the largest real number such that $K \subset H_{l \geqslant \alpha}$. In particular, $H_{l=\alpha}$ is a closed supporting hyperplane of K. By [21, II, §7.1, Corollary to Proposition 1], the hyperplane $H_{l=\alpha}$ contains an extreme point x of K. Moreover, $K - x$ is contained in the positive half-space $H_{l\geqslant 0}$. □

Combining the previous two lemmas, we obtain the following consequence:

Corollary 3.2.6. *Let $C \subset X$ be a locally compact, closed, convex, proper cone with vertex x and let $K \subset X$ be a non-empty compact convex set. Then there exists an extreme point $y \in \mathrm{ex}(K)$ and a closed hyperplane H which separates $C - x$ and $K - y$. Moreover, these two sets only intersect in* 0, *i.e.* $(C - x) \cap (K - y) = \{0\}$.

3.3 Convergence of spectral truncations of the d-torus

The goal of this subsection is to prove that the maps $\tau_\Lambda : \mathrm{C}(\mathbb{T}^d) \to \mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$, given by the compression $\tau_\Lambda(f) := P_\Lambda f P_\Lambda$, and $\sigma_\Lambda : \mathrm{C}(\mathbb{T}^d)^{(\Lambda)} \to \mathrm{C}(\mathbb{T}^d)$, given by $\sigma_\Lambda(T) := \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)}\mathrm{Tr}(\psi^{\mathrm{t}}\psi\alpha(T))$, form a C^1-approximate complete order isomorphism.

3.3.1 A candidate for the C^1-approximate order isomorphism

We begin by checking unitality, complete positivity and Lip-norm contractivity for the maps τ_Λ and σ_Λ.

Lemma 3.3.1. *The map $\tau_\Lambda : \mathrm{C}(\mathbb{T}^d) \to \mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ is ucp and Lip-norm contractive.*

Proof. Unitality is clear as $P_\Lambda \mathbf{1} P_\Lambda \left(\sum_{n\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} a_n e_n \otimes s_n\right) = \sum_{n\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} a_n e_n \otimes s_n$, for all $\sum_{n\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} a_n e_n \otimes s_n \in \mathrm{L}^2(S)_\Lambda$. Also complete positivity is obvious since $\langle PaP\xi, P\xi\rangle_{PH} = \langle aP\xi, P\xi\rangle_H \geqslant 0$, for any Hilbert space H, any projection $P \in \mathcal{B}(H)$ and any positive operator $a \in \mathcal{B}(H)^+$. Contractivity in norm follows from Plancherel's theorem:

$$\|P_\Lambda f P_\Lambda\|^2 = \sup_{\substack{\varphi\in\mathrm{L}^2(S)_\Lambda \\ \|\varphi\|\leqslant 1}} \|P_\Lambda f P_\Lambda \varphi\|^2 \leqslant \sum_{n\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} \left|\widehat{f}(n)\right|^2 \leqslant \sum_{n\in\mathbb{Z}^d} \left|\widehat{f}(n)\right|^2 = \|f\|^2$$

For contractivity in Lip-norm, we first observe that τ_Λ commutes with $[D, \cdot]$ in the following sense, which is an immediate consequence of the fact that D commutes with P_Λ:

$$[D_\Lambda, \tau_\Lambda(f)] = P_\Lambda [D, f] P_\Lambda$$

This immediately gives $\|[D, \cdot]\|$-contractivity:

$$\|[D_\Lambda, \tau_\Lambda(f)]\| = \|P_\Lambda [D, f] P_\Lambda\| \leqslant \|[D, f]\|$$

□

Lemma 3.3.2. *The map $\sigma_\Lambda : \mathrm{C}(\mathbb{T}^d)^{(\Lambda)} \to \mathrm{C}(\mathbb{T}^d)$ is ucp and Lip-norm contractive.*

Proof. Unitality is clear as, for all $x \in \mathbb{T}^d$, we have:

$$\sigma_\Lambda(\mathbf{1})(x) = \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)} \mathrm{Tr}\left(\psi^{\mathrm{t}}\psi\alpha_x(\mathbf{1})\right) = \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)} \mathrm{Tr}\left(\psi^{\mathrm{t}}\psi\mathbf{1}\right) = 1$$

Positivity is also immediate from the definition. Namely, let $T \in (\mathrm{C}(\mathbb{T}^d)^{(\Lambda)})^+$ be a positive operator on $\mathrm{L}^2(S)_\Lambda$ and let $\mathrm{L}^2(S)_\Lambda \ni \zeta \mapsto Q_T(\zeta) := \langle \zeta, T\zeta \rangle$ be its associated quadratic form. For $\zeta = \sum_{n \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} e_n \in \mathrm{L}^2(S)_\Lambda$, we obtain:

$$0 \leqslant \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)} Q_T(\zeta)(x) = \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)} \mathrm{Tr}\left(\psi^{\mathrm{t}}\psi \left(e_{-(m-n)} T_{m-n}\right)_{m,n}\right) = \sigma_\Lambda(T)(x),$$

for all $x \in \mathbb{T}^d$. Complete positivity of σ_Λ is automatic, since its range is a commutative C*-algebra. For contractivity, we compute:

$$|\sigma_\Lambda(T)(x)| \leqslant \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)} \mathrm{Tr}(\psi^{\mathrm{t}}\psi)\|\alpha_x(T)\| = \|T\|$$

For contractivity in Lip-norm, we first observe that σ_Λ commutes with $[D, \cdot]$ in the following sense, which is an easy consequence of (3.5):

$$\begin{aligned}
[D, \sigma_\Lambda(T)] &= -i \sum_{\mu=1}^{d} \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)} \sum_{n \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} i n_\mu T_n e_n \otimes \gamma^\mu \\
&= \sum_{\mu=1}^{d} \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)} \mathrm{Tr}\left(\psi^{\mathrm{t}}\psi\alpha\left(\left((n_\mu - m_\mu)T_{n-m}\right)_{n,m \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}}\right)\right) \otimes \gamma^\mu \\
&= \sigma_\Lambda \otimes \mathbf{1}_{2^{\lfloor \frac{d}{2} \rfloor}}([D_\Lambda, T])
\end{aligned}$$

This immediately gives $\|[D, \cdot]\|$-contractivity:

$$\|[D, \sigma_\Lambda(T)]\| = \|\sigma_\Lambda \otimes \mathbf{1}_{2^{\lfloor \frac{d}{2} \rfloor}}([D_\Lambda, T])\| \leqslant \|[D_\Lambda, T]\|$$

□

We now compute the compositions $\sigma_\Lambda \circ \tau_\Lambda$ and $\tau_\Lambda \circ \sigma_\Lambda$:

Lemma 3.3.3. *The two compositions $\sigma_\Lambda \circ \tau_\Lambda : \mathrm{C}(\mathbb{T}^d) \to \mathrm{C}(\mathbb{T}^d)$ and $\tau_\Lambda \circ \sigma_\Lambda : \mathrm{C}(\mathbb{T}^d)^{(\Lambda)} \to \mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ are given respectively by the Fourier multiplier and by the Schur multiplier with the symbol $\mathfrak{m}_\Lambda$:*

$$\begin{aligned}
\sigma_\Lambda \circ \tau_\Lambda &= \mathcal{F}_{\mathfrak{m}_\Lambda} \\
\tau_\Lambda \circ \sigma_\Lambda &= \mathcal{S}_{\mathfrak{m}_\Lambda}
\end{aligned}$$

Proof. Both identities are just simple computations:

$$\begin{aligned}
\sigma_\Lambda \circ \tau_\Lambda(f)(x) &= \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)} \mathrm{Tr}\left(\psi^{\mathrm{t}}\psi\alpha_x(P_\Lambda f P_\Lambda)\right) \\
&= \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)} \mathrm{Tr}\left(\psi^{\mathrm{t}}\psi\alpha_x\left(\left(\widehat{f}(k-l)\right)_{k,l\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}}\right)\right) \\
&= \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)} \sum_{k,l\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} \widehat{f}(k-l)e^{i(k-l)\cdot x} \\
&= \sum_{n\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}-\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}} \frac{\mathcal{N}_{\mathrm{L}}(\Lambda,n)}{\mathcal{N}_{\mathrm{B}}(\Lambda)} \widehat{f}(n)e_n(x)
\end{aligned}$$

$$\begin{aligned}
\rho_\Lambda \circ \sigma_\Lambda(T) &= \rho_\Lambda\left(\frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)} \mathrm{Tr}\left(\psi^{\mathrm{t}}\psi\alpha_\bullet(T)\right)\right) \\
&= \rho_\Lambda\left(\frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)} \sum_{k,l\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}}} T_{k-l}e_{k-l}\right) \\
&= \rho_\Lambda\left(\sum_{n\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}}-\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}}} \frac{\mathcal{N}_{\mathrm{L}}(\Lambda,n)}{\mathcal{N}_{\mathrm{B}}(\Lambda)} T_n e_n\right) \\
&= \left(\frac{\mathcal{N}_{\mathrm{L}}(\Lambda,m-n)}{\mathcal{N}_{\mathrm{B}}(\Lambda)} T_{m-n}\right)_{m,n\in\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}}}
\end{aligned}$$

□

3.3.2 A transference result

In order to show that the pair $(\tau_\Lambda, \sigma_\Lambda)$ is a C^1-approximate order isomorphism it remains to check that the compositions $\sigma_\Lambda \circ \tau_\Lambda$ and $\tau_\Lambda \circ \sigma_\Lambda$ approximate the identity respectively on $\mathrm{C}(\mathbb{T}^d)$ and $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ in Lip-norm. For this, recall the definition of $\mathfrak{n}_\Lambda^\mu(n)$, for $\mu = 1, \ldots, d$ and $n \in \mathbb{Z}^d$ from (3.2).

Lemma 3.3.4. *For every $f \in \mathrm{C}^\infty(\mathbb{T}^d)$ we have the following equality of bounded operators on the Hilbert space $\mathrm{L}^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^{\lfloor \frac{d}{2} \rfloor}}$:*

$$(f - \sigma_\Lambda \circ \tau_\Lambda(f)) \otimes \mathbf{1}_{2^{\lfloor \frac{d}{2} \rfloor}} = \frac{i}{2}\left(\sum_{\mu=1}^{d} \mathcal{F}_{\mathfrak{n}_\Lambda^\mu} \otimes \{\gamma^\mu, \cdot\}\right)([D, f]),$$

*where $\mathcal{F}_{\mathfrak{n}_\Lambda^\mu}$ is the Fourier multiplier on the *-algebra $\mathrm{C}^\infty(\mathbb{T}^d)$ with symbol $\mathfrak{n}_\Lambda^\mu$.*

Similarly, for every $T \in \mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$, we have the following equality of bounded operators on the Hilbert space $P_\Lambda\mathrm{L}^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^{\lfloor \frac{d}{2} \rfloor}}$:

$$(T - \tau_\Lambda \circ \sigma_\Lambda(T)) \otimes \mathbf{1}_{2^{\lfloor \frac{d}{2} \rfloor}} = \frac{i}{2}\left(\sum_{\mu=1}^{d} \mathcal{S}_{\mathfrak{n}_\Lambda^\mu} \otimes \{\gamma^\mu, \cdot\}\right)([D_\Lambda, T]),$$

where $\mathcal{S}_{\mathfrak{n}^\mu_\Lambda}$ *denotes Schur (i.e. entrywise) multiplication on the operator system* $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ *with symbol* $\mathfrak{n}^\mu_\Lambda$.

Proof. For the first claim, we have:

$$\begin{aligned} f - \sigma_\Lambda \circ \tau_\Lambda(f) = f - \mathcal{F}_{\mathfrak{m}_\Lambda}(f) &= \sum_{n\in\mathbb{Z}^d} (1 - \mathfrak{m}_\Lambda(n))\widehat{f}(n)e_n \\ &= \sum_{\mu=1}^{d} \sum_{n\in\mathbb{Z}^d} (1 - \mathfrak{m}_\Lambda(n))\frac{n_\mu}{\|n\|^2} \cdot n_\mu \widehat{f}(n)e_n \\ &= \sum_{\mu,\nu=1}^{d} \mathcal{F}_{\mathfrak{n}^\mu_\Lambda}(-i\partial_\nu f) \cdot \delta^{\mu\nu}. \end{aligned}$$

The first claimed result then follows by writing $2\delta^{\mu\nu} \cdot \mathbf{1}_{2^{\lfloor \frac{d}{2} \rfloor}} = \{\gamma^\mu, \gamma^\nu\}$.

The computation for the second claim is largely analogous:

$$\begin{aligned} T - \tau_\Lambda \circ \sigma_\Lambda(T) &= T - \mathcal{S}_{\mathfrak{m}_\Lambda}(T) \\ &= \sum_{\mu=1}^{d} \left((1 - \mathfrak{m}_\Lambda(k-l))\frac{k_\mu - l_\mu}{\|k-l\|^2} \cdot (k_\mu - l_\mu)T_{k-l} \right)_{k,l\in\overline{\mathrm{B}}^{\mathbb{Z}}_\Lambda} \\ &= i \sum_{\mu,\nu=1}^{d} \mathcal{S}_{\mathfrak{n}^\mu_\Lambda} \left(((k_\nu - l_\nu)T_{k-l})_{k,l\in\overline{\mathrm{B}}^{\mathbb{Z}}_\Lambda} \right) \cdot \delta^{\mu\nu}. \end{aligned}$$

The result now follows by combining the defining relations for the gamma-matrices as before with the expression (3.5) for the operator $[D_\Lambda, T] \in \mathcal{B}(\mathrm{L}^2(S)_\Lambda)$. ☐

It is a classical result of Bożejko and Fendler [22] that for any discrete group Γ and function $u : \Gamma \to \mathbb{C}$ the cb-norm of Schur multiplication S_u on $\mathcal{B}(\ell^2(\Gamma))$ coincides with the cb-norm of Fourier multiplication $\mathcal{F}_u$ on $\mathrm{C}^*_\mathrm{r}(\Gamma)$ (see also [110, Theorem 6.4] and [23, Proposition D.6]). However, the two linear maps obtained in Lemma 3.3.4 act on the operator subsystems of differential forms on $\mathrm{L}^2(S)$ and $\mathrm{L}^2(S)_\Lambda$, respectively, so the result of Bożejko and Fendler does not apply directly. We prove a variation on it which relates the cb-norms of the two linear maps which appear in Lemma 3.3.4:

$$\mathcal{F}_{\mathfrak{n}_\Lambda} := \frac{i}{2} \sum_{\mu=1}^{d} \mathcal{F}_{\mathfrak{n}^\mu_\Lambda} \otimes \{\gamma^\mu, \cdot\} : [D, \mathrm{C}^\infty(\mathbb{T}^d)] \to \mathrm{C}^\infty(\mathbb{T}^d); \tag{3.7}$$

$$\mathcal{S}_{\mathfrak{n}_\Lambda} := \frac{i}{2} \sum_{\mu=1}^{d} \mathcal{S}_{\mathfrak{n}^\mu_\Lambda} \otimes \{\gamma^\mu, \cdot\} : [D_\Lambda, \mathrm{C}(\mathbb{T}^d)^{(\Lambda)}] \to \mathrm{C}(\mathbb{T}^d)^{(\Lambda)},$$

Here we consider $[D, \mathrm{C}^\infty(\mathbb{T}^d)]$ and $[D_\Lambda, \mathrm{C}(\mathbb{T}^d)^{(\Lambda)}]$ as (dense subsets of) operator subsystems of $\mathcal{B}(\mathrm{L}^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^{\lfloor \frac{d}{2} \rfloor}})$.

Lemma 3.3.5. *For the above two linear maps we have the following norm inequality:*

$$\|\mathcal{S}_{\mathfrak{n}_\Lambda}\| \leqslant \|\mathcal{F}_{\mathfrak{n}_\Lambda}\|$$

Proof. We vary on the proof given in [110, Theorem 6.4]. First identify the Hilbert spaces $\mathrm{L}^2(\mathbb{T}^d)\otimes\mathbb{C}^{2^{\lfloor\frac{d}{2}\rfloor}} \cong \ell^2(\mathbb{Z}^d)\otimes\mathbb{C}^{2^{\lfloor\frac{d}{2}\rfloor}}$ using the Fourier basis $\{e_n\}_{n\in\mathbb{Z}^d}$, and write $H := \ell^2(\mathbb{Z}^d)\otimes\mathbb{C}^{2^{\lfloor\frac{d}{2}\rfloor}}$. Set $V := \mathbb{C}^{2^{\lfloor\frac{d}{2}\rfloor}}$. Consider the unitary operator U defined on $H\otimes H$ by a combination of a shift in Fourier space and a tensor flip in spinor space:

$$U(e_n\otimes v\otimes e_m\otimes v') = e_n\otimes v'\otimes e_{n+m}\otimes v$$

Note that an elementary matrix $E_{kl}\in\mathrm{M}_{\mathbb{Z}^d}$ $(k,l\in\mathbb{Z}^d)$ acts on H as:

$$(E_{kl}\otimes\mathbf{I}^V)(e_n\otimes v) = \delta_{ln}e_k\otimes v,$$

in contrast to a generator e_k in the group C*-algebra $\mathrm{C}^*(\mathbb{Z}^d)\cong\mathrm{C}(\mathbb{T}^d)$, which acts as

$$(e_k\otimes\mathbf{I}^V)(e_n\otimes v) = e_{n+k}\otimes v.$$

Note furthermore that under the identification $\mathrm{L}^2(\mathbb{T}^d)\cong\ell^2(\mathbb{Z}^d)$ we have

$$D(e_n\otimes v) = ne_n\otimes\gamma^\mu v, \qquad (n\in\mathbb{Z}^d, v\in V). \tag{3.8}$$

We then find that

$$\begin{aligned} U(E_{kl}\otimes\mathbf{I}^V\otimes\mathbf{I}^H)U^* &= E_{kl}\otimes\mathbf{I}^V\otimes e_{k-l}\otimes\mathbf{I}^V \\ U(E_{kl}\otimes\gamma^\mu\otimes\mathbf{I}^H)U^* &= E_{kl}\otimes\mathbf{I}^V\otimes e_{k-l}\otimes\gamma^\mu \end{aligned}$$

where γ^μ acts of course on the spinor space V. Note that in view of Equation (3.8) we also have $U([D,E_{kl}\otimes\mathbf{I}^V]\otimes\mathbf{I}^H)U^* = E_{kl}\otimes\mathbf{I}^V\otimes[D,e_{k-l}\otimes\mathbf{I}^V]\in\mathcal{K}(\mathrm{L}^2(S))\otimes[D,\mathrm{C}^\infty(\mathbb{T}^d)]$. Moreover one readily checks that $[D,E_{kl}\otimes\mathbf{I}^V] = \sum_\mu(k_\mu-l_\mu)E_{kl}\otimes\gamma^\mu$.

Using this we may now show

$$\begin{aligned} &U\left((\mathcal{S}_{\mathfrak{n}_\Lambda}\otimes\mathbf{I}^{\mathcal{B}(\mathrm{L}^2(S))})([D,E_{kl}\otimes\mathbf{I}^V]\otimes\mathbf{I}^H)\right)U^* \\ &= \sum_\mu U\left((\mathcal{S}_{\mathfrak{n}_\Lambda}((k-l)_\mu E_{kl}\otimes\gamma^\mu))\otimes\mathbf{I}^H\right)U^* \\ &= i\sum_\mu U\left((\mathfrak{n}^\mu_\Lambda(k-l)(k-l)_\mu E_{kl}\otimes\mathbf{I}^V)\otimes\mathbf{I}^H\right)U^* \\ &= i\sum_\mu E_{kl}\otimes\mathbf{I}^V\otimes\mathfrak{n}^\mu_\Lambda(k-l)(k-l)_\mu e_{k-l}\otimes\mathbf{I}^V \\ &= E_{kl}\otimes\mathbf{I}^V\otimes\mathcal{F}_{\mathfrak{n}_\Lambda}([D,e_{k-l}\otimes\mathbf{I}^V]) \\ &= (\mathbf{I}^{\mathcal{B}(\mathrm{L}^2(S))}\otimes\mathcal{F}_{\mathfrak{n}_\Lambda})\left(U([D,E_{kl}\otimes\mathbf{I}^V]\otimes\mathbf{I}^H)U^*\right). \end{aligned}$$

This extends by linearity to arbitrary $x = \sum_{k,l\in\overline{\mathrm{B}}^{\mathbb{Z}^d}_\Lambda} t_{k-l}E_{kl}\otimes\mathbf{I}^V\in\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ to yield

$$U\left((\mathcal{S}_{\mathfrak{n}_\Lambda}\otimes\mathrm{id}_{\mathcal{B}(\mathrm{L}^2(S))})([D,x]\otimes\mathbf{I}^{\mathcal{H}})\right)U^* = (\mathbf{I}^{\mathcal{B}(\mathrm{L}^2(S))}\otimes\mathcal{F}_{\mathfrak{n}_\Lambda})\left(U([D,x]\otimes\mathbf{I}^H)U^*\right).$$

From this we obtain the following estimate:

$$\begin{aligned}
\|\mathcal{S}_{\mathfrak{n}_\Lambda}([D,x])\| &= \|\mathcal{S}_{\mathfrak{n}_\Lambda}([D,x]) \otimes \mathbf{I}^{\mathrm{L}^2(S)}\| \\
&= \|(\mathcal{S}_{\mathfrak{n}_\Lambda} \otimes \mathrm{id}_{\mathcal{B}(\mathrm{L}^2(S))})([D,x] \otimes \mathbf{I}^{\mathrm{L}^2(S)})\| \\
&= \|U\left((\mathcal{S}_{\mathfrak{n}_\Lambda} \otimes \mathbf{I}^{\mathcal{B}(\mathrm{L}^2(S))})([D,x] \otimes \mathbf{I}^H)\right)U^*\| \\
&= \|(\mathbf{I}^{\mathcal{B}(\mathrm{L}^2(S))} \otimes \mathcal{F}_{\mathfrak{n}_\Lambda})\left(U([D,x] \otimes \mathbf{I}^H)U^*\right)\| \\
&\leqslant \|\mathbf{I}^{\mathcal{K}(\mathrm{L}^2(S))} \otimes \mathcal{F}_{\mathfrak{n}_\Lambda}\|\|U([D,x] \otimes \mathbf{I}^H)U^*\| \\
&= \|\mathcal{F}_{\mathfrak{n}_\Lambda}\|_{\mathrm{cb}}\|[D,x]\|,
\end{aligned}$$

where the penultimate step follows from the fact that

$$U([D,x] \otimes \mathbf{I}^H)U^* \in \mathcal{K}(\mathrm{L}^2(S)) \otimes [D, \mathrm{C}^\infty(\mathbb{T}^d)].$$

This implies that $\|\mathcal{S}_{\mathfrak{n}_\Lambda}\| \leqslant \|\mathcal{F}_{\mathfrak{n}_\Lambda}\|_{\mathrm{cb}}$.

Finally, supposing that $\mathcal{F}_{\mathfrak{n}_\Lambda}$ is a bounded linear map its norm and cb-norm coincide because its range is a subset of a commutative C*-algebra, namely $\mathrm{C}(\mathbb{T}^d)$ (*cf.* [106, Theorem 3.9]). ☐

Our task is thus reduced to computing the norm of the map $\mathcal{F}_{\mathfrak{n}_\Lambda} : [D, \mathrm{C}^\infty(\mathbb{T}^d)] \to \mathrm{C}^\infty(\mathbb{T}^d)$ given in Equation (3.7).

3.3.3 The spectral Fejér kernel

We define $K_{\mathfrak{m}_\Lambda}$ as the convolution kernel corresponding to the Fourier multiplier $\mathcal{F}_{\mathfrak{m}_\Lambda}$ as in (3.6), i.e.

$$K_{\mathfrak{m}_\Lambda} := \check{\mathfrak{m}}_\Lambda.$$

In view of Lemma 3.2.2 it would be desirable to see that $K_{\mathfrak{m}_\Lambda}$ is a *good kernel*. Indeed, by Lemma 3.3.4 this would give precisely the estimate of $\mathcal{F}_{\mathfrak{n}_\Lambda}$ which remains to show.

Lemma 3.3.6. *For every $n \in \mathbb{Z}^d$, we have that $\mathfrak{m}_\Lambda(n) \to 1$, as $\Lambda \to \infty$.*

Proof. This follows from two simple geometric observations. One is that

$$|\mathcal{N}_{\mathrm{B}}(\Lambda) - \mathcal{V}_{\mathrm{B}}(\Lambda)| \leqslant \sqrt{d}\mathcal{A}_{\mathrm{B}}(\Lambda),$$

where $\mathcal{V}_{\mathrm{B}}(\Lambda)$ is the volume of the d-dimensional ball of radius Λ and $\mathcal{A}_{\mathrm{B}}(\Lambda)$ is its surface area. The other is that a lens (i.e. the intersection of two d-dimensional balls of radius Λ, one of them shifted by a parameter $n \in \mathbb{Z}^d$) contains the ball of radius $\Lambda - \|n\|$, if this number is non-negative. Together, these observations yield the following estimates:

$$|1 - \mathfrak{m}_\Lambda(n)| = \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)}|\mathcal{N}_{\mathrm{B}}(\Lambda) - \mathcal{N}_{\mathrm{L}}(\Lambda, n)|$$

$$
\begin{aligned}
&\leqslant \frac{1}{\mathcal{V}_{\mathrm{B}}(\Lambda-\sqrt{d})}|\mathcal{V}_{\mathrm{B}}(\Lambda+\sqrt{d})-\mathcal{V}_{\mathrm{B}}((\Lambda-\sqrt{d}-\|n\|)_+)| \\
&= \frac{1}{(\Lambda-\sqrt{d})^d}\underbrace{|(\Lambda+\sqrt{d})^d-((\Lambda-\sqrt{d}-\|n\|)_+)^d|}_{=\mathcal{O}(\Lambda^{d-1})} \\
&\to 0,
\end{aligned}
$$

as $\Lambda \to \infty$. (Here $t_+ := t$, if $t \geqslant 0$, and $t_+ := 0$, if $t < 0$, for $t \in \mathbb{R}$.) $\square$

Proposition 3.3.7. *The function $K_{\mathfrak{m}_\Lambda}$ is positive and is a* good kernel.

Proof. We begin by showing positivity of $K_{\mathfrak{m}_\Lambda}$. Indeed, $K_{\mathfrak{m}_\Lambda} = \check{\mathfrak{m}}_\Lambda$ is the (inverse) Fourier transform of a convolution-square:

$$
\begin{aligned}
\mathfrak{m}_\Lambda(n) &= \frac{\mathcal{N}_{\mathrm{L}}(\Lambda,n)}{\mathcal{N}_{\mathrm{B}}} \\
&= \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)}\sum_{k\in\mathbb{Z}^d}\chi_{\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}}}(k)\chi_{\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}}(n)}(k) \\
&= \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)}\left(\chi_{\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}}} * \chi_{\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}}}\right)(n)
\end{aligned} \tag{3.9}
$$

Next, we check the total mass of $K_{\mathfrak{m}_\Lambda}$:

$$
\begin{aligned}
\int_{\mathbb{T}^d} K_{\mathfrak{m}_\Lambda}(x)\,\mathrm{d}x &= \frac{1}{\mathcal{N}_{\mathrm{B}}(\Lambda)}\left(\chi_{\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}}} * \chi_{\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}}}\right)(0) \\
&= \frac{\mathcal{N}_{\mathrm{L}}(\Lambda,0)}{\mathcal{N}_{\mathrm{B}}(\Lambda)} = 1
\end{aligned}
$$

Last, we argue that the mass of $K_{\mathfrak{m}_\Lambda}$ becomes concentrated around 0 as $\Lambda \to \infty$. Fix some $\delta > 0$ and let $\varepsilon > 0$ be arbitrary. Let φ be a non-negative trigonometric polynomial such that the following holds:

$$1 - \chi_{\mathrm{B}_\delta} \leqslant \varphi \leqslant 1 - \chi_{\mathrm{B}_{\delta/2}} + \frac{\varepsilon}{2}$$

The existence of such a trigonometric polynomial φ follows from Weierstraß's approximation theorem for trigonometric polynomials since the $\frac{\varepsilon}{4}$-neighborhood of the continuous function ψ, given by

$$
\psi(x) = \begin{cases} \frac{\varepsilon}{4}, \ \text{if } x \in \mathrm{B}_{\frac{\delta}{2}}, \\ \frac{2}{\delta}\|x\| + \frac{\varepsilon}{4} - 1, \ \text{if } x \in \mathrm{B}_\delta \backslash \mathrm{B}_{\frac{\delta}{2}}, \\ 1 + \frac{\varepsilon}{4}, \ \text{if } \mathrm{T}^d \backslash \mathrm{B}_\delta, \end{cases}
$$

lies between $1 - \chi_{\mathrm{B}_\delta}$ and $1 - \chi_{\mathrm{B}_{\frac{\delta}{2}}} + \frac{\varepsilon}{2}$. Let $\Lambda_0 > 0$ be large enough such that, for all $\Lambda \geqslant \Lambda_0$, we have that $\max_{n\in\operatorname{supp}(\widehat{\varphi})}|1-\mathfrak{m}_\Lambda(n)| \leqslant \frac{\varepsilon}{2\|\widehat{\varphi}\|_{\ell^1(\mathbb{Z}^d)}}$. This is possible since the pointwise convergence from Lemma 3.3.6 implies uniform convergence to 1 of the restriction of $\mathfrak{m}_\Lambda$ to the finite set $\operatorname{supp}(\widehat{\varphi})$. Then the following holds:

$$\begin{aligned}
\int_{\mathbb{T}^d \setminus \mathrm{B}_\delta} K_{\mathfrak{m}_\Lambda}(x)\,\mathrm{d}x &= \int_{\mathbb{T}^d} K_{\mathfrak{m}_\Lambda}(x)(1-\chi_{\mathrm{B}_\delta}(x))\,\mathrm{d}x \\
&\leqslant \int_{\mathbb{T}^d} K_{\mathfrak{m}_\Lambda}(x)\varphi(x)\,\mathrm{d}x \\
&\leqslant \left| \int_{\mathbb{T}^d} K_{\mathfrak{m}_\Lambda}(x)\varphi(x)\,\mathrm{d}x - \varphi(0) \right| + \frac{\varepsilon}{2} \\
&= \left| \sum_{n\in\mathbb{Z}^d} \mathfrak{m}_\Lambda(n)\widehat{\varphi}(n) - \sum_{n\in\mathbb{Z}^d} \widehat{\varphi}(n) \right| + \frac{\varepsilon}{2} \\
&\leqslant \|(\mathfrak{m}_\Lambda - 1)\,|_{\operatorname{supp}(\widehat{\varphi})}\|_{\ell^\infty(\mathbb{Z}^d)} \cdot \|\widehat{\varphi}\|_{\ell^1(\mathbb{Z}^d)} + \frac{\varepsilon}{2} \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2},
\end{aligned}$$

for all $\Lambda \geqslant \Lambda_0$. In the fourth step we applied the Plancherel formula and in the fifth step the Hölder inequality. □

Remark 3.3.8. Note that (3.9) shows that the function $K_{\mathfrak{m}_\Lambda}$ is precisely the square of the well-known spherical Dirichlet kernel (*cf.* [62, Definition 3.1.6]). However, by a classical result by du Bois-Reymond the latter is not a *good kernel* (*cf.* [62, Proposition 3.3.5]). The feature of $K_{\mathfrak{m}_\Lambda}$ which is crucial for its good behavior is positivity. We emphasize that our *spectral Fejér kernel* does not coincide with the so-called *circular Fejér kernel* which is investigated in [62, Chapter 3]. Note in particular, that the *circular* Fejér kernel fails to be *good* in dimensions $d \geqslant 3$ which motivates the introduction of Bochner–Riesz summability methods.

Theorem 3.3.9. *Spectral truncations of* $\mathbb{T}^d$ *converge, for all* $d \geqslant 1$. *I.e. we have*

$$\operatorname{dist}^{\mathrm{s}}\Big((\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}, \|[D_\Lambda,\cdot]\|), (\mathrm{C}(\mathbb{T}^d), \|[D,\cdot]\|) \Big) \overset{\Lambda\to\infty}{\longrightarrow} 0.$$

Proof. From $(T - \tau_\Lambda \circ \sigma_\Lambda(T)) \otimes \mathbf{1}_{2^{\lfloor \frac{d}{2} \rfloor}} = \mathcal{S}_{\mathfrak{n}_\Lambda}(T)$ (Lemma 3.3.4) and $\|\mathcal{S}_{\mathfrak{n}_\Lambda}\| \leqslant \|\mathcal{F}_{\mathfrak{n}_\Lambda}\|$ (Lemma 3.3.5), together with $\mathcal{F}_{\mathfrak{n}_\Lambda}(f) = (f - \sigma_\Lambda \circ \tau_\Lambda(f)) \otimes \mathbf{1}_{2^{\lfloor \frac{d}{2} \rfloor}} = (f - K_{\mathfrak{n}_\Lambda} * f) \otimes \mathbf{1}_{2^{\lfloor \frac{d}{2} \rfloor}}$ (again Lemma 3.3.4), we obtain from Lemma 3.2.2 a sequence $\gamma_\Lambda \to 0$, as $\Lambda \to \infty$, such that

$$\begin{aligned}
\|T - \tau_\Lambda \circ \sigma_\Lambda(T)\| &\leqslant \gamma_\Lambda \|[D_\Lambda, T]\| \quad \text{and} \\
\|f - \sigma_\Lambda \circ \tau_\Lambda(f)\| &\leqslant \gamma_\Lambda \|[D, f]\|,
\end{aligned}$$

since the spectral Fejér kernel $K_{\mathfrak{m}_\Lambda}$ is a good kernel (Proposition 3.3.7). Together with Lemma 3.3.1 and Lemma 3.3.2 this shows that $(\tau_\Lambda, \sigma_\Lambda)$ is a C^1-approximate complete order isomorphism which by Proposition 2.3.21 implies our result. □

Remark 3.3.10. We point out that similarly convergence of other kinds of truncations of $\mathbb{T}^d$ can be shown. In particular, by replacing our projections P_Λ with the projections $P_N^\square$ with

$$\operatorname{ran}(P_N^\square) = \{e_n \mid n \in \mathbb{Z}^d, n_i = -N, \ldots, N, \text{ for } i = 1, \ldots, d\},$$

analogous arguments to the ones presented in this chapter yield a new proof that the truncations of $\mathbb{T}^d$ which were considered in [14], converge.

3.4 Minimal C^*-cover and propagation number of the operator system $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$

Recall the definitions of the minimal C*-cover and the propagation number from Section 2.2. For $p \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d} + \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$, we define the following operator in $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)} \subseteq \mathcal{B}(\mathrm{L}^2(S)_\Lambda)$:

$$T_p := \sum_{n \in \overline{L}_\Lambda^{\mathbb{Z}}(p)} E_{n-p,n} = \sum_{n \in \overline{L}_\Lambda^{\mathbb{Z}}(-p)} E_{n,n+p}, \tag{3.10}$$

where $E_{k,l} \in \mathcal{B}(\mathrm{L}^2(S)_\Lambda)$ is the *matrix unit* given by $\langle e_m, E_{k,l} e_n \rangle = \delta_{mk}\delta_{ln}$, for $k, l, m, n \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$. It is not hard to check that $\{T_p\}_{p \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d} + \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}}$ is a basis for the operator system $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$. We have collected some basic facts from convex geometry in Subsection 3.2.4. With these preparations we are in position to treat the minimal C*-cover and propagation number of the operator system $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$.

Proposition 3.4.1. *The minimal* C^**-cover and the propagation number of* $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ *are given by* $\mathrm{C}^*_{\min}(\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}) = \mathcal{B}(\mathrm{L}^2(S)_\Lambda)$ *and* $\operatorname{prop}(\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}) = 2$ *respectively.*

Proof. The matrix order structure on $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ is the one inherited from the inclusion into $\mathcal{B}(\mathrm{L}^2(S)_\Lambda)$. It remains to show that the inclusion $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)} \hookrightarrow \mathcal{B}(\mathrm{L}^2(S)_\Lambda)$ is a C*-extension, i.e. that the operator system $\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ generates $\mathcal{B}(\mathrm{L}^2(S)_\Lambda)$. Indeed, if this is the case, it is clear that $\mathcal{B}(\mathrm{L}^2(S)_\Lambda) \cong \mathrm{C}^*_{\min}(\mathrm{C}(\mathbb{T}^d)^{(\Lambda)})$ since $\mathcal{B}(\mathrm{L}^2(S)_\Lambda)$ is simple.

We will see that $\mathcal{B}(\mathrm{L}^2(S)_\Lambda)$ is in fact spanned by respective products of two basic operators (3.10). To this end, let $p, q \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d} + \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$. Then, the following holds:

$$\begin{aligned} T_p T_q &= \left(\sum_{n \in \overline{L}_\Lambda^{\mathbb{Z}}(p)} E_{n-p,n} \right) \left(\sum_{n \in \overline{L}_\Lambda^{\mathbb{Z}}(-q)} E_{n,n+q} \right) \\ &= \sum_{n \in \overline{L}_\Lambda^{\mathbb{Z}}(p) \cap \overline{L}_\Lambda^{\mathbb{Z}}(-q)} E_{n-p,n+q} \\ &= \sum_{n \in \overline{L}_\Lambda^{\mathbb{Z}}(p+q) \cap \overline{L}_\Lambda^{\mathbb{Z}}(q)} E_{n-p-q,n}, \end{aligned}$$

where we used the fact that $\big(\overline{L}_\Lambda(p) \cap \overline{L}_\Lambda(-q)\big) + q = \overline{L}_\Lambda(p+q) \cap \overline{L}_\Lambda(q)$ which can be easily checked. As a special case, for $l, k \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d} + \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$ such that $l+k \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d} + \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$, we obtain

$$T_{-k}T_{l+k} = \sum_{n \in \overline{L}_\Lambda^{\mathbb{Z}}(l) \cap \overline{L}_\Lambda^{\mathbb{Z}}(l+k)} E_{n-l,n}. \tag{3.11}$$

Note that this generalizes the formula given in the proof of [33, Proposition 4.2] where d was equal to 1. [1]

We need some elementary geometric observations. For $\Lambda' \geqslant 0$, let $K_{\Lambda'} := \mathrm{co}(\overline{\mathrm{B}}_{\Lambda'}^{\mathbb{Z}^d})$ denote the convex hull of the set of $\mathbb{Z}^d$-lattice points in the closed euclidean ball of radius Λ'. Note that $K_{\Lambda'}^{\mathbb{Z}^d} := K_{\Lambda'} \cap \mathbb{Z}^d = \overline{\mathrm{B}}_{\Lambda'}^{\mathbb{Z}^d}$. Furthermore, $K_{\Lambda'}$ is a polytope which is symmetric under reflections along coordinate axes and diagonals (i.e. under changing signs of coordinates and exchanging coordinates). Clearly, all the extreme points of $K_{\Lambda'}$, the set of which is denoted by $\mathrm{ex}(K_{\Lambda'})$, have integer coordinates. Moreover, if $x \in K_{\Lambda'}$ is of norm $\|x\| = \Lambda'$ it is an extreme point, but not necessarily all extreme points of $K_{\Lambda'}$ are of norm Λ' as can be seen in the case $d = 2$, $\Lambda' = 3$ (*cf.* Figure 3.1).

In order to prove the claim it is enough to write every rank-one operator $E_{p,q} \in \mathcal{B}(\mathrm{L}^2(S)_\Lambda)$ as a linear combination of products of the form (3.11), where $p, q \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$ and $l, k \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d} + \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$ such that $l + k \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d} + \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$. To this end, fix $p, q \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$ and set $l := q - p \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d} + \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$. Set $\Lambda' := \|q\|$. We claim that we can find an extreme point $m \in \mathrm{ex}(K_\Lambda)$ such that the following holds:

$$K_{\Lambda'} \cap (K_\Lambda - m + q) = \{q\}. \tag{3.12}$$

To see this, note that q is an extreme point of $K_{\Lambda'}$ and that the smallest cone which contains $K_{\Lambda'}$ (i.e. $[0,\infty) \cdot (K_{\Lambda'} - q) + q$) is locally compact (trivially in this finite-dimensional case), closed, convex, proper and has vertex q. By Corollary 3.2.6 we can find an extreme point $m \in \mathrm{ex}(K_\Lambda)$ such that

$$(K_\Lambda - m) \cap (K_{\Lambda'} - q) = (K_\Lambda - m) \cap ([0,\infty) \cdot (K_{\Lambda'} - q) + q) = \{0\},$$

which is equivalent to (3.12).

Now, fix an extreme point $m \in \mathrm{ex}(K_\Lambda)$ such that (3.12) is satisfied and set

$$k := q - l - m = p - m.$$

Note that for this l and k the product (3.11) makes sense, i.e. $k \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d} + \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$ and $l + k \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d} + \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$. Furthermore, the following holds:

$$\left(\overline{\mathrm{L}}_\Lambda^{\mathbb{Z}^d}(l) \cap \overline{\mathrm{L}}_\Lambda^{\mathbb{Z}^d}(l+k)\right) \cap K_{\Lambda'} = \{q\}$$

[1] Moreover, this formula may be interpreted in a similar way: The operator $T_{-k}T_{l+k}$ can be regarded as matrix (with multi-indexed entries) which has 0-entries everywhere except for the l-th diagonal where its entries are either 1 or 0 depending on the parameter k.

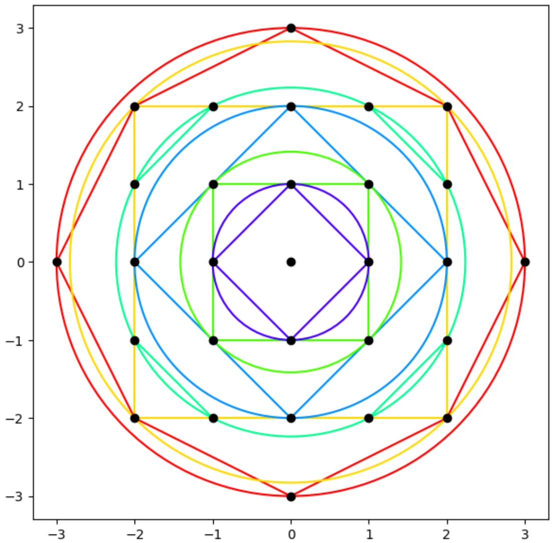

Figure 3.1: A plot of the boundaries of the balls B_Λ and the polytopes K_Λ, for $\Lambda = 0, 1, \sqrt{2}, 2, \sqrt{5}, 2\sqrt{2}, 3$.

The converse inclusion follows from (3.12) together with

$$\overline{\mathrm{L}}_\Lambda^{\mathbb{Z}^d}(l+k) \subseteq \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}(l+k) = K_\Lambda^{\mathbb{Z}^d} + q - m,$$

where $q - m = l + k$ was used.

This shows that, for $l = q - p$ and $k = p - m$, the rank-one operator $E_{p,q}$ is a summand of $T_{-k}T_{l+k}$ as in (3.11) where, for all the other rank-one operators $E_{n-l,n}$ appearing in the sum, we have that $\|n\| > \|q\|$, i.e.

$$E_{p,q} = T_{-k}T_{l+k} - \sum_{\substack{n \in \overline{\mathrm{L}}_\Lambda^{\mathbb{Z}^d}(l+k) \cap \overline{\mathrm{L}}_\Lambda^{\mathbb{Z}^d}(l) \\ \|n\| > \|q\|}} E_{n-l,n}. \tag{3.13}$$

Moreover, for each $E_{n-l,n}$ in the above sum, a similar expression can be obtained, and so forth. Hence, after finitely many steps this gives a finite linear combination of products of the form (3.11) for $E_{p,q}$.

Altogether, this proves that

$$\mathcal{B}(\mathrm{L}^2(S)_\Lambda) \subseteq \mathrm{span}\{T_{-k}T_{l+k} \mid l, k, l+k \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d} + \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}\}$$

which shows that $\mathrm{C}^*_{\min}(\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}) \cong \mathcal{B}(\mathrm{L}^2(S)_\Lambda)$ and $\mathrm{prop}(\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}) \leqslant 2$. Realizing that $E_{0,0} \notin \mathrm{C}(\mathbb{T}^d)^{(\Lambda)}$ it is clear that $\mathrm{prop}(\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}) > 1$. This finishes the proof. $\square$

We illustrate the procedure of expressing elementary matrices $E_{p,q} \in \mathcal{B}(\mathrm{L}^2(S)_\Lambda)$ in terms of products of basic operators of the form (3.11) as described in the above proof in the following two examples.

Example 3.4.2. Let $p, q \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$ such that $\|q\| = \Lambda$. Finding an $m \in \mathrm{ex}(K_\Lambda)$ such that (3.12) holds is particularly easy in this case, namely, set $m := -q$. Then set $k := p - m = p + q$ and, with $l = q - p$, we obtain:

$$E_{p,q} = T_{-k}T_{l+k} = T_{-p-q}T_{q-p+p+q} = T_{-p-q}T_{2q},$$

according to (3.13), since there are no $n \in \overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$ with $\|n\| > \|q\| = \Lambda$.

Example 3.4.3. Let $d = 2$ and $\Lambda = \sqrt{2}$, i.e. $\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d}$ consists of 9 points and $K_\Lambda = \mathrm{co}(\overline{\mathrm{B}}_\Lambda^{\mathbb{Z}^d})$ is the square with side length 2. For $p = (0, 0)$ and $q = (1, 0)$, we want to express the matrix unit $E_{p,q}$ as a linear combination of products of basic operators (3.11) with $l = q - p = (1, 0)$. Set $\Lambda' := \|q\| = 1$. Now, find an extreme point $m \in \mathrm{ex}(K_\Lambda)$ such that (3.12) holds. A valid choice is e.g. $m := (-1, -1)$. Set $k := p - m = (1, 1)$. Then we have:

$$\overline{\mathrm{L}}_\Lambda^{\mathbb{Z}^d}(l) \cap \overline{\mathrm{L}}_\Lambda^{\mathbb{Z}^d}(l + k) = \overline{\mathrm{L}}_{\sqrt{2}}^{\mathbb{Z}}(1, 0) \cap \overline{\mathrm{L}}_{\sqrt{2}}^{\mathbb{Z}}(2, 1) = \{(1, 0), (1, 1)\}$$

Therefore:

$$\begin{aligned} T_{-k}T_{l+k} &= T_{(-1,-1)}T_{(2,1)} \\ &= \sum_{n \in \overline{\mathrm{L}}_{\sqrt{2}}^{\mathbb{Z}}(1,0) \cap \overline{\mathrm{L}}_{\sqrt{2}}^{\mathbb{Z}}(2,1)} E_{n-(1,0),n} \\ &= \underbrace{E_{(0,0),(1,0)}}_{=E_{p,q}} + E_{(0,1),(1,1)} \end{aligned}$$

Note that $\|(1, 1)\| = \sqrt{2} > \|q\| = 1$.

By Example 3.4.2, we have $E_{(0,1),(1,1)} = T_{(-1,-2)}T_{(2,2)}$. Altogether, we obtain:

$$E_{(0,0),(1,0)} = T_{(-1,-1)}T_{(2,1)} - T_{(-1,-2)}T_{(2,2)}$$

Remark 3.4.4. We point out that analogously to the proof of Proposition 3.4.1 one can show that the $\mathrm{C}^*_{\mathrm{env}}(P_\Lambda^K \mathrm{C}(\mathbb{T}^d) P_\Lambda^K) = \mathcal{B}(P_\Lambda^K \mathrm{L}^2(S))$ and $\mathrm{prop}(P_\Lambda^K \mathrm{C}^\infty(\mathbb{T}^d) P_\Lambda^K) = 2$, where K is a convex compact subset of $\mathbb{R}^d$ which is symmetric with respect to reflections along the coordinate axes and diagonals and where $P_\Lambda^K \in \mathcal{B}(\mathrm{L}^2(S))$ is the orthogonal projection with $\mathrm{ran}(P_\Lambda^K) = \{e_n : n \in (\Lambda \cdot K) \cap \mathbb{Z}^d\}$. In particular, this shows that the propagation number of the operator system obtained from truncations which were considered in [14], is also 2.

Chapter 4

Peter–Weyl and Fourier truncations of compact quantum groups

This chapter is based on the preprint [93] which is accepted for publication. Section 4.4 was added.

4.1 Introduction

In this chapter we discuss convergence of truncations of coamenable compact quantum groups. Other than in Chapter 3, we do not truncate for the spectrum of a Dirac operator, but consider the set of finite-dimensional irreducible unitary corepresentations as spectral data. This allows us to generalize techniques used for Peter–Weyl truncations of compact metric groups [56]. We call the truncations under consideration in this chapter *Peter–Weyl truncations.*

Let us give a brief sketch of our line of argument for convergence of these truncations. Given a coamenable compact quantum group $\mathbb{G}$, we assume that its function algebra $A := \mathrm{C}(\mathbb{G})$ is equipped with a Lip-norm L_A to give it the structure of a compact quantum metric space. It is crucial that the Lip-norm is invariant for the left and right coactions by comultiplication of A on itself. The notion of (right) invariance was put forward in [97] and means that $L_A((\mu \otimes \mathbf{I}^A)\Delta(a)) \leqslant L_A(a)$, for all elements $a \in A$ and states $\mu \in \mathcal{S}(A)$. Let $P_\Lambda : \mathrm{L}^2(\mathbb{G}) \to \mathrm{L}^2(\mathbb{G})$ be a projection associated to the Peter–Weyl decomposition of the compact quantum group $\mathbb{G}$. Upon realizing that the comultiplication induces ergodic left and right coactions on the compression $A^{(\Lambda)} := P_\Lambda A P_\Lambda$, we apply one of the main results from [97] to obtain an induced bi-invariant Lip-norm on the operator system $A^{(\Lambda)}$. Note that we do not need to require that the projection P_Λ has finite rank.

Similarly as before, we aim at using the criterion from Proposition 2.3.21 to show complete Gromov–Hausdorff convergence of Peter–Weyl truncations. To this end, we propose for the map $\tau : A \to A^{(\Lambda)}$ the compression map $a \mapsto P_\Lambda a P_\Lambda$ and, similarly as in [56], for the map $\sigma : A^{(\Lambda)} \to A$ the slice map $x \mapsto (\phi \otimes \mathbf{I}^A)\alpha^\tau(x)$, for an appropriate choice of a state $\phi \in \mathcal{S}(A^{(\Lambda)})$, where $\alpha^\tau : A^{(\Lambda)} \to A^{(\Lambda)} \otimes A$ is the above-mentioned coaction induced by the comultiplication. Invariance of the

Lip-norms guarantees that these maps are morphisms of compact quantum metric spaces. Their compositions can be compared to the respective identity maps on A and $A^{(\Lambda)}$ in terms of Lip-norms, by a general argument about slice maps. A density result for states now is enough to satisfy the criterion in Proposition 2.3.21 and thus to prove our main theorem, Theorem 4.3.13, that Peter–Weyl truncations of coamenable compact quantum groups converge in complete Gromov–Hausdorff distance. We also explain in some detail how our result generalizes the one about compact groups from [56].

Peter–Weyl truncations are complementary to Fourier truncations [123]. In fact, the methods employed to show convergence of both are similar, and our preparations in Subsection 2.3.2 and Subsection 4.2.2 put us in position to slightly strengthen [123, Theorem 6.1] and obtain convergence of Fourier truncations of an ergodic coaction of a coamenable (not necessarily matrix) compact quantum group on an operator system (rather than a C*-algebra) in operator (rather than quantum) Gromov–Hausdorff distance.

We point out that the operator systems arising in these two approaches are quite different and, in particular, they are in general not dual, as discussed in more detail in Chapter 5, despite their duality in the case of the circle [33, 50].

The attentive reader will notice some analogy between the maps τ and σ employed in the proofs of our convergence results for tori and compact quantum groups, Theorem 3.3.9 and Theorem 4.3.13; in fact, in both cases τ is the compression map and σ is a slice map, i.e. convolution with an appropriate state. In this sense, we may consider the composition $\sigma \circ \tau$ from (4.12) as Fourier multiplication on the C*-algebra $\mathrm{C}(\mathbb{G})$, and the composition $\tau \circ \sigma$ from (4.13) as Schur multiplication on the operator system $P\mathrm{C}(\mathbb{G})P$.

It is not clear (to us) whether the induced Lip-norm L on $P\mathrm{C}(\mathbb{G})P$ given by

$$L(x) := \sup_{\phi \in \mathcal{S}(P\mathrm{C}(\mathbb{G})P)} L_A((\phi \otimes \mathbf{I})\alpha(x)) \tag{4.1}$$

coincides with $[PDP, \cdot]$ in the case that $\mathbb{G} = G$ is a compact Lie group with a specified spin-structure. Let us elaborate on this. Let D be the spin-Dirac operator on $\mathrm{L}^2(G)$. The group G acts on its function algebra $\mathrm{C}(G)$ by conjugation by the unitaries in $\mathcal{B}(\mathrm{L}^2(G))$ given by $U_g\xi(h) = \xi(g^{-1}h)$, and the Dirac operator is G-equivariant, i.e. $U_gD = DU_g$. Denote by $\tau : \mathrm{C}(G) \to P\mathrm{C}(G)P$ the compression of functions by a spectral projection P for D. Note that $\tau(U_gfU_g^*) = U_g\tau(f)U_g^*$ by G-equivariance of D, so the G-action on the C*-algebra $\mathrm{C}(G)$ induces a G-action on the operator system $P\mathrm{C}(G)P$. We compute the Lip-norm as in (4.1) which is induced by the Lip-norm $\|[D, \cdot]\|$ on $\mathrm{C}(G)$ and the G-action:

$$\begin{aligned}
&\sup_{\phi \in \mathcal{S}(P\mathrm{C}(G)P)} \|[D, g \mapsto \phi(U_gxU_g^*)]\| \\
&= \sup_{\phi \in \mathcal{S}(P\mathrm{C}(G)P)} \|[D, \int_G f(g^{-1}\cdot)\mathrm{d}\tau^*\phi(g)]\| \\
&= \sup_{\phi \in \mathcal{S}(P\mathrm{C}(G)P)} \|\int_G [D, f(g^{-1}\cdot)]\mathrm{d}\tau^*\phi(g)\|,
\end{aligned}$$

for all $f \in \mathrm{C}(G)$ with $\tau(f) = x$. This Lip-norm would be equivalent to the Lipschitz seminorm $\|[PDP, \cdot]\|$ on $P\mathrm{C}(G)P$ if the state spaces of the operator systems $P\mathrm{C}(G)P$ and $[PDP, P\mathrm{C}(G)P]$ coincided, but this is not obvious to us.

4.2 Preliminaries

4.2.1 Compact quantum groups

We consider compact quantum groups in the sense of Woronowicz [145, 146] and summarize some of their main properties, following the exposition in [104]. See also [88] for another standard reference which, however, takes a more (Hopf*-)algebraic approach.

For two C*-algebras A_1 and A_2, we denote by $A_1 \otimes A_2$ their minimal tensor product.

Definition 4.2.1. A *compact quantum group* is a pair (A, Δ), where A is a unital C*-algebra and $\Delta : A \to A \otimes A$ is the *comultiplication map*, i.e. a unital *-homomorphism which is *coassociative*, i.e.

$$(\mathbf{I}^A \otimes \Delta)\Delta = (\Delta \otimes \mathbf{I}^A)\Delta,$$

and such that the *Podleś density* (or *cancellation*) *property* is satisfied:

$$\overline{\mathrm{span}}((A \otimes \mathbf{1}_A)\Delta(A)) = A \otimes A = \overline{\mathrm{span}}((\mathbf{1}_A \otimes A)\Delta(A))$$

We think of the C*-algebra A as the "*function algebra*" $\mathrm{C}(\mathbb{G})$ of a (virtual) compact quantum group $\mathbb{G}$ and will (ab)use this terminology and notation throughout.

Notation 4.2.2. We use Sweedler notation for the comultiplication, i.e. we set

$$a_{(0)} \otimes a_{(1)} := \Delta(a),$$

for all $a \in A$. Coassociativity allows for an unambiguous use of the notations

$$(\mathbf{I}^A \otimes \Delta)\Delta(a) = a_{(0)} \otimes a_{(1)} \otimes a_{(2)} = a_{(-1)} \otimes a_{(0)} \otimes a_{(1)} := (\Delta \otimes \mathbf{I}^A)\Delta(a).$$

If ϕ and ψ are any cb maps with domain A we set

$$a_{(0)} \otimes \phi(a_{(1)}) := (\mathbf{I}^A \otimes \phi)\Delta(a), \text{ and } \psi(a_{(0)}) \otimes a_{(1)} := (\psi \otimes \mathbf{I}^A)\Delta(a),$$

and, by Lemma 2.2.6, we may unambiguously write

$$\psi(a_{(0)}) \otimes \phi(a_{(1)}) := (\psi \otimes \phi)\Delta(a).$$

If ϕ or ψ are functionals, we may of course omit the tensor product "$\otimes$" in this notation.

Fix a compact quantum group $\mathbb{G}$ with function algebra $A = \mathrm{C}(\mathbb{G})$ and comultiplication $\Delta : A \to A \otimes A$.

Definition 4.2.3. A *unitary (right) corepresentation* π of the compact quantum group $\mathbb{G}$ is given by a Hilbert space H_π and a unitary element $U^\pi \in \mathcal{M}(\mathcal{K}(H_\pi) \otimes A)$, such that

$$(\mathbf{I} \otimes \Delta)(U^\pi) = U^\pi_{12} U^\pi_{13}. \tag{4.2}$$

If the Hilbert space H_π is finite-dimensional, the corepresentation π is called *finite-dimensional* and we set $\dim(\pi) := \dim(H_\pi)$.

In (4.2) above, the *-homomorphism

$$\mathbf{I} \otimes \Delta : \mathcal{M}(\mathcal{K}(H_\pi) \otimes A) \to \mathcal{M}(\mathcal{K}(H_\pi) \otimes A \otimes A)$$

denotes the unique extension of the map $\mathbf{I}^{\mathcal{K}(H_\pi)} \otimes \Delta$ on $\mathcal{K}(H_\pi) \otimes A$. Recall that there are two canonical embeddings of $\mathcal{M}(\mathcal{K}(H_\pi) \otimes A)$ into $\mathcal{M}(\mathcal{K}(H_\pi) \otimes A \otimes A)$, which are given by the unique extensions of the maps $\mathcal{K}(H_\pi) \otimes A \to \mathcal{K}(H_\pi) \otimes A \otimes A$ defined by $T \otimes a \mapsto T \otimes a \otimes \mathbf{1}_A$ and $T \otimes a \mapsto T \otimes \mathbf{1}_A \otimes a$ respectively. The elements $U^\pi_{12}, U^\pi_{13} \in \mathcal{M}(\mathcal{K}(H_\pi) \otimes A \otimes A)$ denote the respective images of U^π under these two canonical embeddings. See also [98] for more background on this definition. Note that in the finite-dimensional case $\mathcal{M}(\mathcal{K}(H_\pi) \otimes A) = \mathcal{B}(H_\pi) \otimes A$.

Every finite-dimensional unitary corepresentation π induces an *isometric comodule map* $\delta^\pi : H_\pi \to H_\pi \otimes A$, given by $\delta^\pi(\xi) := U^\pi(\xi \otimes \mathbf{1}_A)$. Being a comodule map means that δ^π satisfies the *comodule property*

$$(\mathbf{I}^H \otimes \Delta)\delta^\pi = (\delta^\pi \otimes \mathbf{I}^A)\delta^\pi,$$

and being *isometric* means

$$\delta^\pi(\xi)^* \delta^\pi(\eta) = \langle \xi, \eta \rangle_{H_\pi} \mathbf{1}_A,$$

for all vectors $\xi, \eta \in H_\pi$. Conversely, every isometric comodule map $\delta : H \to H \otimes A$ on a finite-dimensional Hilbert space H gives rise to a finite-dimensional unitary corepresentation [43, Lemma 1.7].

An *intertwiner* of two finite-dimensional unitary corepresentations π, ρ is an operator $T : H_\pi \to H_\rho$ such that $(T \otimes \mathbf{1}_A)U^\pi = U^\rho(T \otimes \mathbf{1}_A)$. The set of all intertwiners of the corepresentations π and ρ is denoted by $\mathrm{Mor}(\pi, \rho)$. If the set $\mathrm{Mor}(\pi, \rho)$ contains a unitary element, the corepresentations π and ρ are called *unitarily equivalent*. The set $\mathrm{End}(\pi) := \mathrm{Mor}(\pi, \pi)$ is a C*-algebra and the corepresentation π is called *irreducible* if $\mathrm{End}(\pi) = \mathbb{C}\mathbf{I}^{H_\pi}$. *Schur's lemma* states that two finite-dimensional irreducible unitary corepresentations π, ρ are either unitarily equivalent and $\dim(\mathrm{Mor}(\pi, \rho)) = 1$, or that $\dim(\mathrm{Mor}(\pi, \rho)) = 0$. We denote the set of unitary equivalence classes of finite-dimensional irreducible unitary corepresentations by $\widehat{\mathbb{G}}$.

There is a unique left and right invariant state $h_A \in \mathcal{S}(A)$, i.e. a state which satisfies

$$a_{(0)} h_A(a_{(1)}) = h_A(a_{(0)}) a_{(1)} = h_A(a) \mathbf{1}_A,$$

for all elements $a \in A$. The state $h_A \in \mathcal{S}(A)$ is called the *Haar state* of the compact quantum group $\mathbb{G}$.

Denote by $H := \mathrm{L}^2(\mathbb{G})$ the Hilbert space of the GNS-representation $\pi_{h_A} : A \to \mathcal{B}(H)$ of the C*-algebra A induced by the Haar state h_A. Denote the GNS-map by $\Lambda : A \to H$. Assume moreover, that the C*-algebra A is faithfully represented on a Hilbert space H_0 and denote the inclusion of A into $\mathcal{B}(H_0)$ by ι. There are two unitary operators $W \in \mathcal{M}(\mathcal{K}(H) \otimes A)$ and $V \in \mathcal{M}(A \otimes \mathcal{K}(H))$ which satisfy

$$\begin{aligned} W(\Lambda(a) \otimes \xi) &= (\pi_{h_A} \otimes \iota)(\Delta(a))(\Lambda(\mathbf{1}_A) \otimes \xi), \\ V(\xi \otimes \Lambda(a)) &= (\iota \otimes \pi_{h_A})(\Delta(a))(\xi \otimes \Lambda(\mathbf{1}_A)), \end{aligned}$$

for all elements $a \in A$ and $\xi \in H_0$. The unitaries W and V define unitary (respectively right and left) corepresentations of the compact quantum group $\mathbb{G}$ and are usually referred to as the *multiplicative unitaries*. In particular, they implement the comultiplication Δ as follows:

$$\begin{aligned} W(\pi_{h_A}(a) \otimes \mathbf{1}_{\mathcal{B}(H_0)})W^* &= (\pi_{h_A} \otimes \iota)\Delta(a), \\ V(\mathbf{1}_{\mathcal{B}(H_0)} \otimes \pi_{h_A}(a))V^* &= (\iota \otimes \pi_{h_A})\Delta(a), \end{aligned}$$

for all elements $a \in A$. For more details, see [104, Section 1.5], [88, Section 11.3.6].

For a finite-dimensional unitary corepresentation π and vectors $\xi, \eta \in H_\pi$, we denote by $\omega^\pi_{\xi,\eta}$ the functional on $\mathcal{K}(H_\pi)$ given by $T \mapsto \langle \xi, T\eta \rangle_{H_\pi}$. Then the elements $(\omega^\pi_{\xi,\eta} \otimes \mathbf{I}^A)(U^\pi) \in A$, for $\xi, \eta \in H_\pi$, are called the *matrix coefficients* of the corepresentation π. Denote by $\mathcal{A} = \mathcal{O}(\mathbb{G})$ the linear span of all the matrix coeffients of all finite-dimensional irreducible unitary corepresentations. The set $\mathcal{A}$ is a *Hopf*-algebra*, i.e. a unital *-algebra with a coassociative *comultiplication* map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, an *antipode* $S : \mathcal{A} \to \mathcal{A}$ and a *counit* $\epsilon : \mathcal{A} \to \mathbb{C}$, which satisfy $S(a_{(0)})a_{(1)} = a_{(0)}S(a_{(1)}) = \epsilon(a)\mathbf{1}_{\mathcal{A}}$ and the *counit property*

$$\epsilon(a_{(0)})a_{(1)} = a_{(0)}\epsilon(a_{(1)}) = a,$$

for all $a \in \mathcal{A}$. The Hopf*-algebra $\mathcal{A}$ is dense in the C*-algebra A and its unit and comultiplication are those inherited from A. We call $\mathcal{A}$ the *coordinate algebra* of the compact quantum group $\mathbb{G}$.

There is a quantum group version of *Peter–Weyl theory* which is crucial for our purposes. It states that every unitary corepresentation decomposes into a direct sum of finite-dimensional irreducible unitary corepresentations [104, Theorem 1.5.4]. For the multiplicative unitaries W, V this gives an orthogonal decomposition of the GNS Hilbert space

$$H = \bigoplus_{\gamma \in \widehat{\mathbb{G}}} H_\gamma \otimes H_\gamma^*, \tag{4.3}$$

which is respected by the multiplicative unitaries. I.e. W and V restrict to a unitary operators on $H_\gamma \otimes H_\gamma^* \otimes H_0$ and $H_0 \otimes H_\gamma \otimes H_\gamma^*$ respectively, for all finite-dimensional irreducible unitary corepresentations $\gamma \in \widehat{\mathbb{G}}$. More concretely, for any finite-dimensional irreducible unitary corepresenation γ, one can define a bilinear map $\beta : H_\gamma \times H_\gamma^* \to H$, given by $(\xi, \eta^*) \mapsto \dim_{\mathrm{q}}(\gamma)^{1/2}(\omega^\gamma_{Q_\gamma^{1/2}\eta,\xi} \otimes \Lambda)(U^\gamma)$, *cf.* the end of Section 1.5 of [104]. Then, for a fixed vector $\eta \in H_\gamma$, the induced linear map $\ell_\eta^\gamma := \beta(\cdot, \eta^*) : H_\gamma \to H$ intertwines the corepresentations γ and W. If $\eta \in H_\gamma$ is a unit vector, the map ℓ_η^γ is an isometry, and if the vectors $\eta, \eta' \in H_\gamma$ are orthogonal, so are the images of the maps ℓ_η^γ and $\ell_{\eta'}^\gamma$. The span of the images $\ell_\eta^\gamma(\xi)$, for all vectors $\xi, \eta \in H_\gamma$ and all finite-dimensional irreducible unitary corepresentations $\gamma \in \widehat{\mathbb{G}}$, in the GNS space H is equal to the image of the coordinate algebra under the GNS map, so a dense subspace of H [104, Corollary 1.5.5].

The function algebra A of the compact quantum group $\mathbb{G}$ can come in different versions. On the one hand, the *universal* function algebra $A_\mathrm{u} = \mathrm{C_u}(\mathbb{G})$ is given by the universal C*-completion of the coordinate algebra $\mathcal{A}$ and the comultipliation $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ extends to a *-homomorphism $A_\mathrm{u} \to A_\mathrm{u} \otimes A_\mathrm{u}$ which is still coassociative and satisfies the Podleś density property. Also the counit $\epsilon : \mathcal{A} \to \mathbb{C}$ extends to a bounded map $A_\mathrm{u} \to \mathbb{C}$ satisfying the counit property. On the other hand, the *reduced* function algebra $A_\mathrm{r} = \mathrm{C_r}(\mathbb{G})$ is given by the image $\pi_{h_A}(A) \subseteq \mathcal{B}(H)$ under the GNS representation. The cyclic vector ξ_{h_A} for the GNS representation of the function algebra A induces a bi-invariant state $\langle \xi_{h_A}, \cdot \xi_{h_A} \rangle$ on the reduced function algebra A_r which is still called the *Haar state* and which turns out to be faithful. Moreover, the Haar state h_A is faithful on the coordinate algebra $\mathcal{A}$, so that we may regard the reduced function algebra A_r as a completion of the coordinate algebra. In particular, the comultiplication map on $\mathcal{A}$ extends to A_r. The universal and reduced function algebra come with *-homomorphisms

$$A_\mathrm{u} \xrightarrow{\pi_\mathrm{u}} A \xrightarrow{\pi_\mathrm{r}} A_\mathrm{r},$$

which extend the identity maps on the coordinate algebra $\mathcal{A}$. The existence of the *-homomorphism π_u is guaranteed by universality of the C*-algebra A_u and the *-homomorphism π_r is the GNS representation.

In general, neither the counit $\epsilon : \mathcal{A} \to \mathbb{C}$ extends to a bounded map on the reduced function algebra A_r, nor does the Haar state h_A extend to a faithful state on the universal function algebra A_u. It turns out, however, that both is true if and only if the *-homomorphism $\pi_\mathrm{r} \circ \pi_\mathrm{u} : A_\mathrm{u} \to A_\mathrm{r}$ is an isomorphism [12]. In that case the compact quantum group $\mathbb{G}$ is called *coamenable*.

We end this section by noting that the dual A^* can be given an algebra structure as follows:

$$\mu * \nu(a) := (\mu \otimes \nu)\Delta(a),$$

for all functionals $\mu, \nu \in A^*$ and elements $a \in A$. This restricts to a semigroup structure on the state space $\mathcal{S}(A)$. If the counit $\epsilon : A \to \mathbb{C}$ is bounded (in particular, if compact quantum group $\mathbb{G}$ is coamenable), the counit is a state on A and it is the unit for the convolution product $*$.

4.2.2 Coactions

For operator systems X and Y, we denote by $X \otimes Y$ their minimal tensor product. Since we take an operator system point of view throughout, we collect the relevant notions of coactions on operator systems. Much of this is algebraic in nature, i.e. it can be seen in the setting of coactions on order-unit spaces. See [128] for this point of view. Coactions of compact quantum groups à la Woronowicz go back to Podleś [112, 113] and a thorough treatment can be found in [43].

For the theory of coactions on operator systems, we follow [45, 44]. In this subsection, we fix a compact quantum group $\mathbb{G}$ and denote its reduced function algebra by $A := \mathrm{C}_{\mathrm{r}}(\mathbb{G})$ and the comultiplication by $\Delta : A \to A \otimes A$. We furthermore fix an operator system X.

Definition 4.2.4. A *right coaction* α of the function algebra A on the operator system X is a unital complete order embedding $\alpha : X \to X \otimes A$ such that the *coaction property*

$$(\alpha \otimes \mathbf{I}^A)\alpha = (\mathbf{I}^X \otimes \Delta)\alpha \tag{4.4}$$

and the Podleś density condition

$$\overline{\mathrm{span}}((\mathbf{1}_X \otimes A)\alpha(X)) = X \otimes A$$

are satisfied. A *left coaction* $\beta : X \to A \otimes X$ is defined analogously.

We say that a right coaction α and a left coaction β *cocommute* if the following holds:

$$(\beta \otimes \mathbf{I}^A)\alpha = (\mathbf{I}^A \otimes \alpha)\beta \tag{4.5}$$

Notation 4.2.5. We use Sweedler notation whenever convenient, i.e. for an element $x \in X$, we write

$$x_{(0)} \otimes x_{(1)} := \alpha(x) \in X \otimes A,$$

as well as

$$x_{(0)} \otimes x_{(1)} \otimes x_{(2)} \in X \otimes A \otimes A,$$

for any of the two maps in (4.4) applied to x. Similarly,

$$x_{(-1)} \otimes x_{(0)} \otimes x_{(1)} \in A \otimes X \otimes A,$$

for any of the two maps in (4.5) applied to x.

Remark 4.2.6. Coactions on operator systems generalize reduced coactions of the reduced function algebra on C*-algebras. Indeed, a reduced (C*-algebraic) coaction $\alpha : B_{\mathrm{r}} \to B_{\mathrm{r}} \otimes A_{\mathrm{r}}$, with A_{r} the reduced function algebra of the compact quantum group $\mathbb{G}$, is an injective *-homomorphism [97, Proposition 3.4]. By [18, Corollary II.2.2.9] it follows that the map α is an isometry and arguing similarly for the matrix amplifications $\alpha^{(n)} = \alpha \otimes \mathbf{1}_n$, it follows that α is unital completely isometric, whence a unital complete order embedding.

In particular, if the compact quantum group $\mathbb{G}$ is coamenable, every C*-algebraic coaction of its function algebra on a unital C*-algebra is a coaction in the operator system sense. The comultiplication $\Delta : A \to A \otimes A$ on the reduced function algebra A is an example of both, a right and left coaction of A on the operator system A.

Conversely, if $\alpha : X \to X \otimes A$ is a coaction in the operator system sense and if X is a unital C*-algebra, the map α is a *-homomorphism [45, Proposition 3.7] and hence a C*-algebraic coaction.

For the remainder of this subsection, we fix a right coaction $\alpha : X \to X \otimes A$.

Remark 4.2.7. The coaction α gives the dual X^* the structure of a right module for the convolution algebra $(A^*, *)$, which we denote as follows:

$$\phi \lhd \mu(x) := (\phi \otimes \mu)\alpha(x) = \phi(x_{(0)})\mu(x_{(1)}),$$

for all elements $x \in X$ and functionals $\phi \in X^*$, $\mu \in A^*$. The right action $\lhd$ restricts to a right action of the semigroup $(\mathcal{S}(A), *)$ on $\mathcal{S}(X)$.

If Y is another operator system with a coactions $\beta : Y \to Y \otimes A$ and if there is a ucp onto map $\Phi : X \to Y$ which is equivariant for the coaction α and β, i.e.

$$(\Phi \otimes \mathbf{I}^A)\alpha = \beta\Phi,$$

then, for all states $\psi \in \mathcal{S}(Y)$, $\mu \in \mathcal{S}(A)$, we have

$$\psi(\Phi(x_{(0)}))\mu(x_{(1)}) = \psi(\Phi(x)_{(0)})\mu(\Phi(x)_{(1)}),$$

for all elements $x \in X$. In other words,

$$(\Phi^*\psi) \lhd \mu = \Phi^*(\psi \lhd \mu) \in \mathcal{S}(X),$$

where $\Phi^* : Y^* \to X^*$ denotes the pullback map which restricts to a map between the state spaces.

A convenient feature of the requirement that coactions on operator systems be unital complete order embeddings (rather than just ucp maps) is the following:

Lemma 4.2.8. *Assume that the compact quantum group $\mathbb{G}$ is coamenable. Then the* counit property *also holds for the coaction α, i.e.*

$$(\mathbf{I}^X \otimes \epsilon)\alpha = \mathbf{I}^X,$$

or, in Sweedler notation,

$$x_{(0)}\epsilon(x_{(1)}) = x,$$

for all elements $x \in X$.

Proof. From the counit and coaction properties, together with the Fubini theorem Lemma 2.2.6, we obtain

$$\alpha(\mathbf{I}^X \otimes \epsilon)\alpha = (\mathbf{I}^X \otimes \mathbf{I}^X \otimes \epsilon)(\alpha \otimes \mathbf{I}^A)\alpha = (\mathbf{I}^X \otimes \mathbf{I}^X \otimes \epsilon)(\mathbf{I}^X \otimes \Delta)\alpha = \alpha.$$

Since the map α is a unital complete order embedding, it is in particular injective, so the claim follows. ☐

Definition 4.2.9. A *fixed point* for the coaction α is an element $x \in X$ which satisfies

$$x_{(0)} \otimes x_{(1)} = x \otimes \mathbf{1}_A.$$

The set of fixed points is denoted by $X^\alpha := \{x \in X \mid \alpha(x) = x \otimes \mathbf{1}_A\}$.

The coaction α is called *ergodic* if its only fixed points are multiples of the unit, i.e. $X^\alpha = \mathbb{C}\mathbf{1}_X$.

Example 4.2.10. The comultiplication $\Delta : A \to A \otimes A$ is an ergodic coaction.

Definition 4.2.11. Let π be a finite-dimensional unitary corepresentation of the compact quantum group $\mathbb{G}$. An *intertwiner* of π and the coaction α is a linear map $T : H_\pi \to X$ such that

$$\alpha T = (T \otimes \mathbf{I}^A)\delta^\pi,$$

where the map $\delta^\pi : H_\pi \to H_\pi \otimes A$ is the isometric comodule map associated to the corepresentation π by $\delta^\pi(\xi) := U^\pi(\xi \otimes \mathbf{1}_A)$. The set of all intertwiners of π and α is denoted by $\mathrm{Mor}(\pi, \alpha)$.

For the corepresentation π, the *isotypical component* is defined by

$$X^\pi := \{T\xi \in X \mid T \in \mathrm{Mor}(\pi, \alpha), \xi \in H_\pi\} \subseteq X.$$

See Example 4.2.12 for consistency of the notation X^π and X^α. We denote the linear span of all isotypical components in X of finite-dimensional irreducible unitary corepresentations by

$$\mathcal{X} := \sum_{\gamma \in \widehat{\mathbb{G}}} X^\gamma \subseteq X.$$

The set $\mathcal{X}$ is called the *algebraic core* of the operator system X for the coaction α.

Example 4.2.12. The isotypical component $X^{\mathbf{1}}$ for the trivial corepresentation $\mathbf{1} = \mathbf{1}_A \in \mathcal{B}(\mathbb{C}) \otimes A$ coincides with the set of fixed points X^α. Indeed, by definition, a linear map $T : \mathbb{C} \to X$ is an intertwiner of the corepresentation $\mathbf{1}$ and the coaction α if and only if $\alpha(T(\lambda)) = (T \otimes \mathbf{I}^A)\mathbf{1}(\lambda)$, for all $\lambda \in H_{\mathbf{1}} = \mathbb{C}$, which we may rewrite as $(T(\lambda))_{(0)} \otimes (T(\lambda))_{(1)} = T(\lambda) \otimes \mathbf{1}_A$. Hence $T \in \mathrm{Mor}(\mathbf{1}, \alpha)$ if and only if $T(\lambda) \in X$ is a fixed point for the coaction α.

Remark 4.2.13. For $X = A$, $\alpha = \Delta$ and π a finite-dimensional irreducible unitary corepresentation, the isotypical component A^π coincides with the *coalgebra* $\mathcal{A}^\pi$ spanned by the matrix coefficients of π, see e.g. [43]. In fact, given a matrix coefficient $(\omega^\pi_{\xi,\eta} \otimes \mathbf{I}^A)(U^\pi) \in \mathcal{A}^\pi$, we may define $T_\eta : H_\pi \to A$ by $T_\eta \xi := (\omega^\pi_{\xi,\eta} \otimes \mathbf{I}^A)(U^\pi)$. We check that $T_\eta \in \mathrm{Mor}(\pi, \Delta)$:

$$\begin{aligned}
\Delta(T_\eta \xi) &= \Delta((\omega^\pi_{\xi,\eta} \otimes \mathbf{I}^A)(U^\pi)) \\
&= (\omega^\pi_{\xi,\eta} \otimes \mathbf{I}^A \otimes \mathbf{I}^A)(\mathbf{I}^{\mathcal{B}(H_\pi)} \otimes \Delta)(U^\pi) \\
&= (\omega^\pi_{\xi,\eta} \otimes \mathbf{I}^A \otimes \mathbf{I}^A)(U^\pi_{12} U^\pi_{13}) \\
&= (T_\eta \otimes \mathbf{I}^A) U^\pi(\xi \otimes \mathbf{1}_A) \\
&= (T_\eta \otimes \mathbf{I}^A)\delta_\pi(\xi),
\end{aligned}$$

for all $\xi \in H_\pi$, by (4.2), showing that $T_\eta \xi \in \mathrm{Mor}(\pi, \Delta) H_\pi = A^\pi$ as claimed.

Conversely, $T\xi \in A^\pi$ with $T \in \mathrm{Mor}(\pi, \Delta)$ an intertwiner and $\xi \in H_\pi$. We need to show that $T\xi$ is a matrix coefficient of π, i.e. $T\xi \in \mathcal{A}^\pi$. To this end, as in (3.1) of [43] one can find an element $\chi_\pi \in \mathcal{A}^\pi$ such that $h_A(((\omega^\pi_{\xi,\eta} \otimes \mathbf{I}^A)(U^\pi))\chi^*_\pi) = \langle \xi, \eta \rangle$, for all $\xi, \eta \in H_\pi$. Note that $\chi_\pi \in \mathcal{A}^\pi$ implies that $h_A(a\chi_\pi^*) = 0$, for all $a \in \mathcal{A}^{\pi'}$, $\pi' \in \widehat{\mathbb{G}}\backslash\{\pi\}$, by the orthogonality relations [104, Theorem 1.4.3]. Given $\xi \in H_\pi$, we obtain

$$\begin{aligned} T\xi &= T(\eta^* \mapsto \langle \eta, \xi \rangle) \\ &= (T \otimes \mathbf{I}^A)\left(\eta \mapsto h_A(((\omega^\pi_{\eta,\xi} \otimes \mathbf{I}^A)(U^\pi))\chi_\pi^*)\right) \\ &= (\mathbf{I}^A \otimes h_A)\left((T \otimes \mathbf{I}^A)(\delta_\pi(\xi))(\mathbf{1}_A \otimes \chi_\pi^*)\right) \\ &= (\mathbf{I}^A \otimes h_A)\left(\Delta(T\xi)(\mathbf{1}_A \otimes \chi_\pi^*)\right) \in \mathcal{A}^\pi. \end{aligned}$$

In the second line we used the property of χ_π mentioned above, in the third line we used the identification $\delta_\pi(\xi) = (\omega^\pi_{\cdot,\xi} \otimes \mathbf{I}^A)(U^\pi)$, and in the last line we applied the assumption that $T \in \mathrm{Mor}(\pi, \Delta)$.

Altogether, we have shown that $A^\pi = \mathcal{A}^\pi$, for all $\pi \in \widehat{\mathbb{G}}$. The fact that $\mathcal{A}^\pi$ is indeed a coalgebra now follows from part (4) of Lemma 4.2.16 below.

Definition 4.2.14. A state $\phi \in \mathcal{S}(X)$ is called *invariant* for the coaction α if

$$\phi \lhd \mu = \mu(\mathbf{1}_A)\phi,$$

for all functionals $\mu \in A^*$.

Lemma 4.2.15. *The following properties hold:*

(1) The set of fixed points X^α is an operator subsystem of X.

(2) The map $E_\alpha : X \to X^\alpha$ given by $E_\alpha(x) = x_{(0)}h_A(x_{(1)})$, for all $x \in X$, is a ucp conditional expectation.

(3) If the coaction α is ergodic, there is a unique invariant state h_X on X defined by

$$E_\alpha(x) = h_X(x)\mathbf{1}_X, \tag{4.6}$$

for all $x \in X$.

For parts (2) and (3) of the proof, we follow the arguments in [20, Lemma 4].

Proof. (1) Note that the unit $\mathbf{1}_X$ is a fixed point for the coaction α by unitality of α. Moreover, the set of fixed points X^α is $*$-closed, since $\alpha(x^*) = \alpha(x)^* = x^* \otimes \mathbf{1}_A$, for $x \in X^\alpha$. This shows that X^α is an operator system.

(2) Note that E_α is a ucp map being the composition of the unital complete order embedding α and the ucp map $\mathbf{I}^X \otimes h_A$. If $x \in X^\alpha$ is a fixed point, we have $E_\alpha(x) = x_{(0)}h_A(x_{(1)}) = xh(\mathbf{1}_A) = x\alpha$. For $x \in X$ we obtain by invariance of the Haar state h_A,

$$\begin{aligned} \alpha(E_\alpha(x)) &= \alpha\left((\mathbf{I}^X \otimes h_A)\alpha(x)\right) \\ &= x_{(0)} \otimes x_{(1)}h_A(x_{(2)}) \\ &= x_{(0)} \otimes h_A(x_{(1)})\mathbf{1}_A \end{aligned}$$

$$= E_\alpha(x) \otimes \mathbf{1}_A,$$

which shows that $E_\alpha(X) \subseteq X^\alpha$.

(3) By ergodicity, the range of E_α is $X^\alpha = \mathbb{C}\mathbf{1}_X$. In particular, the map $h_X : X \to \mathbb{C}$ defined by (4.6) is ucp, whence a state. We check that the state h_X is invariant. To this end, let $\phi \in \mathcal{S}(X)$ be an arbitrary state and $\mu \in A^*$ a functional. Note that we then have $h_X(x) = \phi(E_\alpha(x)) = \phi(x_{(0)})h_A(x_{(1)})$, for all $x \in X$. With this and using invariance of the Haar state h_A, we obtain

$$\begin{aligned}(h_X \lhd \mu)(x) &= h_X(x_{(0)})\mu(x_{(1)}) \\ &= \phi(x_{(0)})h_A(x_{(1)})\mu(x_{(2)}) \\ &= \phi(x_{(0)})h_A(x_{(1)})\mu(\mathbf{1}_A) \\ &= h_X(x)\mu(\mathbf{1}_A)\end{aligned}$$

for all $x \in X$. This shows invariance of the state h_X.

To show uniqueness of the invariant state h_X, let $\phi \in \mathcal{S}(X)$ be another invariant state, i.e. a state which satisfies $\phi(x_{(0)})\mu(x_{(1)}) = \phi(x)\mu(\mathbf{1}_A)$, for all functionals $\mu \in A^*$. Then the following holds:

$$\phi(x) = \phi(x)h_A(\mathbf{1}_A) = \phi(x_{(0)})h_A(x_{(1)}) = \phi(E_\alpha(x)) = h_X(x)$$

for all $x \in X$. □

Lemma 4.2.16. *Let $\pi \in \widehat{\mathbb{G}}$ be a finite-dimensional irreducible unitary corepresentation of $\mathbb{G}$. The following properties hold:*

(1) There is an idempotent map $E_\pi : X \to X^\pi$.

(2) The isotypical component X^π is a closed subspace of X.

(3) The algebraic core $\mathcal{X}$ is a dense operator subsystem of X.

(4) The coaction α restricts to the isotypical component, i.e.

$$\alpha(X^\pi) \subseteq X^\pi \otimes \mathcal{A}^\pi,$$

where $\mathcal{A}^\pi$ is the coalgebra of matrix coefficients of the corepresentation π. In particular, the coaction α restricts to a Hopf-algebra coaction $\alpha : \mathcal{X} \to \mathcal{X} \otimes \mathcal{A}$, i.e. that in addition to having the* coaction property*, the map α is* *-preserving *and* counital*, i.e. $x_{(0)}\epsilon(x_{(1)}) = x$, for all $x \in \mathcal{X}$ and where $\epsilon : \mathcal{A} \to \mathbb{C}$ is the counit of the Hopf*-algebra $\mathcal{A}$.*

(5) The ucp conditional expectation $E_\alpha : X \to X^\alpha$ is faithful, i.e., for all positive elements $x \in X^+$, if $E_\alpha(x) = 0$ it follows that $x = 0$.

Proof. Most of the claims are proven in [45, Proposition 3.4], see also [43, Section 3] for more details (with the apparent modifications for coactions on operator systems rather than C*-algebras). For the fact that the algebraic core $\mathcal{X}$ is an operator system, i.e. unital and *-closed, we refer to [43, Theorem 3.16].

To see (4), let $\xi \in H_\pi$ and $T \in \mathrm{Mor}(\pi, \alpha)$, and note that $\delta^\pi(\xi) = U^\pi(\xi \otimes \mathbf{1}_A) \in H_\pi \otimes A$ can be canonically identified with the linear map

$$H_\pi^* \ni \eta^* \mapsto (\eta^* \otimes \mathbf{1}_A)U^\pi(\xi \otimes \mathbf{1}_A) = (\omega^\pi_{\eta,\xi} \otimes \mathbf{I}^A)(U^\pi) \in \mathcal{A}^\pi.$$

Thus $\delta^\pi(\xi)$ is indeed an element of $H^\pi \otimes \mathcal{A}^\pi$. It follows that we have $\alpha(T\xi) = (T \otimes \mathbf{I}^A)\delta^\pi(\xi) \in X^\pi \otimes \mathcal{A}^\pi$.

The proof of faithfulness of the ucp conditional expectation E_α is as in [43, Lemma 3.19]. □

The following lemma is an important tool for our later arguments. The proof of the ergodicity statement is inspired by the proof of [56, Proposition 9].

Lemma 4.2.17. *Let $\tau : X \to Y$ be a ucp map onto an operator system Y. Assume that there is a well-defined unital complete order embedding $\alpha^\tau : Y \to Y \otimes A$ such that the following holds:*

$$(\tau \otimes \mathbf{I}^A)\alpha = \alpha^\tau \tau \tag{4.7}$$

Then the map α^τ is a coaction on the operator system Y. Moreover, if the coaction α is ergodic, so is the coaction α^τ.

Notation 4.2.18. We write (4.7) in Sweedler notation as

$$\tau(x_{(0)}) \otimes x_{(1)} = (\tau(x))_{(0)} \otimes (\tau(x))_{(1)} = y_{(0)} \otimes y_{(1)} \in Y \otimes A, \tag{4.8}$$

for all elements $x \in X$ and $y \in Y$ with $\tau(x) = y$.

Proof. The coaction property for α^τ follows from that for α:

$$\begin{aligned}(\alpha^\tau \otimes \mathbf{I}^A)\alpha^\tau\tau &= (\alpha^\tau \otimes \mathbf{I}^A)(\tau \otimes \mathbf{I}^A)\alpha \\ &= (\tau \otimes \mathbf{I}^A \otimes \mathbf{I}^A)(\alpha \otimes \mathbf{I}^A)\alpha \\ &= (\tau \otimes \mathbf{I}^A \otimes \mathbf{I}^A)(\mathbf{I}^A \otimes \Delta)\alpha \\ &= (\mathbf{I}^Y \otimes \Delta)(\tau \otimes \mathbf{I}^A)\alpha \\ &= (\mathbf{I}^Y \otimes \Delta)\alpha^\tau\tau\end{aligned}$$

Also the Podleś density property $\overline{\mathrm{span}}((\mathbf{1}_Y \otimes A)\alpha^\tau(Y)) = Y \otimes A$ follows from that of α. Indeed the span of elements of the form

$$(\mathbf{1}_Y \otimes a)\alpha^\tau(\tau(x)) = (\mathbf{1}_Y \otimes a)\left((\tau \otimes \mathbf{I}^A)(\alpha(x))\right) = (\tau \otimes \mathbf{I}^A)\left((\mathbf{1}_A \otimes a)\alpha(x)\right),$$

with $a \in A$, $x \in X$, is dense in $Y \otimes A$, since τ is onto.

We assume now that the coaction α is ergodic. Recall from Lemma 4.2.15 that there is a unique state h_X on X which is invariant for the coaction α, and which can be defined by $h_X(x)\mathbf{1}_X = x_{(0)}h_A(x_{(1)})$, for all elements $x \in X$. For any fixed point $y \in Y^{\alpha^\tau}$, i.e. y satisfies $\alpha^\tau(y) = y \otimes \mathbf{1}_A$, and any element $x \in X$ with $\tau(x) = y$, the following holds:

$$y = \tau(x) = \tau(x)h_A(\mathbf{1}_A) = \tau(x_{(0)})h_A(x_{(1)}) = \tau(h_X(x)\mathbf{1}_X) = h_X(x)\mathbf{1}_Y$$

Therefore, the fixed point y is an element of $\mathbb{C}\mathbf{1}_Y$ and thus the induced coaction α^τ is ergodic. □

4.2.3 Invariant Lip-norms

Throughout this section, let $\mathbb{G}$ be a compact quantum group with reduced function algebra $A := \mathrm{C_r}(\mathbb{G})$. Denote the comultiplication on A by Δ and fix an operator system X with a right coaction $\alpha : X \to X \otimes A$. We use Sweedler notation throughout. Much of the terminology and results presented in this section are due to [97] in the setting of coactions on C*-algebras, whereas here we consider coactions on operator systems. We point out that in *loc.cit.* a *right* $\mathrm{C}(\mathbb{G})$*-coaction* is considered as a *left* $\mathbb{G}$*-action*, so the reader has to make the according adjustments in terminology when referring back.

Definition 4.2.19. We say that a seminorm $L_X : X \to [0, \infty]$ is (right) *invariant* for the right coaction α if

$$L_X(x_{(0)}\mu(x_{(1)})) \leqslant L_X(x),$$

for all elements $x \in X$ and states $\mu \in \mathcal{S}(A)$. (Left) *invariance* for a left coaction is defined analogously.

Similarly, a seminorm $L_A : A \to [0, \infty]$ is called *right* (respectively *left*) *invariant* if it is invariant for the right (respectively left) coaction Δ. The seminorm L_A is called *bi-invariant* if it is both right and left invariant.

Example 4.2.20. For a compact group G with a metric d which is left invariant, i.e. $d(gh, g'h) = d(g, g')$, for all elements $g, g', h \in G$, the Lipschitz constant Lip is a right invariant Lip-norm on the C*-algebra of continuous functions on the group G (with domain the Lipschitz functions on G); see [56].

Conversely, assume that L is a Lip-norm on $\mathrm{C}(G)$, which is invariant for the right coaction

$$\mathrm{C}(G) \ni f \mapsto \big((g,h) \mapsto f(gh) =: \rho_h(f)(g)\big) \in \mathrm{C}(G \times G) \cong \mathrm{C}(G) \otimes \mathrm{C}(G).$$

Then the induced Monge–Kantorovich distance d^L is left invariant for the action of G on the state space $\mathcal{S}(\mathrm{C}(G))$ given by pullback of ρ, i.e. $\rho_h^*\mu(f) := \mu(\rho_h(f))$. Indeed, by right invariance of L, for all elements $g \in G$, it holds that $L(f) \leqslant 1$ if and only if $L(\rho_{g^{-1}}(f)) \leqslant 1$. Therefore,

$$\begin{aligned} d^L(\rho_g^*\mu, \rho_g^*\nu) &= \sup_{L(f)\leqslant 1} |\rho_g^*\mu(f) - \rho_g^*\nu(f)| \\ &= \sup_{L(\rho_{g^{-1}}(f))\leqslant 1} |\mu(f) - \nu(f)| \\ &= \sup_{L(f)\leqslant 1} |\mu(f) - \nu(f)| \\ &= d^L(\mu, \nu). \end{aligned}$$

Definition 4.2.21. A Lipschitz seminorm L_X on X is called *regular* if L_X is finite on the algebraic core $\mathcal{X} := \bigoplus_{\gamma \in \widehat{\mathbb{G}}} X^\gamma \subseteq X$.

The following proposition is treated for coactions on unital C*-algebras in [97, Theorem 1.4]. See [128, Section 2.5] for an order unit space version. All arguments adapt to operator systems. We find it convenient to split the statements into the first three basic observations and the last main result.

Proposition 4.2.22. *Let $\mathbb{G}$ be a compact quantum group with reduced function algebra $A := \mathrm{C_r}(\mathbb{G})$. Let X be an operator system and $\alpha : X \to X \otimes A$ a coaction. Assume that the function algebra A is equipped with a seminorm L_A. For all elements $x \in X$, set*

$$L_X^\alpha(x) := \sup_{\phi \in \mathcal{S}(X)} L_A(\phi(x_{(0)})x_{(1)}). \tag{4.9}$$

The following properties hold:

(1) The function $L_X^\alpha : X \to [0, \infty]$ is a seminorm on X.

(2) If L_A is a regular Lipschitz seminorm, so is the induced seminorm L_X^α.

(3) If the seminorm L_A is right invariant, the induced seminorm L_X^α is invariant for the right coaction α.

(4) Assume that the compact quantum group $\mathbb{G}$ is coamenable. If the seminorm L_A is a regular Lip-norm and the coaction α is ergodic, the induced seminorm L_X^α is a Lip-norm.

Proof. (1) The fact that L_X^α is a seminorm is immediate from the seminorm properties of L_A.

(2) The fact that the seminorm L_X^α is *-invariant follows from *-invariance of the seminorm L_A together with the identity

$$\phi((x^*)_0)(x^*)_{(1)} = (\phi \otimes \mathbf{I}^A)\alpha(x^*) = ((\phi \otimes \mathbf{I}^A)\alpha(x))^* = \phi(x_{(0)})(x_{(1)})^*,$$

for all $x \in X$, $\phi \in \mathcal{S}(X)$. Moreover, unitality of slice maps, i.e. $\phi((\mathbf{1}_X)_{(0)})(\mathbf{1}_X)_{(1)} = \mathbf{1}_A$, implies that $\mathbb{C}\mathbf{1}_X \subseteq \ker(L_X^\alpha)$. Last, observe that, since the coaction α restricts to a Hopf*-algebra coaction $\mathcal{X} \to \mathcal{X} \otimes \mathcal{O}(\mathbb{G})$, we have $\phi(x_{(0)})x_{(1)} \in \mathcal{O}(\mathbb{G})$, for all $x \in \mathcal{X}$. By regularity of L_A, we have $\mathcal{O}(\mathbb{G}) \subseteq \mathrm{Dom}(L_A)$. We conclude that L_X^α is finite on $\mathcal{X}$ and thus a regular Lipschitz seminorm.

(3) Right invariance of the seminorm L_X^α is a direct computation:

$$\begin{aligned} L_X^\alpha(x_{(0)}\mu(x_{(1)})) &= \sup_{\phi \in \mathcal{S}(X)} L_A(\phi(x_{(0)})x_{(1)}\mu(x_{(2)})) \\ &\leqslant \sup_{\phi \in \mathcal{S}(X)} L_A(\phi(x_{(0)})x_{(1)}) \\ &= L_X^\alpha(x), \end{aligned}$$

for all elements $x \in X$ and states $\mu \in \mathcal{S}(A)$, where we applied the Fubini theorem for slice maps and right invariance of L_A.

(4) To establish that L_X^α is a Lip-norm it remains to show that (X, L_X^α) has finite radius and that the subset $\overline{\mathrm{B}}_1^{\|\cdot\|, L_X^\alpha} \subseteq X$ is totally bounded. We refrain from going through the entire argument here, but point to [97, Section 8], in particular Lemma

8.5, Lemma 8.6 and Lemma 8.7 therein, and [128, Section 2.5] for details. However, we can deduce our claim from the results in [128]. In fact, by [128, Lemma 2.19], the order-unit quantum metric space $(X_{\mathrm{sa}}, (L_X^\alpha)_{\mathrm{sa}})$ has radius at most $2r_{A_{\mathrm{sa}}}$, where $r_{A_{\mathrm{sa}}}$ is the radius of $(A_{\mathrm{sa}}, (L_A)_{\mathrm{sa}})$. Applying the triangle inequality to the decomposition of x into its real and imaginary part yields that the radius of (X, L_X^α) is at most $4r_{A_{\mathrm{sa}}}$. Moreover, it follows from the proof of [128, Proposition 2.18] that the subset $\overline{\mathrm{B}}_1^{\|\cdot\|_{\mathrm{sa}},(L_X^\alpha)_{\mathrm{sa}}}$ of X_{sa} is totally bounded, from which we conclude that the closed subset $\overline{\mathrm{B}}_1^{\|\cdot\|,L_X^\alpha}$ of the totally bounded subset $\overline{\mathrm{B}}_1^{\|\cdot\|_{\mathrm{sa}},(L_X^\alpha)_{\mathrm{sa}}} + i\overline{\mathrm{B}}_1^{\|\cdot\|_{\mathrm{sa}},(L_X^\alpha)_{\mathrm{sa}}}$ of the operator system $X_{\mathrm{sa}} + iX_{\mathrm{sa}} = X$ is totally bounded. □

Remark 4.2.23. Any seminorm L_A on the function algebra A can be turned into a right invariant seminorm. Indeed, as in [97, Proposition 8.9], we set

$$L'_A(a) := \sup_{\mu\in\mathcal{S}(A)} L_A(a_{(0)}\mu(a_{(1)})),$$

and check for right invariance:

$$\begin{aligned} L'_A(a_{(0)}\nu(a_{(0)})) &= L'_A((\mathbf{I}^A\otimes\nu)\Delta(a)) \\ &= \sup_{\mu\in\mathcal{S}(A)} L_A((\mathbf{I}^A\otimes\mu)\Delta(\mathbf{I}^A\otimes\nu)\Delta(a)) \\ &= \sup_{\mu\in\mathcal{S}(A)} L_A((\mathbf{I}^A\otimes\mu\otimes\nu)(\Delta\otimes\mathbf{I}^A)\Delta(a)) \\ &= \sup_{\mu\in\mathcal{S}(A)} L_A(a_{(0)}(\mu*\nu)(a_{(1)})) \\ &\leqslant \sup_{\mu\in\mathcal{S}(A)} L_A(a_{(0)}\mu(a_{(1)})) \\ &= L'_A(a), \end{aligned}$$

for all $a \in A$, $\nu \in \mathcal{S}(A)$.

Similarly, setting

$$L''_A(a) := \sup_{\mu\in\mathcal{S}(A)} L_A(\mu(a_{(0)})a_{(1)})$$

and

$$L'''_A := \max\{L'_A, L''_A\}$$

gives respectively left and bi-invariant seminorms on A.

Remark 4.2.24. If the compact quantum group $\mathbb{G}$ is coamenable, it is clear that $L_A \leqslant L'_A$, where L'_A is the induced seminorm from Remark 4.2.23. Indeed,

$$L_A(a) = L_A(a_{(0)}\epsilon(a_{(1)})) \leqslant \sup_{\mu\in\mathcal{S}(A)} L_A(a_{(0)}\mu(a_{(1)})) = L'_A(a),$$

for all elements $a \in A$. Conversely, if L_A is right invariant to begin with, we have that $L'_A \leqslant L_A$, so that in this case $L_A = L'_A$.

Analogous statements hold for the induced left, respectively bi-invariant seminorms L''_A and L'''_A.

Remark 4.2.25. As pointed out in [97, Remark 8.2], it follows from [117, Proposition 1.1] that, if the function algebra A is separable, it admits a regular Lip-norm. Together with Remark 4.2.23 this shows that, if A is the function algebra of a coamenable compact quantum group and if A is separable, it admits a bi-invariant regular Lip-norm [97, Corollary 8.10].

For similar observations as the following, *cf.* also the proofs of [97, Lemma 8.7] and [56, Proposition 15].

Proposition 4.2.26. *Let $\mathbb{G}$ be a compact quantum group with reduced function algebra $A := \mathrm{C}_{\mathrm{r}}(\mathbb{G})$. Let X be an operator system and $\alpha : X \to X \otimes A$ a coaction. Let L_A be a Lipschitz seminorm on A with $\ker(L_A) = \mathbb{C}\mathbf{1}_A$, and let L_X^α be the induced seminorm (4.9) on X. Let $\mu, \nu \in \mathcal{S}(A)$ be states on A and consider the induced slice maps $X \to X$, given by $x \mapsto x_{(0)}\mu(x_{(1)})$ and $x \mapsto x_{(0)}\nu(x_{(1)})$ respectively. Then the following holds, for all $x \in X$:*

$$\|x_{(0)}\mu(x_{(1)}) - x_{(0)}\nu(x_{(1)})\| \leqslant 2d^{L_A}(\mu,\nu)L_X^\alpha(x)$$

Similarly, if $\beta : X \to A \otimes X$ is a left coaction and L_X^β the induced seminorm on X, the following holds, for all $x \in X$:

$$\|\mu(x_{(-1)})x_{(0)} - \nu(x_{(-1)})x_{(0)}\| \leqslant 2d^{L_A}(\mu,\nu)L_X^\beta(x)$$

Proof. For all elements $x \in X$ and any functional $\rho \in X^*$, the following holds:

$$\|x_{(0)}\rho(x_{(1)})\| \leqslant 2 \sup_{\phi\in\mathcal{S}(X)} |\phi(x_{(0)}\rho(x_{(1)}))| = 2 \sup_{\phi\in\mathcal{S}(X)} |\rho(\phi(x_{(0)})x_{(1)})|, \tag{4.10}$$

by the Kadison function representation and the Fubini theorem for slice maps. Recall from the definition of the Monge–Kantorovich distance that $\frac{|(\mu-\nu)(a)|}{L_A(a)} \leqslant d^{L_A}(\mu,\nu)$, for all $a \in A\backslash\ker(L_A) = A\backslash\mathbb{C}\mathbf{1}_A$, and therefore

$$|\mu(a) - \nu(a)| \leqslant d^{L_A}(\mu,\nu)L_A(a),$$

for all $a \in A$. By applying (4.10), the definition of the Monge–Kantorovich distance and the definition of the seminorm L_X^α, we obtain the result:

$$\begin{aligned}\|x_{(0)}\mu(x_{(1)}) - x_{(0)}\nu(x_{(1)})\| &\leqslant 2 \sup_{\phi\in\mathcal{S}(X)} |(\mu-\nu)(\phi(x_{(0)})x_{(1)})| \\ &\leqslant 2 \sup_{\phi\in\mathcal{S}(X)} d^{L_A}(\mu,\nu)L_A(\phi(x_{(0)})x_{(1)}) \\ &= 2d^{L_A}(\mu,\nu)L_X^\alpha(x)\end{aligned}$$

The proof of the statement for the left coaction β is analogous. □

With the right and left coactions α and β respectively replaced by the comultiplication Δ, and the seminorms L_X^α and L_X^β respectively replaced by the invariant seminorms L_A'' and L_A' from Remark 4.2.23, we obtain the following corollary.

Corollary 4.2.27. *Assume that L_A is a Lipschitz seminorm on A with $\ker(L_A) = \mathbb{C}\mathbf{1}_A$, and let L'_A and L''_A be the induced right and left invariant seminorms as in Remark 4.2.23. Then, for all states $\mu, \nu \in \mathcal{S}(A)$ and elements $a \in A$, the following inequalities hold:*

$$\|a_{(0)}\mu(a_{(1)}) - a_{(0)}\nu(a_{(1)})\| \leqslant 2d^{L_A}(\mu,\nu)L''_A(a), \text{ and}$$
$$\|\mu(a_{(-1)})a_{(0)} - \nu(a_{(-1)})a_{(0)}\| \leqslant 2d^{L_A}(\mu,\nu)L'_A(a).$$

4.3 Peter–Weyl truncations of a compact quantum group

Let $\mathbb{G}$ be a compact quantum group and denote by $\widehat{\mathbb{G}}$ its set of unitary equivalence classes of finite-dimensional unitary corepresentations. Write $A := \mathrm{C_r}(\mathbb{G})$ for the reduced function algebra, $\Delta := \Delta_\mathrm{r}$ for the comultiplication thereon and $H := \mathrm{L}^2(\mathbb{G}, h_A)$ for the GNS-space. Throughout this subsection fix a subset $\Lambda \subseteq \widehat{\mathbb{G}}$. This gives a closed subspace $H_\Lambda := \bigoplus_{\gamma\in\Lambda} H_\gamma \otimes H_\gamma^*$ of the Hilbert space H in the Peter–Weyl decomposition (4.3).

Denote by $P_\Lambda \in \mathcal{B}(\mathrm{L}^2(\mathbb{G}))$ the orthogonal projection onto the subspace H_Λ. The multiplicative unitaries $W, V \in \mathcal{B}(H \otimes H)$ commute with the projections $P_\Lambda \otimes \mathbf{I}^H$ and $\mathbf{I}^H \otimes P_\Lambda$ in $\mathcal{B}(H\otimes H)$ respectively, since $W((H_\gamma \otimes H_\gamma^*)\otimes H) \subseteq (H_\gamma\otimes H_\gamma^*)\otimes H$ and $V(H \otimes (H_\gamma \otimes H_\gamma^*)) \subseteq H \otimes (H_\gamma \otimes H_\gamma^*)$, for all $\gamma \in \widehat{\mathbb{G}}$.

Definition 4.3.1. We denote by $\tau_\lambda : \mathcal{B}(H) \to \mathcal{B}(H_\Lambda)$ the *compression map*, given by

$$\tau_\Lambda(T) := P_\Lambda T P_\Lambda,$$

for all $T \in \mathcal{B}(H)$, and write $A^{(\Lambda)} := \tau_\Lambda(A) \subseteq \mathcal{B}(H_\Lambda)$ for the image of the function algebra A under the compression map.

Notation 4.3.2. Throughout this section we may drop the subindex Λ of the projection P_Λ and the compression map τ_Λ whenever convenient.

Remark 4.3.3. Note that $A^{(\Lambda)}$ is an operator system and the compression map $\tau : A \to A^{(\Lambda)}$ is ucp onto.

Proposition 4.3.4. *There are unique ergodic cocommuting right and left coactions $\alpha^\tau : A^{(\Lambda)} \to A^{(\Lambda)} \otimes A$ and $\beta^\tau : A^{(\Lambda)} \to A \otimes A^{(\Lambda)}$ which satisfy*

$$(\tau \otimes \mathbf{I}^A)\Delta = \alpha^\tau \tau \text{ and } (\mathbf{I}^A \otimes \tau)\Delta = \beta^\tau \tau, \tag{4.11}$$

respectively.

Proof. The claim follows from Lemma 4.2.17 once we know that (4.11) well-defines maps α^τ and β^τ which are furthermore unital complete order embeddings. To this end, let $n \geqslant 1$ be a positive integer and let $(a_{ij})_{i,j} \in \mathrm{M}_n(A) \subseteq \mathcal{B}(H \otimes \mathbb{C}^n)$ be an $n \times n$ matrix with entries in A. Then, from the fact that the multiplicative unitary W commutes with the projection $P \otimes \mathbf{I}^H$ and from unitarity of W, we obtain:

$$\begin{aligned}
&\|\left((\tau \otimes \mathbf{I}^A)\Delta(a_{ij})\right)_{i,j}\| \\
&= \|\left((P \otimes \mathbf{I}^H)W(a_{ij} \otimes \mathbf{1}_A)W^*(P \otimes \mathbf{I}^H)\right)_{i,j}\| \\
&= \|(W \otimes \mathbf{I}^{\mathbb{C}^n})\left((P \otimes \mathbf{I}^H)(a_{ij} \otimes \mathbf{1}_A)(P \otimes \mathbf{I}^H)\right)_{i,j}(W^* \otimes \mathbf{I}^{\mathbb{C}^n})\| \\
&= \|\left((P \otimes \mathbf{I}^H)(a_{ij} \otimes \mathbf{1}_A)(P \otimes \mathbf{I}^H)\right)_{i,j}\| \\
&= \|\left(Pa_{ij}P \otimes \mathbf{1}_{\mathcal{B}(H)}\right)_{i,j}\| \\
&= \|(\tau(a_{ij}))_{i,j}\|
\end{aligned}$$

This shows in particular $\ker(\tau) = \ker((\tau \otimes \mathbf{I}^A)\Delta)$, whence α^τ is well-defined. Moreover, we have proven that α^τ is unital completely isometric, whence a unital complete order embedding. The proof that β^τ is a well-defined unital complete order embedding is analogous by exchanging the multiplicative unitary W for V. □

For the rest of this section, we assume that the compact quantum group $\mathbb{G}$ is coamenable with separable function algebra $A = \mathrm{C}(\mathbb{G})$.

Let L_A be a right/left/bi-invariant regular Lip-norm on the function algebra A, *cf.* Remark 4.2.25. Recall from Proposition 4.2.22 that the Lip-norm L_A induces a right/left/bi-invariant Lip-norm on $A^{(\Lambda)}$:

Corollary 4.3.5. *The operator system $A^{(\Lambda)}$ equipped with any of the induced Lip-norms $L^{\alpha^\tau}_{A^{(\Lambda)}}$, $L^{\beta^\tau}_{A^{(\Lambda)}}$ or $L^{\alpha^\tau,\beta^\tau}_{A^{(\Lambda)}} := \max\{L^{\alpha^\tau}_{A^{(\Lambda)}}, L^{\beta^\tau}_{A^{(\Lambda)}}\}$ is a compact quantum metric space.*

Lemma 4.3.6. *Let L_A be a left invariant regular Lip-norm on the function algebra A and let $L^{\alpha^\tau}_{A^{(\Lambda)}}$ be the induced Lip-norm on the operator system $A^{(\Lambda)}$. Then the compression map $\tau_\Lambda : A \to A^{(\Lambda)}$ is a morphism of compact quantum metric spaces. Analogous statements hold if L_A is right or bi-invariant.*

Proof. We already noted in Remark 4.3.3 that the compression map τ is ucp. For Lip-norm contractivity, observe that the following holds, for all $a \in A$:

$$\begin{aligned}
L^{\alpha^\tau}_{A^{(\Lambda)}}(\tau(a)) &= \sup_{\phi \in \mathcal{S}(A^{(\Lambda)})} L_A(\phi(\tau(a)_{(0)})a_{(1)}) \\
&= \sup_{\phi \in \mathcal{S}(A^{(\Lambda)})} L_A(\tau^*\phi(a_{(0)})a_{(1)}) \\
&= \sup_{\tau^*\phi \in \tau^*\mathcal{S}(A^{(\Lambda)})} L_A(\tau^*\phi(a_{(0)})a_{(1)}) \\
&\leqslant L_A(a),
\end{aligned}$$

by left invariance of the Lip-norm L_A, where we used that $\tau^* : \mathcal{S}(A^{(\Lambda)}) \to \mathcal{S}(A)$ is an injection. □

Definition 4.3.7. Let L_A be a bi-invariant regular Lip-norm on the function algebra A and let $L^{\alpha^\tau,\beta^\tau}_{A^{(\Lambda)}}$ be the induced bi-invariant Lip-norm on the operator system $A^{(\Lambda)}$ as in Corollary 4.3.5. We call the compact quantum metric space $(A^{(\Lambda)}, L^{\alpha^\tau,\beta^\tau}_{A^{(\Lambda)}})$ the (bi-invariant) *Peter–Weyl truncation* of the compact quantum group $\mathbb{G}$.

In order to compare the Peter–Weyl truncations $(A^{(\Lambda)}, L^{\alpha^\tau,\beta^\tau}_{A^{(\Lambda)}})$ with the original compact quantum metric space (A, L_A) using the criterion in Proposition 2.3.21 we need morphisms $\Phi : A \to A^{(\Lambda)}$ and $\Psi : A^{(\Lambda)} \to A$ whose compositions approximate the respective identity maps on A and $A^{(\Lambda)}$ in Lip-norm. We take the compression map $\tau : A \to A^{(\Lambda)}$ as the morphism Φ, so that it remains to find an appropriate candidate for the map Ψ in the converse direction. In earlier works on compact quantum metric spaces [120], [128], [77], [136], [94], these maps were inspired by Berezin quantization [15], see also [90]. For our purposes, rather than working with the adjoint of the compression map τ for a choice of inner products on A and $A^{(\Lambda)}$, we follow the approach taken in [56] to give a whole family of candidates for maps $\Psi : A^{(\Lambda)} \to A$, which we then show to have a member that satisfies the assumptions of Proposition 2.3.21. We keep the subset $\Lambda \subseteq \widehat{\mathbb{G}}$ fixed.

Definition 4.3.8. Let $\phi \in \mathcal{S}(A^{(\Lambda)})$ be any state. We denote the associated slice map by $\sigma^\phi_\Lambda : A^{(\Lambda)} \to A$, i.e.

$$\sigma^\phi_\Lambda(x) := \phi(x_{(0)})x_{(1)} = (\phi \otimes \mathbf{I}^A)\alpha^\tau(x),$$

for all $x \in A^{(\Lambda)}$. We call the map σ^ϕ_Λ a *symbol map*.

Notation 4.3.9. As we did for the compression map τ, we will drop the subindex Λ of the symbol map σ^ϕ_Λ, whenever convenient.

Lemma 4.3.10. *Let L_A be a regular Lip-norm on the function algebra A and let $L^{\alpha^\tau}_{A^{(\Lambda)}}$ be the induced Lip-norm on the operator system $A^{(\Lambda)}$. Then, for every state $\phi \in \mathcal{S}(A^{(\Lambda)})$, the symbol map $\sigma^\phi : A^{(\Lambda)} \to A$ is a morphism of compact quantum metric spaces.*

Analogous statements hold if the operator system $A^{(\Lambda)}$ is equipped with one of the induced Lip-norms $L^{\beta^\tau}_{A^{(\Lambda)}}$ or $L^{\alpha^\tau,\beta^\tau}_{A^{(\Lambda)}}$.

Proof. The symbol map σ^ϕ, being the composition of the unital complete order embedding α^τ and the ucp map $\phi \otimes \mathbf{I}^A$, is ucp. Lip-norm contractivity of σ^ϕ follows from the definition of the induced Lip-norm:

$$L_A(\sigma^\phi(x)) = L_A(\phi(x_{(0)})x_{(1)}) \leqslant \sup_{\psi \in \mathcal{S}(A^{(\Lambda)})} L_A(\psi(x_{(0)})x_{(1)}) = L^{\alpha^\tau}_{A^{(\Lambda)}}(x)$$

□

Before we can apply Proposition 2.3.21, we compute the compositions of the compression and symbol maps:

$$\sigma^\phi\tau(a) = \phi(\tau(a)_{(0)})\tau(a)_{(1)} = \tau^*\phi(a_{(0)})a_{(1)}, \tag{4.12}$$

for all $a \in A$, and

$$\tau\sigma^{\phi}(x) = \phi(x_{(0)})\tau(x_{(1)}) = \tau^*\phi(a_{(0)})\tau(a_{(1)}), \tag{4.13}$$

for all $x \in A^{(\Lambda)}$ and $a \in A$ with $\tau(a) = x$, where we used (4.8) in the last step.

Recall that we are assuming that the compact quantum group $\mathbb{G}$ is coamenable with separable function algebra A.

Proposition 4.3.11. *Let $\phi \in \mathcal{S}(A^{(\Lambda)})$ be a state. Assume that L_A is a regular Lipschitz seminorm on the function algebra A with $\ker(L_A) = \mathbb{C}\mathbf{1}_A$. Let L'_A be the induced right invariant regular Lipschitz seminorm on A and let $L^{\beta^\tau}_{A^{(\Lambda)}}$ be the induced regular Lipschitz seminorm on $A^{(\Lambda)}$. Then the following inequalities hold:*

$$\|\sigma^{\phi}\tau(a) - a\| \leqslant 2d^{L_A}(\tau^*\phi, \epsilon)L'_A(a),$$

and

$$\|\tau\sigma^{\phi}(x) - x\| \leqslant 2d^{L_A}(\tau^*\phi, \epsilon)L^{\beta^\tau}_{A^{(\Lambda)}}(x),$$

for all elements $a \in A$ and $x \in A^{(\Lambda)}$, where we recall that $\epsilon \in \mathcal{S}(A)$ is the counit of the compact quantum group $\mathbb{G}$.

Proof. The first inequality follows immediately from (4.12) and Corollary 4.2.27. Indeed, we have

$$\|\sigma^{\phi}\tau(a) - a\| = \|\tau^*\phi(a_{(0)})a_{(1)} - \epsilon(a_{(0)})a_{(1)}\| \leqslant 2d^{L_A}(\tau^*\phi, \epsilon)L'_A(a).$$

As for the second inequality, observe that, for all $a \in A$ with $\tau(a) = x$, we obtain the following, using (4.13), the Kadison function representation and the Fubini theorem for slice maps:

$$\begin{aligned}
\|\tau\sigma^{\phi}(x) - x\| &= \|\tau^*\phi(a_{(0)})\tau(a_{(1)}) - \tau(a)\| \\
&\leqslant 2 \sup_{\psi \in \mathcal{S}(A^{(\Lambda)})} \left|\psi\left(\tau^*\phi(a_{(0)})\tau(a_{(1)}) - \epsilon(a_{(0)})\tau(a_{(1)})\right)\right| \\
&= 2 \sup_{\psi \in \mathcal{S}(A^{(\Lambda)})} \left|\tau^*\psi\left(\tau^*\phi(a_{(0)})a_{(1)} - \epsilon(a_{(0)})a_{(1)}\right)\right| \\
&= 2 \sup_{\psi \in \mathcal{S}(A^{(\Lambda)})} |(\tau^*\phi - \epsilon)(a_{(0)}\tau^*\psi(a_{(1)}))| \\
&\leqslant 2 \sup_{\psi \in \mathcal{S}(A^{(\Lambda)})} d^{L_A}(\tau^*\phi, \epsilon)L_A(a_{(0)}\tau^*\psi(a_{(1)})) \\
&= 2 \sup_{\psi \in \mathcal{S}(A^{(\Lambda)})} d^{L_A}(\tau^*\phi, \epsilon)L_A(x_{(-1)}\psi(x_{(0)})) \\
&= 2d^{L_A}(\tau^*\phi, \epsilon)L^{\beta^\tau}_{A^{(\Lambda)}}(x).
\end{aligned}$$

Note that in the penultimate line we used that $a_{(0)} \otimes a_{(1)} = a_{(-1)} \otimes a_{(0)}$ and that $a_{(-1)}\tau^*\psi(a_{(0)}) = x_{(-1)}\psi(x_{(0)})$. ☐

Corollary 4.3.12. *Assume that L_A is a right invariant regular Lip-norm on the function algebra A and let $L^{\beta^\tau}_{A^{(\Lambda)}}$ be the induced regular Lip-norm on $A^{(\Lambda)}$. Then for every positive real number $\varepsilon > 0$, there is a finite subset $\Lambda \subseteq \widehat{\mathbb{G}}$ and a state $\phi \in \mathcal{S}(A^{(\Lambda)})$ such that the following inequalities hold:*

$$\begin{aligned}\|\sigma^\phi_\Lambda \tau_\Lambda(a) - a\| &\leqslant \varepsilon L_A(a), \text{ and}\\ \|\tau_\Lambda \sigma^\phi_\Lambda(x) - x\| &\leqslant \varepsilon L^{\beta^\tau}_{A^{(\Lambda)}}(x),\end{aligned}$$

for all $a \in A$ and $x \in A^{(\Lambda)}$.

Proof. Using Proposition 4.3.11 together with the fact that $L_A = L'_A$ by right invariance, the claim follows from the assumption that d^{L_A} metrizes the weak*-topology on $\mathcal{S}(A)$, together with the weak*-density of the subset of liftable states in $\mathcal{S}(A)$ as in Lemma 2.2.3. □

For the main theorem of this chapter recall that, for any set $\Lambda \subseteq \widehat{\mathbb{G}}$ of finite-dimensional irreducible unitary corepresentations, we denote by

$$P_\Lambda : \mathrm{L}^2(\mathbb{G}) \to \bigoplus_{\gamma \in \Lambda} H_\gamma \otimes H^*_\gamma$$

the orthogonal projection and by

$$\tau : \mathrm{C}(\mathbb{G}) \to P_\Lambda \mathrm{C}(\mathbb{G}) P_\Lambda =: \mathrm{C}(\mathbb{G})^{(\Lambda)}$$

the compression. The induced right and left coactions of $\mathrm{C}(\mathbb{G})$ on the operator system $\mathrm{C}(\mathbb{G})^{(\Lambda)}$ are denoted α^τ and β^τ respectively.

Theorem 4.3.13. *Let $\mathbb{G}$ be a coamenable compact quantum group with separable function algebra $A = \mathrm{C}(\mathbb{G})$. Assume that L_A is a bi-invariant regular Lip-norm on A and denote by $L^{\alpha^\tau,\beta^\tau}_{A^{(\Lambda)}}$ the induced bi-invariant Lip-norm on the operator system $A^{(\Lambda)}$. Let $\varepsilon > 0$ be a positive real number. Then there exists a finite subset $\Lambda \subseteq \widehat{\mathbb{G}}$ of finite-dimensional irreducible unitary corepresentations such that*

$$\mathrm{dist}^{\mathrm{s}}\left((A^{(\Lambda)}, L^{\alpha^\tau,\beta^\tau}_{A^{(\Lambda)}}), (A, L_A)\right) \leqslant \varepsilon.$$

Proof. By Lemma 4.3.6 and Lemma 4.3.10, the compression map $\tau : A \to A^{(\Lambda)}$ and the symbol maps $\sigma^\phi : A^{(\Lambda)} \to A$ are morphisms of compact quantum metric spaces. By Corollary 4.3.12, there is a finite subset $\Lambda \subseteq \widehat{\mathbb{G}}$ of finite-dimensional irreducible unitary corepresentations such that $\|\sigma^\phi_\Lambda \tau_\Lambda(a) - a\| \leqslant \varepsilon L_A(a)$ and $\|\tau_\Lambda \sigma^\phi_\Lambda(x) - x\| \leqslant \varepsilon L^{\alpha^\tau,\beta^\tau}_{A^{(\Lambda)}}(x)$, for all $a \in A$ and $x \in A^{(\Lambda)}$. Thus, by Proposition 2.3.21, we obtain $\mathrm{dist}^{\mathrm{s}}\left((A^{(\Lambda)}, L^{\alpha^\tau,\beta^\tau}_{A^{(\Lambda)}}), (A, L_A)\right) \leqslant \varepsilon$ as claimed. □

By Corollary 2.3.24, the same convergence result holds also in the distances $\mathrm{dist}_{\mathrm{GH}}$, $\mathrm{dist}^{\mathrm{q}}_{\mathrm{GH}}$, $\mathrm{dist}^{\mathrm{s}}_n$, for all $n \in \mathbb{N}$, and $\mathrm{dist}^{\mathrm{op}}$.

4.3.1 The case of a compact group

We compare our setup with the compact group case as in [56]. To this end, let G be a second countable compact group with a bi-invariant metric d, i.e. $d(gh, gp) = d(hg, pg) = d(h, p)$, for all elements $g, h, p \in G$. Recall that we denote the comultiplication $\Delta : \mathrm{C}(G) \to \mathrm{C}(G \times G)$, $\Delta(f)(g, h) := f(gh)$, by α or β whenever considered as a right or left coaction respectively. Recall furthermore that the Lipschitz constant Lip_d is a Lip-norm on $\mathrm{C}(G)$ and observe that it is bi-invariant in the sense of Li. Indeed, for any function $f \in \mathrm{C}(G)$ and state $\mu \in \mathcal{S}(\mathrm{C}(G))$ (i.e. μ is a probability measure on G), we have

$$\begin{aligned}\mathrm{Lip}_d((\mathbf{I}^{\mathrm{C}(G)} \otimes \mu)\alpha(f)) &= \mathrm{Lip}_d\left(g \mapsto \int_G f(gh)\mathrm{d}\mu(h)\right)\\ &= \sup_{g,p\in G} \frac{|\int_G f(gh) - f(ph)\mathrm{d}\mu(h)|}{d(g,p)}\\ &\leqslant \int_G \sup_{g,p\in G} \frac{|f(gh) - f(ph)|}{d(g,p)}\mathrm{d}\mu(h)\\ &\leqslant \mathrm{Lip}_d(f),\end{aligned}$$

by right invariance of the metric d and the fact that μ is a probability measure on G. Similarly, one shows that $\mathrm{Lip}_d((\mu \otimes \mathbf{I}^{\mathrm{C}(G)})\beta(f)) \leqslant \mathrm{Lip}_d(f)$.

Denote by U and V the respective left and right regular representation of G on the Hilbert space $\mathrm{L}^2(G)$, given by $U_g\xi(h) := \xi(g^{-1}h)$ and $V_g\xi(h) := \xi(hg)$ respectively, for all elements $g, h \in G$ and $\xi \in \mathrm{L}^2(G)$. We write λ, ρ for the strong*-continuous left and right G-actions on $\mathcal{B}(\mathrm{L}^2(G))$ by conjugation with the left and right regular representation respectively, i.e. $\lambda_g(T) := U_gTU_g^*$ and $\rho_g(T) := V_gTV_g^*$, for all elements $g \in G$ and operators $T \in \mathcal{B}(\mathrm{L}^2(G))$. It is straightforward to check that $\lambda_g(f)(h) = f(g^{-1}h)$ and $\rho_g(f)(h) = f(hg)$, for all elements $g, h \in G$ and functions $f \in \mathrm{C}(G)$ viewed as operators on the Hilbert space $\mathrm{L}^2(G)$ by pointwise multiplication. For all $T \in \mathcal{B}(\mathrm{L}^2(G))$, the authors of [56] set

$$\|T\|_\lambda := \sup_{g\in G\setminus\{e\}} \frac{\|\lambda_g(T) - T\|}{d(g,e)}, \quad \|T\|_\rho := \sup_{g\in G\setminus\{e\}} \frac{\|\rho_g(T) - T\|}{d(g,e)},$$

and

$$\|T\|_{\lambda,\rho} := \max\{\|T\|_\lambda, \|T\|_\rho\}.$$

It is straightforward to check that the Lipschitz constant of a function $f \in \mathrm{C}(G)$ coincides with the Lipschitz constants of the $\mathrm{C}(G)$-valued functions $g \mapsto \lambda_g(f)$ and $g \mapsto \rho_g(f)$, i.e.

$$\mathrm{Lip}_d(f) = \|f\|_\lambda = \|f\|_\rho = \|f\|_{\lambda,\rho},$$

for all $f \in \mathrm{C}(G)$.

Let $\Lambda \subseteq \widehat{G}$ be a set of equivalence classes of finite-dimensional irreducible unitary representations of G and let $P : \mathrm{L}^2(G) \to \bigoplus_{\gamma\in\Lambda} H_\gamma \otimes H_\gamma$ be the associated orthogonal projection. The actions λ, ρ commute with the compression map

$\tau : \mathcal{B}(\mathrm{L}^2(G)) \ni T \mapsto PTP \in \mathcal{B}(P\mathrm{L}^2(G))$, so that we obtain G-actions on the operator system $P\mathrm{C}(G)P$ which we still denote by λ and ρ respectively. Note that, respectively being the composition of a norm-continuous G-action on $\mathrm{C}(G)$ and the compression map, these actions are norm-continuous.

Denote by $\mathrm{Lip}_{P\mathrm{C}(G)P}^{\alpha^\tau,\beta^\tau}$ the bi-invariant Lip-norm on $P\mathrm{C}(G)P$ induced from the Lipschitz constant Lip_d by the coactions $\alpha^\tau : P\mathrm{C}(G)P \to P\mathrm{C}(G)P \otimes \mathrm{C}(G)$ and $\beta^\tau : P\mathrm{C}(G)P \to \mathrm{C}(G) \otimes P\mathrm{C}(G)P$ from Lemma 4.2.17 in the sense of Li. I.e.

$$\begin{aligned}&\mathrm{Lip}_{P\mathrm{C}(G)P}^{\alpha^\tau,\beta^\tau}(x)\\&:= \max\left\{\sup_{\phi\in\mathcal{S}(P\mathrm{C}(G)P)} \mathrm{Lip}_d(\phi(\alpha^\tau_\bullet(x))),\ \sup_{\phi\in\mathcal{S}(P\mathrm{C}(G)P)} \mathrm{Lip}_d(\phi(\beta^\tau_\bullet(x)))\right\},\end{aligned}$$

for all $x \in P\mathrm{C}(G)P$. This Lip-norm is equivalent to the seminorm $\|\cdot\|_{\lambda,\rho}$ on $P\mathrm{C}(G)P$:

Lemma 4.3.14. *For all $x \in P\mathrm{C}(G)P$, the following holds:*

$$\frac{1}{2}\|x\|_{\lambda,\rho} \leqslant \mathrm{Lip}_{P\mathrm{C}(G)P}^{\alpha^\tau,\beta^\tau}(x) \leqslant \|x\|_{\lambda,\rho}.$$

Proof. Let $x \in P\mathrm{C}(G)P$. We show that

$$\frac{1}{2}\|x\|_{\rho} \leqslant \mathrm{Lip}_{P\mathrm{C}(G)P}^{\alpha^\tau}(x) \leqslant \|x\|_{\rho}, \tag{4.14}$$

where $\mathrm{Lip}_{P\mathrm{C}(G)P}^{\alpha^\tau}(x) := \sup_{\phi\in\mathcal{S}(P\mathrm{C}(G)P)} \mathrm{Lip}_d(\phi(\alpha^\tau_\bullet(x)))$. To this end, using the fact that $\mathrm{Lip}_d(f) = \|f\|_\lambda$, for all $f \in \mathrm{C}(G)$, we have:

$$\begin{aligned}\mathrm{Lip}_{P\mathrm{C}(G)P}^{\alpha^\tau}(x) &= \sup_{\phi\in\mathcal{S}(P\mathrm{C}(G)P)} \|\phi(\alpha^\tau_\bullet(x))\|_\lambda\\&= \sup_{\phi\in\mathcal{S}(P\mathrm{C}(G)P)}\ \sup_{g\in G\setminus\{e\}} \frac{\|\lambda_g(\phi(\alpha^\tau_\bullet(x))) - \phi(\alpha^\tau_\bullet(x))\|_\infty}{d(g,e)}\\&= \sup_{\phi\in\mathcal{S}(P\mathrm{C}(G)P)}\ \sup_{g\in G\setminus\{e\}}\ \sup_{h\in G} \frac{|\phi(\alpha^\tau_{g^{-1}h}(x)) - \phi(\alpha^\tau_h(x))|}{d(g,e)}\end{aligned}$$

By the Kadison function representation, this last quantity is bounded below and above by respectively $\frac{1}{2}$ and 1 times

$$\sup_{g\in G\setminus\{e\}}\ \sup_{h\in G} \frac{\|\alpha^\tau_{g^{-1}h}(x) - \alpha^\tau_h(x)\|}{d(g,e)} = \|\alpha^\tau_\bullet(x)\|_\lambda.$$

Note that, for all $g, h \in G$, $x \in P\mathrm{C}(G)P$ and $f \in \mathrm{C}(G)$ with $\tau(f) = x$, we have

$$\begin{aligned}\alpha_{g^{-1}h}(x) &= ((\tau \otimes \mathbf{I}^{\mathrm{C}(G)})\Delta(f))(g^{-1}h)\\&= \tau(p \mapsto f(pg^{-1}h))\\&= \tau(p \mapsto \rho_{g^{-1}h}(f)(p))\\&= \rho_{g^{-1}h}(x).\end{aligned}$$

This implies that

$$\|\alpha^{\tau}_{\bullet}(x)\|_{\lambda} = \sup_{g\in G\setminus\{e\}} \sup_{h\in G} \frac{\|\rho_{g^{-1}h}(x) - \rho_h(x)\|}{d(g,e)} = \|x\|_{\rho},$$

by invariance of the metric d. Altogether we obtain (4.14).

Similarly, we can show

$$\frac{1}{2}\|x\|_{\lambda} \leqslant \mathrm{Lip}^{\beta^{\tau}}_{PC(G)P}(x) \leqslant \|x\|_{\lambda}.$$

Together with (4.14) this yields the claim. □

We now obtain [56, Theorem 17] as a corollary of our Theorem 4.3.13:

Corollary 4.3.15. *Let G be a compact group Assume that G is equipped with a bi-invariant metric d. Let $\varepsilon > 0$ be a positive real number. Then there exists a finite subset $\Lambda \subseteq \widehat{G}$ of finite-dimensional irreducible unitary representations such that* $\mathrm{dist}_{\mathrm{GH}}\left((\mathcal{S}(P_\Lambda \mathrm{C}(G) P_\Lambda), d^{\|\cdot\|_{\lambda,\rho}}), (\mathcal{S}(\mathrm{C}(G)), d^{\mathrm{Lip}})\right) \leqslant \varepsilon$.

Proof. By Lemma 4.3.14 and Theorem 4.3.13, there exists a set Λ as claimed such that $\mathrm{dist}^{\mathrm{op}}\left((P_\Lambda \mathrm{C}(G) P_\Lambda, \|\cdot\|_{\lambda,\rho}), (\mathrm{C}(G), \mathrm{Lip})\right) \leqslant \varepsilon$, which implies the claim, by Corollary 2.3.24. □

Remark 4.3.16. Note that the two seminorms $L^{\alpha^{\tau},\rho^{\tau}}_{PC(G)P}$ and $\|\cdot\|_{\lambda,\rho}$ coincide on the self-adjoint subspace $(PC(G)P)_{\mathrm{sa}}$. Indeed, this follows from the equality $\|x\| = \sup_{\phi\in\mathcal{S}(X)}|\phi(x)|$, for all self-adjoint elements of an operator system X, by the Kadison function representation.

4.4 Comparison with Fourier truncations

We now compare our Peter–Weyl truncations with Rieffel's Fourier truncations [123].

As before, let $\mathbb{G}$ be a coamenable compact quantum group with separable function algebra $A = \mathrm{C}(\mathbb{G})$. Let $\Lambda \subseteq \widehat{\mathbb{G}}$ be a (finite) set of finite-dimensional irreducible unitary corepresentations and assume Λ is closed under taking conjugate corepresentations and contains the trivial corepresentation. Note that by $A_{(\Lambda)}$ the direct sum of the coalgebras A^{γ}, $\gamma \in \Lambda$, i.e. the isotypical components of the coaction Δ of A on itself. Clearly $A_{(\Lambda)}$ is a *-closed unital subspace of A, so an operator system which we call the *Fourier system*.

Assume A is equipped with a right invariant regular Lip-norm L. Then, with L_Λ the restriction of L to the Fourier system, the pair $(A_{(\Lambda)}, L_\Lambda)$ is a compact quantum metric space which we call a *Fourier truncation* of $\mathbb{G}$. In order to show convergence $A_{(\Lambda)} \to A$, as Λ exhausts all of $\widehat{\mathbb{G}}$ by applying Proposition 2.3.21, we need uniformly (in Λ) Lip-norm bounded ucp maps $E_\Lambda : A \to A_{(\Lambda)}$ and $\iota_\Lambda : A_{(\Lambda)} \to A$, the compositions of which approximate the identities on A and $A_{(\Lambda)}$ in Lip-norm. It turns out that taking ι_Λ to be the inclusion maps and E_Λ slice maps, for appropriate states $\nu_\Lambda \in \mathcal{S}(A)$, works.

Moreover, since in this setting only right invariance of L_A is required, we can consider an ergodic coaction $\alpha : X \to X \otimes A$ on an operator system X. In this case, we set $X_{(\Lambda)} := \bigoplus_{\gamma \in \Lambda} X^\gamma$. From Proposition 4.2.22 we obtain an α-invariant Lip-norm L_X^α on X which restricts to a Lip-norm L_Λ on $X_{(\Lambda)}$. We call the resulting compact quantum metric space $(X_{(\Lambda)}, L_\Lambda)$ a *Fourier truncation* of X.

Theorem 4.4.1. *Let $\mathbb{G}$ be a coamenable compact quantum group with separable function algebra $A = \mathrm{C}(\mathbb{G})$. Let X be an operator system and $\alpha : X \to X \otimes A$ an ergodic coaction. Assume that A is equipped with a right invariant regular Lip-norm L_A. Then, for every $\varepsilon > 0$, there is a finite collection $\Lambda \subseteq \widehat{\mathbb{G}}$ of irreducible unitary corepresentations, which is closed under taking conjugate corepresentations and contains the trivial corepresentation, such that*

$$\mathrm{dist}^{\mathrm{s}}\big((X_{(\Lambda)}, L_\Lambda), (X, L_X^\alpha)\big) \leqslant \varepsilon.$$

Proof. Let $\varepsilon > 0$. By the Peter–Weyl decomposition (4.3), the subspace $K \subseteq \mathrm{L}^2(\widehat{\mathbb{G}})$ spanned by $H_\gamma \otimes \overline{H_\gamma}$, $\gamma \in \widehat{\mathbb{G}}$, is dense. Since the weak*-topology agrees with the metric topology for the Monge–Kantorovich distance d^{L_A} on $\mathcal{S}(A)$, there is a state

$$\mu \in \mathrm{co}\{\langle \xi, \cdot \xi \rangle \mid \xi \in K, \|\xi\| = 1\} = \mathrm{co}\{\frac{1}{\|\Lambda(a)\|^2} \langle \Lambda(a), \cdot \Lambda(a) \rangle \mid a \in \mathcal{A}\} \subseteq \mathcal{S}(A),$$

for $\mathcal{A} = \mathcal{O}(\mathbb{G})$ the coordinate algebra, such that $d^{L_A}(\epsilon, \mu) \leqslant \frac{\varepsilon}{2}$, where $\epsilon \in \mathcal{S}(A)$ denotes the counit. Let $\Lambda \subseteq \widehat{\mathbb{G}}$ be a finite collection of irreducible unitary corepresentations which is closed under conjugation and contains the trivial corepresentation such that $\mu(\mathcal{A}^\gamma) = 0$, for all $\gamma \in \Lambda$.

Denote the slice map for μ on X by $E_\mu := (\mathbf{I}^X \otimes \mu)\alpha$. Since $\alpha(X^\gamma) \subseteq X^\gamma \otimes \mathcal{A}^\gamma$, for all $\gamma \in \widehat{\mathbb{G}}$, by part (4) of Lemma 4.2.16, it follows that $E_\mu(X^\gamma) = 0$, for all $\gamma \in \widehat{\mathbb{G}} \backslash \Lambda$. Hence, E_μ is a ucp map $X \to X_{(\Lambda)}$. Moreover, E_μ is Lip-norm contractive, i.e. $L_X^\alpha(E_\mu(x)) \leqslant L_X^\alpha(x)$, for all $x \in X$, since L_X^α is invariant by invariance of L_A together with part (3) of Proposition 4.2.22. By the slice map lemma, Proposition 4.2.26, we obtain

$$\|x - E_\mu(x)\| \leqslant 2 d^{L_A}(\epsilon, \mu) L_X^\alpha(x) \leqslant \varepsilon L_X^\alpha(x),$$

for all $x \in X$. Now, Proposition 2.3.21 implies the claim. ☐

It should be emphasized that the above theorem is only a slightly strengthened version of [123, Theorem 6.1] and most ingredients of the proof appear in a similar form in [97, 123]. Note that Theorem 4.4.1 only requires right invariance of L_A, rather than bi-invariance as in Theorem 4.3.13.

Chapter 5

An operator system point of view at the extension problem for positive semi-definite functions on a discrete group

The results in this chapter were obtained in collaboration with Evgenios T.A. Kakariadis, Ivan G. Todorov, and Walter D. van Suijlekom. See Section 2.2 for the preparations on operator systems which are relevant for this chapter.

5.1 Introduction

As seen in the previous chapter, operator systems arise in a constrained resolution approach to the spectral point of view at noncommutative geometry [33] in two ways in particular. On the one hand as compressions PAP of a C*-algebra $A \subseteq \mathcal{B}(H)$ by a projection $P \in \mathcal{B}(H)$ (e.g. a spectral projection for a Dirac operator on H); and on the other hand, as certain operator subsystems of A. In the case of the C*-algebra of continuous functions on the circle $A = \mathrm{C}(\mathrm{S}^1)$, the operator systems of interest are the *Toeplitz system* $\mathrm{C}(\mathrm{S}^1)^{(\{1,\dots,N\})}$ consisting of $N \times N$ Toeplitz matrices $(T_{l-k})_{l,k=1}^{N}$ and the *Fourier system* $\mathrm{C}(\mathrm{S}^1)_{(\{-N+1,\dots,N-1\})}$ which is the operator subsystem of $\mathrm{C}^*(\mathbb{Z}) \cong \mathrm{C}(\mathrm{S}^1)$ spanned by finitely supported sequences $(\dots, 0, a_{-N+1}, \dots, a_0, \dots, a_{N-1}, 0, \dots)$. Using the matrix-valued Fejér–Riesz lemma it was shown that the Toeplitz and the Fourier system are unitally completely order isomorphic to each other's operator system duals [33, 50]. As we will explain shortly, it seems appropriate to consider this duality of operator systems as a property of the group $\mathbb{Z}$ and the subset $\{1, \dots, N\}$; it is then natural to ask, for which discrete groups Γ and finite subsets Σ analogous results hold.

To this end, for a discrete group Γ and a finite subset $\Sigma \subseteq \Gamma$, we denote by $\mathrm{C}^*(\Gamma)^{(\Sigma)}$ the *Toeplitz system*, i.e. the operator system consisting of Toeplitz matrices $(T_{st^{-1}})_{s,t\in\Sigma} \in \mathrm{M}_\Sigma \cong \mathcal{B}(\ell^2(\Sigma))$. Moreover, denote by $\mathrm{C}^*(\Gamma)_{(\Sigma\Sigma^{-1})}$ the *Fourier system*, i.e. the operator subsystem of $\mathrm{C}^*(\Gamma)$ spanned by $\delta_{st^{-1}}$, $s, t \in \Sigma$, where δ_s,

$s \in \Gamma$, are the unitary generators of the group C^*-algebra. The Toeplitz system $\mathrm{C}^*(\Gamma)^{(\Sigma)}$ arises both as the compression of the (reduced) group C^*-algebra by the orthogonal projection $P : \ell^2(\Gamma) \to \ell^2(\Sigma)$ and as the image of the restriction map $\rho : \mathrm{B}(\Gamma) \to \mathrm{M}_\Sigma$ of functions in the Fourier–Stieltjes algebra $\mathrm{B}(\Gamma)$ to the set $\Sigma\Sigma^{-1}$. We adopt the latter point of view. Note that the Toeplitz and the Fourier system both have dimension $|\Sigma\Sigma^{-1}|$.

It is not hard to see that Toeplitz matrices and sums of squares of (matrices over) elements of $\mathrm{span}\{\delta_s \mid s \in \Delta\}$ are dual to each other. This duality was already exploited by Rudin [127] in relating the extension problem for partially defined positive semi-definite functions to a Fejér–Riesz type factorization property. See for instance [131, 130, 10] for extensive treatments and surveys, and [8, 9]. We capture the sums-of-squares concept by introducing the *sums-of-squares system* $\mathrm{SOS}(\Sigma)$ and we show that the Toeplitz system $\mathrm{C}^*(\Gamma)^{(\Sigma)}$ is its operator system dual. As vector spaces the sums of squares system and the Fourier system are canonically isomorphic, and in fact the canonical inclusion map $\iota : \mathrm{SOS}(\Sigma) \to \mathrm{C}^*(\Gamma)$ has image the Fourier system $\mathrm{C}^*(\Gamma)_{(\Sigma\Sigma^{-1})}$ and can be easily seen to be ucp. This map is a complete order embedding precisely if the set $\Sigma \subseteq \Gamma$ has a Fejér–Riesz type complete factorization property.

The Fourier–Stieltjes algebra $\mathrm{B}(\Gamma)$ is the operator space and the matrix-ordered vector space dual of the group C^*-algebra $\mathrm{C}^*(\Gamma)$. Moreover, the restriction map $\rho : \mathrm{B}(\Gamma) \to \mathrm{C}^*(\Gamma)^{(\Sigma)}$ is the dual map of the canonical inclusion map ι. We exploit this to show that the canonical inclusion map ι is a complete order embedding if and only if the restriction map ρ is a quotient map of matrix-ordered vector spaces in Theorem 5.3.18; the latter means that the set Σ has a Rudin type complete extension property. We will see that these properties characterize when the Toeplitz and the Fourier system are each other's operator system duals.

Note that if the group Γ is amenable, its group C^*-algebra $\mathrm{C}^*(\Gamma) \cong \mathrm{C}^*_{\mathrm{r}}(\Gamma)$ (with comultiplication $\mathrm{C}^*_{\mathrm{r}}(\Gamma) \ni \lambda(s) \mapsto \lambda(s) \otimes \lambda(s) \in \mathrm{C}^*_{\mathrm{r}}(\Gamma) \otimes \mathrm{C}^*_{\mathrm{r}}(\Gamma)$) is a coamenable compact quantum group in the sense of Woronowicz [145], and the Toeplitz and Fourier systems are the operator systems arising respectively as Peter–Weyl [93] and Fourier truncations [123]. In this way, we provide a connection between these two types of truncations. As a byproduct we obtain the fact that minimal and maximal tensor products of the operator system of Toeplitz matrices with itself do not agree, as a corollary of Rudin's result that squares in $\mathbb{Z}^d$ do not have the extension property [127], thereby simplifying the argument in [50].

More generally, the problem of extending positive semi-definite (operator-valued) functions is usually phrased for *positivity domains*, i.e. subsets $\Delta \subseteq \Gamma$ with $\Delta^{-1} = \Delta \ni e$. A matrix-valued function $u : \Delta \to \mathrm{M}_n$ is called *positive semi-definite* if the Toeplitz matrix $(u(st^{-1}))_{s,t\in\Sigma} \in \mathrm{M}_\Sigma(\mathrm{M}_n) \cong \mathcal{B}(\ell^2(\Sigma)^n)$ is positive semi-definite, for all finite subsets $\Sigma \subseteq \Gamma$ such that $\Sigma\Sigma^{-1} \subseteq \Delta$. A positivity domain Δ is said to have the *complete extension property* if every positive semi-definite function $u : \Delta \to \mathrm{M}_n$ admits an extension to a positive semi-definite function on Γ. We observe that positive semi-definite functions on a positivity domain span a matrix-ordered vector space $\mathrm{B}(\Delta)$ which we identify as the dual of an operator system $\mathrm{P}(\Delta)$. The latter can be built up from (the positive matrix-cones of) sums-of-squares systems.

Similarly as for Toeplitz matrices and sums of squares, we relate the extension property for positive semi-definite functions to a factorization property in Theorem 5.4.16. As an example we discuss a positivity domain, depicted in Figure 5.3, consisting of five points in $\mathbb{Z}^2$, and show that the operator system $\mathrm{P}(\Delta)$ is the amalgamated direct sum of two copies of the sums-of-squares systems generated by $\delta_0, \delta_1 \in \mathbb{C}[\mathbb{Z}]$. This allows us to compute the minimal C*-cover of the operator system $\mathrm{P}(\Delta)$, *viz.* $\mathrm{C}^*(\mathbb{F}_2)$. On the other hand, the minimal C*-cover of the Fourier system $\mathrm{C}^*(\mathbb{Z}^2)_{(\Delta)}$ is $\mathrm{C}^*(\mathbb{Z}^2)$; we conclude that the operator systems $\mathrm{C}^*(\mathbb{Z}^2)_{(\Delta)}$ and $\mathrm{P}(\Delta)$ are not completely order isomorphic, so this positivity domain Δ does not have the complete factorization and the complete extension properties.

$$\mathrm{B}(\Gamma) \twoheadrightarrow \mathrm{B}(\Gamma)|_{\Delta} \longrightarrow \mathrm{B}(\Delta) \longrightarrow \mathrm{C}^*(\Gamma)^{(\Sigma)}$$

$$\mathrm{C}^*(\Gamma) \hookleftarrow \mathrm{C}^*(\Gamma)_{(\Delta)} \longleftarrow \mathrm{P}(\Delta) \longleftarrow \mathrm{SOS}(\Sigma)$$

Figure 5.1: The operator systems (in the second line) and their respective matrix-ordered vector space duals (in the first line), together with the canonical cp maps, which we discuss in this chapter; here Γ is a discrete group, $\Delta \subseteq \Gamma$ a positivity domain and $\Sigma \subseteq \Gamma$ a finite subset with $\Sigma\Sigma^{-1} \subseteq \Delta$.

5.2 Operator-valued positive semi-definite functions, their restrictions and the Fourier system

Let Γ be a discrete group and H a Hilbert space. We describe the natural generalization of positive semi-definiteness to the operator-valued case.

Notation 5.2.1. We denote by δ_s, $s \in \Gamma$, the unitary generators of the universal group C*-algebra $\mathrm{C}^*(\Gamma)$. For a subset $\Sigma \subseteq \Gamma$, we write $\mathrm{S}[\Sigma] := \operatorname{span}\{\delta_s \mid s \in \Sigma\}$.

Notation 5.2.2. Let $X \subseteq \mathcal{B}(H)$ be an operator space. For a finite subset $\Sigma \subseteq \Gamma$, we denote by $\mathrm{M}_\Sigma(X)$ the set of matrices over X indexed by Σ, which may be identified with the space of operators $X \otimes \mathcal{B}(\ell^2(\Sigma)) \subseteq \mathcal{B}(\ell^2(\Sigma, H))$. We set $\mathrm{M}_\Sigma := \mathrm{M}_\Sigma(\mathbb{C})$.

Definition 5.2.3. A function $u : \Gamma \to \mathcal{B}(H)$ is called *positive semi-definite* if the Toeplitz matrix $(u(st^{-1}))_{s,t\in\Sigma} \in M_\Sigma(\mathcal{B}(H))$ is positive semi-definite, for all finite subsets $\Sigma \subseteq \Gamma$ and $s, t \in \Sigma$. We denote by $\mathrm{B}(\Gamma, H)^+$ the cone of all positive semi-definite $\mathcal{B}(H)$-valued functions, and by $\mathrm{B}(\Gamma, H)$ the linear span of $\mathrm{B}(\Gamma, H)^+$ inside the *-vector space of all $\mathcal{B}(H)$-valued functions defined on Γ. We call $\mathrm{B}(\Gamma, H)$ the *$\mathcal{B}(H)$-valued Fourier–Stieltjes algebra* of Γ. Clearly, if $H = \mathbb{C}$ we obtain the usual *Fourier–Stieltjes algebra*, denoted by $\mathrm{B}(\Gamma)$.

The following proposition is probably well-known.

Proposition 5.2.4. *Let Γ be a discrete group.*

(i) If $u \in \mathrm{B}(\Gamma, H)^+$ is a positive semi-definite function there exists a unique completely positive map $\phi_u : \mathrm{C}^*(\Gamma) \to \mathcal{B}(H)$ such that $\phi_u(\delta_s) = u(s)$, for all $s \in \Gamma$.

(ii) If $\phi : \mathrm{C}^*(\Gamma) \to \mathcal{B}(H)$ is a completely positive map there exists a unique positive semi-definite function $u_\phi \in \mathrm{B}(\Gamma, H)^+$ such that $u_\phi(s) = \phi(\delta_s)$, for all $s \in \Gamma$.

Proof. (i) Given elements $s_i \in \Gamma$ and complex coefficients $c_i \in \mathbb{C}$, $i = 1, \dots, k$, and $a = \sum_{i=1}^k c_i \delta_{s_i} \in \mathbb{C}[\Gamma]$, we denote by $\phi_u(a) := \sum_{i=1}^k c_i u(s_i)$ the linear extension of u from Γ to $\mathbb{C}[\Gamma]$. Let $A \in \mathrm{M}_n(\mathbb{C}[\Gamma])$, and write $A = \sum_{i=1}^k C_i \otimes \delta_{s_i}$, where $C_i \in \mathrm{M}_n$, $i = 1, \dots, k$. We have that $AA^* = \sum_{i,j=1}^k C_i C_j^* \otimes \delta_{s_i s_j^{-1}}$, and hence

$$\phi_u^{(n)}(AA^*) = \sum_{i,j=1}^k C_i C_j^* \otimes u(s_i s_j^{-1}).$$

Since u is positive semi-definite, we have that $(u(s_i s_j^{-1}))_{i,j} \in \mathrm{M}_k(\mathcal{B}(H))^+$; on the other hand, clearly, $(C_i C_j^*)_{i,j} \in \mathrm{M}_k(\mathrm{M}_n)^+$. It is now straightforward to see that $\phi_u^{(n)}(AA^*) \in \mathrm{M}_n(\mathcal{B}(H))^+$. By density of $\mathbb{C}[\Gamma]$ it follows that ϕ_u extends to a cp map $\mathrm{C}^*(\Gamma) \to \mathcal{B}(H)$.

(ii) Conversely, let $u : \Gamma \to \mathcal{B}(H)$ be given by $u(s) = \phi(\delta_s)$. Given $s_i \in \Gamma$, $i = 1, \dots, k$, we have that $(\delta_{s_i s_j^{-1}})_{i,j} \in \mathrm{M}_k(\mathrm{C}^*(\Gamma))^+$; thus, $(u(s_i s_j^{-1}))_{i,j} \in \mathrm{M}_k(\mathcal{B}(H))^+$, that is, u is positive semi-definite. Since ϕ and ϕ_u agree on $\mathbb{C}[\Gamma]$, they coincide by density.

The uniqueness statements in (i) and (ii) follow since the assignments $u \mapsto \phi_u$ and $\phi \mapsto u_\phi$ are inverse to each other. □

The above proposition gives a canonical identification of the cones

$$\mathrm{B}(\Gamma, H)^+ \cong \mathcal{CP}(\mathrm{C}^*(\Gamma), \mathcal{B}(H)) \tag{5.1}$$

implying that the family of cones $(\mathrm{B}(\Gamma, \mathbb{C}^n)^+)_{n\in\mathbb{N}}$ is compatible; in particular the pair $(\mathrm{B}(\Gamma), (\mathrm{B}(\Gamma, \mathbb{C}^n)^+)_{n\in\mathbb{N}})$ is a matrix-ordered vector space.

Remark 5.2.5. We define a norm on $\mathrm{B}(\Gamma, H)$ by $\|u\|_{\mathrm{B}(\Gamma,H)} := \|\phi_u\|_{\mathcal{CB}(\mathrm{C}^*(\Gamma),\mathcal{B}(H))}$. In the case that $H = \mathbb{C}^n$, this can be viewed as a norm on $\mathrm{M}_n(\mathrm{B}(\Gamma))$ and in fact the Fourier–Stieltjes algebra with the matrix norms $\|\cdot\|_{\mathrm{M}_n(\mathrm{B}(\Gamma))} := \|\cdot\|_{\mathrm{B}(\Gamma,\mathbb{C}^n)}$ is the operator space dual of $\mathrm{C}^*(\Gamma)$. Together with the above established matrix-order structure this turns $\mathrm{B}(\Gamma)$ into a dual matrix-ordered operator space, as studied e.g. in [72].

We now study restrictions of $\mathcal{B}(H)$-valued positive semi-definite functions to positivity domains.

Definition 5.2.6. A subset $\Delta \subseteq \Gamma$ is called a *positivity domain* if it is symmetric, i.e. $\Delta^{-1} := \{s^{-1} \mid s \in \Delta\} = \Delta$, and unital, i.e. $e \in \Delta$.

Note that if $\Delta \subseteq \Gamma$ is a positivity domain, the vector space $\mathrm{S}[\Delta] := \operatorname{span}\{\delta_s \mid s \in \Delta\}$ comes with an involution $\delta_s^* := \delta_{s^{-1}}$.

Example 5.2.7. Let $\Sigma \subseteq \Gamma$ be a subset and set $\Sigma\Sigma^{-1} := \{st^{-1} \mid s, t \in \Sigma\}$. Then the set $\Sigma\Sigma^{-1}$ is a positivity domain. However, see Figure 5.3 for an example of a positivity domain in $\mathbb{Z}^2$ which is not of the form $\Sigma\Sigma^{-1}$, for any subset $\Sigma \subseteq \mathbb{Z}^2$.

Definition 5.2.8. Let $\Delta \subseteq \Gamma$ be a positivity domain. The *Fourier system* is the operator subsystem $\mathrm{C}^*(\Gamma)_{(\Delta)} \subseteq \mathrm{C}^*(\Gamma)$ spanned by δ_s, for all $s \in \Delta$. I.e. the positive cones of the Fourier system are given by $\mathrm{M}_n(\mathrm{C}^*(\Gamma)_{(\Delta)})^+ = \mathrm{M}_n(\mathrm{C}^*(\Gamma))^+ \cap \mathrm{M}_n(\mathrm{S}[\Delta])$ and the matrix order-unit by $e_{\mathrm{C}^*(\Gamma)_{(\Delta)}} = \delta_e$.

For a positivity domain Δ, we denote by $\mathrm{B}(\Gamma, H)^+|_\Delta$ the cone given by the restrictions of all positive semi-definite functions $\Gamma \to \mathcal{B}(H)$ to Δ, and similarly we denote by $\mathrm{B}(\Gamma, H)|_\Delta$ the restrictions of functions in $\mathrm{B}(\Gamma, H)$ to Δ. Note that $\mathrm{B}(\Gamma, H)|_\Delta = \operatorname{span}(\mathrm{B}(\Gamma, H)^+|_\Delta)$. We set $\mathrm{B}(\Gamma)|_\Delta := \mathrm{B}(\Gamma, \mathbb{C})|_\Delta$.

Proposition 5.2.9. *Let Γ be a discrete group and $\Delta \subseteq \Gamma$ a positivity domain.*

(i) If $u \in \mathrm{B}(\Gamma, H)^+|_\Delta$ is the restriction of a positive semi-definite function on Γ to Δ then there exists a unique completely positive map $\phi_u : \mathrm{C}^*(\Gamma)_{(\Delta)} \to \mathcal{B}(H)$ such that $\phi_u(\delta_s) = u(s)$, for all $s \in \Delta$.

(ii) If $\phi : \mathrm{C}^*(\Gamma)_{(\Delta)} \to \mathcal{B}(H)$ is a completely positive map then there exists a unique function $u : \Delta \to \mathcal{B}(H)$ which extends to a positive semi-definite function on Γ such that $\phi = \phi_u$.

Proof. (i) Let $u \in \mathrm{B}(\Gamma, H)^+|_\Delta$ and let $\tilde{u} \in \mathrm{B}(\Gamma, H)^+$ be such that $\tilde{u}|_\Delta = u$. By part (i) of Proposition 5.2.4 there is a unique completely positive map $\phi_{\tilde{u}} \in \mathcal{CP}(\mathrm{C}^*(\Gamma), \mathcal{B}(H))$ such that $\phi_{\tilde{u}}(\delta_s) = \tilde{u}(s)$, for all $s \in \Gamma$. Restricting $\phi_{\tilde{u}}$ to the Fourier system $\mathrm{C}^*(\Gamma)_{(\Delta)}$ gives the desired completely positive map $\phi_u \in \mathcal{CP}(\mathrm{C}^*(\Gamma)_{(\Delta)}, \mathcal{B}(H))$. If $\tilde{u}_1, \tilde{u}_2 \in \mathrm{B}(\Gamma, H)^+$ are extensions of u, then $\phi_{\tilde{u}_1}(\delta_s) = \phi_{\tilde{u}_2}(\delta_s)$, for all $s \in \Delta$, which implies uniqueness of ϕ_u.

(ii) Let $\phi \in \mathcal{CP}(\mathrm{C}^*(\Gamma)_{(\Delta)}, \mathcal{B}(H))$ be a completely positive map. By Arveson's extension theorem, we may extend ϕ to a completely positive map $\tilde{\phi} : \mathrm{C}^*(\Gamma) \to \mathcal{B}(H)$. By part (ii) of Proposition 5.2.4 we obtain a positive semi-definite function $u_{\tilde{\phi}} : \Gamma \to \mathcal{B}(H)$, such that $u_{\tilde{\phi}}(s) = \tilde{\phi}(\delta_s)$, for all $s \in \Gamma$. Restricting $u_{\tilde{\phi}}$ to Δ gives the desired function $u_\phi : \Delta \to \mathcal{B}(H)$. If $\tilde{\phi}_1, \tilde{\phi}_2 \in \mathcal{CP}(\mathrm{C}^*(\Gamma), \mathcal{B}(H))$ are extensions of ϕ, then $u_{\tilde{\phi}_1}(s) = u_{\tilde{\phi}_2}(s)$, for all $s \in \Delta$, which implies uniqueness of u_ϕ. □

Similarly as in (5.1) the above proposition gives a canonical identification of the cones

$$\mathrm{B}(\Gamma, H)^+|_\Delta \cong \mathcal{CP}(\mathrm{C}^*(\Gamma)_{(\Delta)}, \mathcal{B}(H)),$$

implying the following:

Corollary 5.2.10. *The pair $(\mathrm{B}(\Gamma)|_\Delta, (\mathrm{B}(\Gamma, \mathbb{C}^n)^+|_\Delta)_{n \in \mathbb{N}})$ is a matrix-ordered vector space.*

Moreover, the identification

$$\begin{aligned} \mathrm{B}(\Gamma)|_\Delta &= \operatorname{span}(\mathrm{B}(\Gamma, H)^+|_\Delta) \\ &\cong \operatorname{span}(\mathcal{CP}(\mathrm{C}^*(\Gamma)_{(\Delta)}, \mathcal{B}(H))) \\ &= \mathcal{CB}(\mathrm{C}^*(\Gamma)_{(\Delta)}, \mathcal{B}(H)) \end{aligned}$$

turns $\mathrm{B}(\Gamma)|_\Delta$ into a (matrix-ordered) operator space.

5.3 Sums of squares and Toeplitz matrices

In this section we discuss the well-known fact that Toeplitz matrices are dual to sums of squares in terms of duality of operator systems. Throughout, Γ will be a discrete group and $\Sigma \subseteq \Gamma$ a finite subset. We begin by defining an operator system spanned by the unitaries $\delta_{st^{-1}}$ in $\mathrm{C}^*(\Gamma)$, for $s, t \in \Sigma$; we let its positive matrix-cones consist of those elements which admit a decomposition into sums of squares, rather than the positive matrix cones inherited from $\mathrm{C}^*(\Gamma)$. The resulting operator system will be denoted $\mathrm{SOS}(\Sigma)$ and one may think of it as the operator system structure on $\mathrm{S}[\Sigma\Sigma^{-1}] = \operatorname{span}\{\delta_{st^{-1}} \mid s, t \in \Sigma\}$, for which a Fejér–Riesz type property is enforced.

5.3.1 The sums of squares system and the complete Fejér–Riesz property

The following sets will constitute the positive matrix cones for our operator system $\mathrm{SOS}(\Sigma)$.

Definition 5.3.1. For $n \in \mathbb{N}$, set

$$\begin{aligned} \mathcal{Q}_n(\Sigma) := &\left\{ \sum_{i=1}^r y_i y_i^* \mid r \in \mathbb{N}, y_i = \sum_{s\in\Sigma} A_s^i \otimes \delta_s, A_s^i \in \mathrm{M}_n \right\} \\ &\subseteq \mathrm{M}_n(\mathrm{S}[\Sigma\Sigma^{-1}])_{\mathrm{h}}. \end{aligned}$$

Lemma 5.3.2. *The following identities hold:*

$$\mathcal{Q}_n(\Sigma) = \left\{ yy^* \mid y = \sum_{s\in\Sigma} A_s \otimes \delta_s, A_s \in \mathrm{M}_{n,k}, k \in \mathbb{N} \right\} \tag{5.2}$$

$$= \left\{ \sum_{s,t\in\Sigma} A_{s,t} \otimes \delta_{st^{-1}} \mid (A_{s,t})_{s,t\in\Sigma} \in (\mathrm{M}_\Sigma(\mathrm{M}_n))^+ \right\}. \tag{5.3}$$

Proof. Let $x \in \mathcal{Q}_n(\Sigma)$, i.e. $x = \sum_{i=1}^r y_i y_i^*$ with $y_i = \sum_{s\in\Sigma} A_s^i \otimes \delta_s \in \mathrm{M}_n \otimes \mathrm{S}[\Sigma]$; we want to show that x is an element of the right-hand side of (5.2). To this end note that we have $x = \sum_{i=1}^r y_i y_i^* = (y_1, \ldots, y_r)(y_1^*, \ldots, y_r^*)^{\mathrm{t}}$. It follows that $x = yy^*$ with $y = (y_1, \ldots, y_r) \in \mathrm{M}_{1,r}(\mathrm{M}_n \otimes \mathrm{S}[\Sigma]) \cong \mathrm{M}_{n,nr} \otimes \mathrm{S}[\Sigma]$; so x is indeed an element of the right-hand side of (5.2).

To show the inclusion of the right-hand side of (5.2) in the set displayed in (5.3), fix $y = \sum_{s\in\Sigma} A_s \otimes \delta_s$, for $A_s \in \mathrm{M}_{n,k}$ with $k \in \mathbb{N}$, as in (5.2). Note that $yy^* = \sum_{s,t\in\Sigma} A_s A_t^* \otimes \delta_{st^{-1}}$ and $(A_s A_t^*)_{s,t\in\Sigma} \in (\mathrm{M}_\Sigma(\mathrm{M}_n))^+$. Setting $A_{s,t} := A_s A_t^* \in \mathrm{M}_\Sigma(\mathrm{M}_n)$ we see that $yy^* = \sum_{s,t\in\Sigma} A_{s,t} \otimes \delta_{st^{-1}}$ is indeed an element of the set displayed in (5.3).

To check the remaining inclusion, i.e. the set displayed in (5.3) is contained in $\mathcal{Q}_n(\Sigma)$, let $x = \sum_{s,t\in\Sigma} A_{s,t} \otimes \delta_{st^{-1}}$ with $(A_{s,t})_{s,t\in\Sigma} \in (\mathrm{M}_\Sigma(\mathrm{M}_n))^+$. Then there is a matrix $B = (B_{s,t})_{s,t\in\Sigma} \in \mathrm{M}_\Sigma(\mathrm{M}_n)$ such that $A = BB^* = (\sum_{r\in\Sigma} B_{s,r}(B^*)_{r,t})_{s,t\in\Sigma}$. Note that $(B^*)_{r,t} = (B_{t,r})^* \in \mathrm{M}_n$, for all $r, t \in \Sigma$. It follows that

$$x = \sum_{s,t\in\Sigma} A_{s,t} \otimes \delta_{st^{-1}} = \sum_{s,t\in\Sigma} \sum_{r\in\Sigma} B_{s,r}(B^*)_{r,t} \otimes \delta_{st^{-1}}$$

$$= \sum_{r\in\Sigma} \left(\sum_{s\in\Sigma} B_{s,r} \otimes \delta_s \right) \left(\sum_{t\in\Sigma} B_{t,r} \otimes \delta_t \right)^* \in \mathcal{Q}_n(\Sigma).$$

This finishes the proof. □

Lemma 5.3.3. *The family $(\mathcal{Q}_n(\Sigma))_{n\in\mathbb{N}}$ is a compatible family of proper convex cones respectively contained in $\mathrm{M}_n(\mathrm{C}^*(\Gamma))^+$. It is the smallest such family whose n-th level contains the set $\{(\delta_{s_i s_j^{-1}})_{i,j=1}^n \mid s_1, \ldots, s_n \in \Sigma\}$. The element δ_e is a matrix order-unit for the family $(\mathcal{Q}_n(\Sigma))_{n\in\mathbb{N}}$. In particular, the triple $(\mathrm{S}[\Sigma\Sigma^{-1}], (\mathcal{Q}_n(\Sigma))_{n\in\mathbb{N}}, \delta_e)$ is a matrix order-unit space and its canonical inclusion into $\mathrm{C}^*(\Gamma)$ is ucp.*

Proof. It is clear that sums of squares are positive, whence $\mathcal{Q}_n(\Sigma) \subseteq \mathrm{M}_n(\mathrm{C}^*(\Gamma))^+$. The fact that $(\mathcal{Q}_n(\Sigma))_{n\in\mathbb{N}}$ is a compatible family of proper convex cones then follows from (5.3).

Note that $(\delta_{s_i s_j^{-1}})_{i,j=1}^n = (\sum_{i=1}^n E_i \otimes \delta_{s_i})(\sum_{i=1}^n E_i \otimes \delta_{s_i})^* \in \mathcal{Q}_n(\Sigma)$, for E_i the matrix units in $\mathrm{M}_{n,1}$ and $s_1, \ldots, s_n \in \Sigma$; thus the matrix $(\delta_{s_i s_j^{-1}})_{i,j=1}^n$ is an element of $\mathcal{Q}_n(\Sigma)$. Conversely, if $(\mathcal{C}_n)_{n\in\mathbb{N}}$ is any compatible family of proper convex cones with $(\delta_{s_i s_j^{-1}})_{i,j=1}^n \in \mathcal{C}_n$, for all $s_1, \ldots, s_n \in \Sigma$, then the cone $\mathcal{C}_n$ must contain $A \otimes (\delta_{s_i s_j^{-1}})_{i,j=1}^k$, for all $A \in \mathrm{M}_l^+$, $s_1, \ldots, s_k \in \Sigma$ and $kl = n$. By the identification $\mathrm{M}_l \otimes \mathrm{M}_k(\mathrm{S}[\Sigma\Sigma^{-1}]) \cong \mathrm{M}_n \otimes \mathrm{S}[\Sigma\Sigma^{-1}]$, for every $A \in \mathrm{M}_l^+$, there is a matrix $(A_{i,j})_{i,j} \in (\mathrm{M}_k(\mathrm{M}_l))^+$ such that $A \otimes (\delta_{s_i s_j^{-1}})_{i,j=1}^k = \sum_{i,j=1}^k A_{i,j} \otimes \delta_{s_i s_j^{-1}}$. Hence $\mathcal{C}_n \supseteq \mathcal{Q}_n(\Sigma)$. So $\mathcal{Q}_n(\Sigma)$ is indeed the smallest compatible family of matrix cones containing the matrix $(\delta_{s_i s_j^{-1}})_{i,j=1}^n$ at the n-th level.

To see that $(\delta_e)_n = \mathbf{1}_n \otimes \delta_e$ is an order-unit for $\mathcal{Q}_n(\Sigma)$, let $s \in \Sigma$ and $A \in \mathrm{M}_n$, and define a hermitian element $x \in \mathrm{M}_n(\mathrm{S}[\Sigma\Sigma^{-1}])_\mathrm{h}$ by

$$x := x(s, A) := A \otimes \delta_s + A^* \otimes \delta_{s^{-1}}. \tag{5.4}$$

Set $r(x) := 2\|A\|$. Write $A = V|A|$ in its polar decomposition and set $A_1 := -V|A|^{\frac{1}{2}}$ and $A_2 := |A|^{\frac{1}{2}}$. Then

$$\begin{aligned}
&r(x) \cdot (\mathbf{1}_n \otimes \delta_e) - x \\
&\geqslant (V|A|V^* + |A|) \otimes \delta_e - (-V|A|^{\frac{1}{2}}|A|^{\frac{1}{2}} \otimes \delta_s - |A|^{\frac{1}{2}}|A|^{\frac{1}{2}}V \otimes \delta_{s^{-1}}) \\
&= (A_1A_1^* + A_2A_2^*) \otimes \delta_e + A_1A_2^* \otimes \delta_s + A_2A_1^* \otimes \delta_{s^{-1}} \\
&= (A_1 \otimes \delta_s + A_2 \otimes \delta_e)\,(A_1 \otimes \delta_s + A_2 \otimes \delta_e)^* \in \mathcal{Q}_n(\Sigma).
\end{aligned}$$

Now, if $y \in \mathrm{M}_n(\mathrm{S}[\Sigma\Sigma^{-1}])_{\mathrm{h}}$ is any hermitian element we have $y = \sum_{s\in\Sigma} x(s, A_s)$, with $x(s, A_s)$ as in (5.4). Define the real number $r(y) := 2\sum_{s\in\Sigma}\|A_s\|$. It follows that $r(y) \cdot (\mathbf{1}_n \otimes \delta_e) - y \in \mathcal{Q}_n(\Sigma)$. This shows that $(\delta_e)_n$ is an order-unit for the ordered vector space $(\mathrm{M}_n(\mathrm{S}[\Sigma\Sigma^{-1}]), \mathcal{Q}_n(\Sigma))$. □

The assumption that Σ is finite allows us to show that δ_e is in fact an archimedean matrix order-unit for the matrix-ordered vector space $(\mathrm{S}[\Sigma\Sigma^{-1}], (\mathcal{Q}_n(\Sigma))_{n\in\mathbb{N}})$.

Proposition 5.3.4. *The matrix order-unit space* $(\mathrm{S}[\Sigma\Sigma^{-1}], (\mathcal{Q}_n(\Sigma))_{n\in\mathbb{N}}, \delta_e)$ *is an operator system.*

The proof is a matrix version of the proof of [127, Lemma 1.3].

Proof. We show that the cones $\mathcal{Q}_n(\Sigma)$ are norm-closed in $\mathrm{M}_n(\mathrm{C}^*(\Gamma))$. Denote by $d := |\Sigma\Sigma^{-1}|$ the number of elements of the set $\Sigma\Sigma^{-1}$. Every $x \in \mathcal{Q}_n(\Sigma)$ is a sum of at most n^2d squares in $\mathrm{M}_n \otimes \mathrm{S}[\Sigma\Sigma^{-1}]$; to see this, let $x = \sum_{i=1}^{r} y_iy_i^* \in \mathcal{Q}_n(\Sigma)$ and assume that $r > n^2d$. Since $\mathcal{Q}_n(\Sigma) - \mathcal{Q}_n(\Sigma) \subseteq \mathrm{M}_n(\mathrm{S}[\Sigma\Sigma^{-1}])_{\mathrm{h}}$, we have $\dim_{\mathbb{R}}(\mathcal{Q}_n(\Sigma) - \mathcal{Q}_n(\Sigma)) \leqslant \dim_{\mathbb{R}}(\mathrm{M}_n(\mathrm{S}[\Sigma\Sigma^{-1}]))_{\mathrm{h}} = n^2d$; thus there are $\alpha_1, \dots, \alpha_r \in \mathbb{R}\backslash\{0\}$ such that $\sum_{i=1}^{r} \alpha_iy_iy_i^* = 0$. We may assume that $\alpha_i \leqslant \alpha_{i+1}$, for all $i = 1, \dots, r-1$. It follows that $x = \sum_{i=1}^{r-1}(1 - \frac{\alpha_i}{\alpha_r})y_iy_i^*$. Hence x is a sum of at most n^2d squares.

Now, let $x^k = \sum_{i=1}^{n^2d} y_i^k(y_i^k)^* \in \mathcal{Q}_n(\Sigma)$ be a sequence with $\lim_k x^k = x \in \mathrm{M}_n\otimes\mathrm{S}[\Sigma\Sigma^{-1}]$. For every $i = 1, \dots, n^2d$, we have that $\|y_i^k\| \leqslant \|x^k\| \leqslant \sup_k\|x^k\| < \infty$. So the sequences $(y_i^k)_k$ are bounded and, since $\dim(\mathrm{M}_n \otimes \mathrm{S}[\Sigma\Sigma^{-1}]) < \infty$, we may choose convergent subsequences $(y_i^{k_j})_j$ and set $\lim_j y_i^{k_j} =: y_i$. It follows that

$$x = \lim_k x^k = \lim_k \sum_{i=1}^{n^2d} y_i^k(y_i^k)^* = \lim_j \sum_{i=1}^{n^2d} y_i^{k_j}(y_i^{k_j})^* = \sum_{i=1}^{n^2d} y_iy_i^*.$$

This shows that $x \in \mathcal{Q}_n(\Sigma)$, so $\mathcal{Q}_n(\Sigma)$ is norm closed in $\mathrm{M}_n(\mathrm{C}^*(\Gamma))$.

To see that δ_e is an archimedean matrix order-unit, recall from [108] that the order topology is a locally convex vector space topology, whence equivalent to the norm topology, since $\mathrm{S}[\Sigma\Sigma^{-1}]$ is finite-dimensional. Thus the cones $\mathcal{Q}_n(\Sigma)$ are closed in the order topology, and, by [108, Theorem 2.30, Remark 3.4], this implies that $(\delta_e)_n$ is an archimedean order-unit for $\mathrm{M}_n(\mathrm{S}[\Sigma\Sigma^{-1}])$. □

Definition 5.3.5. We call the operator system $(\mathrm{S}[\Sigma\Sigma^{-1}], (\mathcal{Q}_n(\Sigma))_{n\in\mathbb{N}}, \delta_e)$ the *sums-of-squares system* and denote it by $\mathrm{SOS}(\Sigma)$.

Note that Lemma 5.3.3 and Proposition 5.3.4 show in particular that the triple $V := (\mathrm{S}[\Sigma\Sigma^{-1}], \mathcal{Q}_1(\Sigma), \delta_e)$ is an archimedean order-unit space. By minimality of the compatible family of cones $(\mathcal{Q}_n(\Sigma))_{n\in\mathbb{N}}$ (Lemma 5.3.3) it follows that $\mathrm{SOS}(\Sigma)$ is the maximal operator system structure $\mathrm{OMAX}(V)$ on V in the terminology of [107].

Given two discrete groups Γ_1, Γ_2 and finite subsets $\Sigma_1 \subseteq \Gamma_1$, $\Sigma_2 \subseteq \Gamma_2$, we identify $\mathrm{S}[\Sigma_1\Sigma_1^{-1} \times \Sigma_2\Sigma_2^{-1}] \cong \mathrm{S}[\Sigma_1\Sigma_1^{-1}]\otimes\mathrm{S}[\Sigma_2\Sigma_2^{-1}]$ canonically. One checks that the cone $\mathcal{Q}_1(\Sigma_1 \times \Sigma_2)$ is identified with the cone

$$\mathcal{D}_1^{\max} := \{\sum_{i=1}^{r} A_i \otimes B_i \mid A_i \in \mathcal{Q}_1(\Sigma_1), B_i \in \mathcal{Q}_1(\Sigma_2), r \in \mathbb{N}\}.$$

It follows that the triple $V_1 \otimes V_2 := (\mathrm{S}[\Sigma_1\Sigma_1^{-1}] \otimes \mathrm{S}[\Sigma_2\Sigma_2^{-1}], \mathcal{D}_1^{\max}, \delta_{e_1} \otimes \delta_{e_2})$ is an archimedean order-unit space. By [107, Proposition 5.13] we have

$$\mathrm{OMAX}(V_1 \otimes V_2) = \mathrm{OMAX}(V_1) \otimes_{\max} \mathrm{OMAX}(V_2),$$

which shows the following proposition:

Proposition 5.3.6. *Let Γ_1 and Γ_2 be discrete groups and $\Sigma_1 \subseteq \Gamma_1$ and $\Sigma_2 \subseteq \Gamma_2$ be finite subsets. Then we have*

$$\mathrm{SOS}(\Sigma_1 \times \Sigma_2) \cong \mathrm{SOS}(\Sigma_1) \otimes_{\max} \mathrm{SOS}(\Sigma_2).$$

Recall from Lemma 5.3.3 that $\mathcal{Q}_n(\Sigma) \subseteq \mathrm{M}_n(\mathrm{C}^*(\Gamma))^+$. In other words, the canonical map $\iota : \mathrm{SOS}(\Sigma) \to \mathrm{C}^*(\Gamma)$ is ucp.

Definition 5.3.7. We say that $\Sigma \subseteq \Gamma$ has the *complete Fejér–Riesz property* if the inclusion of cones $\mathrm{M}_n(\mathrm{C}^*(\Gamma))^+ \cap \mathrm{M}_n(\mathrm{S}[\Sigma\Sigma^{-1}]) \subseteq \mathcal{Q}_n(\Sigma)$ holds. In other words, the set Σ has the complete Fejér–Riesz property if and only if the canonical inclusion map $\iota : \mathrm{SOS}(\Sigma) \to \mathrm{C}^*(\Gamma)$ is a complete order embedding.

Note that since $\iota(\mathrm{SOS}(\Sigma)) = \mathrm{C}^*(\Gamma)_{(\Sigma\Sigma^{-1})}$, the set Σ has the complete Fejér–Riesz property if and only if the sums-of-squares and the Fourier system are (canonically, unitally) completely order isomorphic.

Example 5.3.8. By the operator-valued Fejér–Riesz lemma for the circle, the subset $\Sigma_N := \{0, \ldots, N\} \subseteq \mathbb{Z}$ has the complete Fejér–Riesz property. Note that, in this case, actually every positive matrix $x \in \mathrm{M}_n(\mathrm{C}^*(\mathbb{Z})_{(\Sigma_N\Sigma_N^{-1})})^+$ can be factorized as a *single* square $x = yy^*$, for some $y \in \mathrm{M}_n(\mathrm{S}[\Sigma])$.

Example 5.3.9. In contrast to the previous example, the subsets Σ_N do not have the (complete) Fejér-Riesz property when considered in the finite cyclic groups C_m. Indeed, consider the operator system $\mathrm{C}^*(C_m)_{(\{-N,\ldots,N\})}$ which corresponds to $\Sigma = \{0, 1, \ldots, N\} \subset C_m$. Via finite Fourier transform, this operator system can be identified as the operator subsystem of $\mathrm{C}(C_m)$ consisting of functions of the form

$$f = \left(C_m \ni l \mapsto \sum_{k=-N}^{N} \widehat{f}_k e^{2\pi i l k} \right). \tag{5.5}$$

If the (complete) Fejér-Riesz property did hold, it would imply that all such functions f which are positive (as functions in $\mathrm{C}(C_m)$) allow for a factorization $\widehat{f} = \widehat{g} * \widehat{g}^*$, with $\widehat{g} \in \mathrm{C}^*(C_m)_{(\{0,\ldots,N\})}$. But that would imply that also

$$\theta \mapsto \sum_{k=-N}^{N} \widehat{f}_k e^{ik\theta} = \sum_{k=-N}^{N} (\widehat{g} * \widehat{g}^*)_k e^{ik\theta} = \left| \sum_{k=0}^{N} \widehat{g}_k e^{ik\theta} \right|^2 ; \qquad (\theta \in [0, 2\pi))$$

is a positive function on all of S^1. This cannot be true as there may exist functions in $\mathrm{C}(C_m)$ with support in Fourier restricted to $\{-N, \ldots, N\}$ which are positive as functions on C_m, while the corresponding trigonometric polynomial fails to be positive on S^1; an example is given in Figure 5.2.

Fourier systems in the group C^*-algebras of finite cyclic groups were investigated in the Bachelor's thesis [103], including the remarkable observation that the minimal C^*-covers of their dual operator systems are not commutative.

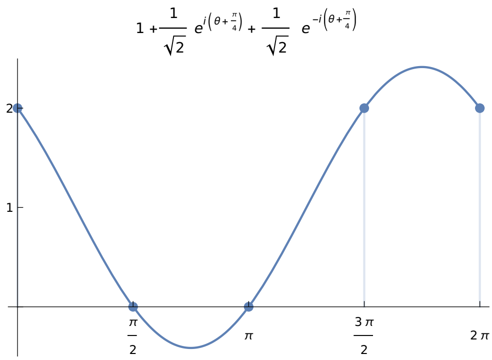

Figure 5.2: A function defined by the Fourier coefficients $\widehat{f}_1 = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}$, $\widehat{f}_0 = 1$, and $\widehat{f}_{-1} = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}}$. The function f on C_4 as given in (5.5) is positive but the corresponding trigonometric polynomial $1 + \sqrt{2}\cos(\theta + \frac{\pi}{4})$ on S^1 is not.

Example 5.3.10. By [127, 129], for $N \geqslant 2$, the set $\{0, \ldots, N\}^2 \subseteq \mathbb{Z}^2$ does not have the complete Fejér–Riesz property.

5.3.2 The Toeplitz system and the complete Σ-extension property

We now study the operator system of Toeplitz matrices associated to a finite subset Σ of a discrete group Γ. We fix a separable Hilbert space H.

Definition 5.3.11. A $\mathcal{B}(H)$*-valued Toeplitz matrix* T is any operator on $\ell^2(\Sigma, H)$ of the form $(T_{st^{-1}})_{s,t\in\Sigma} \in \mathrm{M}_\Sigma(\mathcal{B}(H))$. A function $u : \Sigma\Sigma^{-1} \to \mathcal{B}(H)$ is called *positive semi-definite with respect to* Σ if the associated Toeplitz matrix $(u(st^{-1}))_{s,t\in\Sigma} \in \mathrm{M}_\Sigma(\mathcal{B}(H))$ is positive semi-definite. We call the set of all ($\mathbb{C}$-valued) Toeplitz matrices $(T_{st^{-1}})_{s,t\in\Sigma} \in \mathrm{M}_\Sigma$ the *Toeplitz system* and denote it $\mathrm{C}^*(\Gamma)^{(\Sigma)}$.

By definition the Toeplitz system $\mathrm{C}^*(\Gamma)^{(\Sigma)}$ is an operator subsystem of M_Σ. Moreover, the operator system of $\mathcal{B}(H)$-valued Toeplitz matrices is given by the minimal tensor product $\mathrm{C}^*(\Gamma)^{(\Sigma)} \otimes_{\min} \mathcal{B}(H) \subseteq \mathcal{B}(\ell^2(\Sigma, H))$. From its spatial implementation, it is immediate that the Toeplitz system of a product of two sets Σ_1 and Σ_2 is the minimal tensor product of the respective associated Toeplitz systems:

Proposition 5.3.12. *Let Γ_1 and Γ_2 be discrete groups and $\Sigma_1 \subseteq \Gamma_1$, $\Sigma_2 \subseteq \Gamma_2$ finite subsets. Then we have*

$$\mathrm{C}^*(\Gamma_1 \times \Gamma_2)^{(\Sigma_1 \times \Sigma_2)} = \mathrm{C}^*(\Gamma_1)^{(\Sigma_1)} \otimes_{\min} \mathrm{C}^*(\Gamma_2)^{(\Sigma_2)}.$$

Our interest in the Toeplitz system (and the notation $\mathrm{C}^*(\Gamma)^{(\Sigma)}$) came from its appearance in spectral truncations in noncommutative geometry [33, 94, 93] and its analysis in [50]. It is straightforward to check that $\mathrm{C}^*(\Gamma)^{(\Sigma)} = P\mathrm{C}^*_{\mathrm{r}}(\Gamma)P$, for $\mathrm{P} : \ell^2(\Gamma) \to \ell^2(\Sigma)$ the canonical orthogonal projection. However, here we rather consider the Toeplitz system as arising from the restriction map $\rho : \mathrm{B}(\Gamma) \to \mathrm{C}^*(\Gamma)^{(\Sigma)}$, $u \mapsto (u(st^{-1}))_{s,t\in\Sigma}$; we emphasize that the map ρ is ucp by construction.

Definition 5.3.13. A finite subset $\Sigma \subseteq \Gamma$ is said to have the *cp Σ-extension property* if for every positive semi-definite Toeplitz matrix $T = (T_{st^{-1}})_{s,t\in\Sigma} \in \mathrm{M}_n(\mathrm{C}^*(\Gamma)^{(\Sigma)})^+$ there is a positive semi-definite function $u \in \mathrm{B}(\Gamma, \mathbb{C}^n)$ such that $T_{st^{-1}} = u(st^{-1})$, for all $s, t \in \Sigma$.

Definition 5.3.14. Let $\Sigma \subseteq \Gamma$ be a finite subset. We say that Σ has the *cb Σ-extension property* if, for every Toeplitz matrix $T = (T_{st^{-1}})_{s,t\in\Sigma} \in \mathrm{M}_n(\mathrm{C}^*(\Gamma)^{(\Sigma)})^+$, there is a function $u \in \mathrm{B}(\Gamma, \mathbb{C}^n)$ such that $T_{st^{-1}} = u(st^{-1})$, for all $s, t \in \Sigma$, and $\|T\| = \|\phi_u\|_{\mathrm{cb}}$, where ϕ_u denotes the cb map $\mathrm{C}^*(\Gamma) \to \mathrm{M}_n$ induced by linearly extending u from Γ to $\mathrm{C}^*(\Gamma)$.

Remark 5.3.15. In other words, a finite subset $\Sigma \subseteq \Gamma$ has the *cp/cb Σ-extension property* if and only if the restriction map $\rho : \mathrm{B}(\Gamma) \to \mathrm{C}^*(\Gamma)^{(\Sigma)}$, $u \mapsto (u(st^{-1}))_{s,t\in\Sigma}$ is an **MVS**-/**OSp**-quotient map.

5.3.3 Duality of the sums of squares and Toeplitz system

As above, let $\Sigma \subseteq \Gamma$ be a finite subset of a discrete group Γ and let H be a separable Hilbert space. We now establish the duality of the Toeplitz and sums-of-squares systems.

Proposition 5.3.16. *Let Γ be a discrete group and $\Sigma \subseteq \Gamma$ a finite subset.*

(i) If $T = (T_{st^{-1}})_{s,t\in\Sigma} \in (\mathrm{C}^*(\Gamma)^{(\Sigma)} \otimes_{\min} \mathcal{B}(H))^+$ is a positive semi-definite $\mathcal{B}(H)$-valued Toeplitz matrix, then there exists a unique completely positive map $\phi_T : \mathrm{SOS}(\Sigma) \to \mathcal{B}(H)$ such that $\phi_T(\delta_{st^{-1}}) = T_{st^{-1}}$, for all $s, t \in \Sigma$.

(ii) If $\phi : \mathrm{SOS}(\Sigma) \to \mathcal{B}(H)$ is a completely positive map then there exists a unique Toeplitz matrix $T_\phi \in \mathrm{C}^*(\Gamma)^{(\Sigma)} \otimes_{\min} \mathcal{B}(H)$ which is positive semi-definite and such that $(T_\phi)_{st^{-1}} = \phi(\delta_{st^{-1}})$, for all $s, t \in \Sigma$.

Proof. (i) Let $T = (T_{st^{-1}})_{s,t\in\Sigma} \in (\mathrm{C}^*(\Gamma)^{(\Sigma)} \otimes_{\min} \mathcal{B}(H))^+$ and define ϕ_T as in the proposition. By the characterization of the cones $\mathrm{M}_n(\mathrm{SOS}(\Sigma))^+ = \mathcal{Q}_n(\Sigma)$ in Lemma 5.3.3 as the smallest compatible family of positive matrix cones containing the matrices $(\delta_{s_i s_j^{-1}})_{i,j=1}^n$, for $s_i \in \Sigma$, at the n-th level, it is enough to check that $\phi_T^{(n)}((\delta_{s_i s_j^{-1}})_{i,j=1}^n) \in \mathcal{B}(H)$ is positive, to obtain complete positivity of ϕ_T. This follows from positive semi-definiteness of T and the following computation:

$$\phi_T^{(n)}((\delta_{s_i s_j^{-1}})_{i,j=1}^n) = (\phi_T(\delta_{s_i s_j^{-1}}))_{i,j=1}^n = (T_{s_i s_j^{-1}})_{i,j=1}^n \tag{5.6}$$

(ii) Conversely, let $\phi \in \mathcal{CP}(\mathrm{SOS}(\Sigma), \mathcal{B}(H))$ be a completely positive map and define T_ϕ as in the proposition. Replacing ϕ_T by ϕ and T by T_ϕ in (5.6) and reading backwards, we see that all principal minors of T_ϕ are positive semi-definite, implying the claim that T_ϕ is positive semi-definite.

The uniqueness statements hold, since the assignments $T \mapsto \phi_T$ and $\phi \mapsto T_\phi$ are inverse to each other. □

Corollary 5.3.17. *The following map is a complete order isomorphism of the Toeplitz system and the dual operator system of the sums-of-squares system:*

$$\begin{aligned} \varphi : \mathrm{C}^*(\Gamma)^{(\Sigma)} &\to \mathrm{SOS}(\Sigma)^{\mathrm{d}} \\ T &\mapsto \phi_T \end{aligned}$$

Moreover, its dual map $\varphi^{\mathrm{d}} : \mathrm{SOS}(\Sigma) \to (\mathrm{C}^*(\Gamma)^{(\Sigma)})^{\mathrm{d}}$ *is a complete order isomorphism from the sums-of-squares system to the dual of the Toeplitz system, and we have*

$$\varphi(\mathbf{1}_{\mathrm{M}_\Sigma}) = \left(\delta_{st^{-1}} \mapsto \begin{cases} 1, \text{ if } s = t \\ 0, \text{ otherwise} \end{cases} \right) \text{ and } \varphi^{\mathrm{d}}(\delta_e) = (T \mapsto \frac{1}{|\Sigma|}\mathrm{Tr}(T)).$$

Proof. The fact that the maps φ and φ^{d} are complete order isomorphisms and the computation of φ evaluated at the unit matrix follow directly from Proposition 5.3.16. To see that $\varphi^{\mathrm{d}}(\delta_e)$ is the normalized trace on the Toeplitz system, note that

$$\varphi^{\mathrm{d}}(\delta_{st^{-1}})(T) = \phi_T(\delta_{st^{-1}}) = T_{st^{-1}},$$

and the claim follows. □

Theorem 5.3.18. *The following statements are equivalent:*

(1) The set Σ *has the cp* Σ*-extension property.*

(2) The set Σ *has the cb* Σ*-extension property.*

(3) The set Σ *has the complete Fejér–Riesz property.*

(4) The Toeplitz system $\mathrm{C}^*(\Gamma)^{(\Sigma)}$ *is completely order isomorphic to the dual operator system of the Fourier system* $\mathrm{C}^*(\Gamma)_{(\Sigma\Sigma^{-1})}$.

We refer to Figure 5.1 as an illustration of the operator systems and matrix-ordered vector spaces with the canonical maps which are relevant in the proof below.

Proof. Note that the dual map of the canonical inclusion map $\iota : \mathrm{SOS}(\Sigma) \to \mathrm{C}^*(\Gamma)$ is the restriction map $\rho : \mathrm{B}(\Gamma) \to \mathrm{C}^*(\Gamma)^{(\Sigma)}$, $u \mapsto (u(st^{-1}))_{s,t\in\Sigma}$ where $\mathrm{B}(\Gamma)$ and $\mathrm{C}^*(\Gamma)^{(\Sigma)}$ are at the same time equipped with the structures of the matrix-ordered vector space and operator space duals of $\mathrm{C}^*(\Gamma)$ and $\mathrm{SOS}(\Sigma)$. Recall from Remark 5.3.15 that the set Σ has the cp/cb Σ-extension property if and only if the restriction map ρ is an **MVS**-/**OSp**-quotient map. By Proposition 2.2.7, both of these statements are equivalent to the statement that the map ι is a complete order embedding, i.e. that Σ has the Fejér–Riesz property. This shows the equivalences (1)⇔(2)⇔(3).

For the implication (3)⇒(4), first note that by finite-dimensionality the dual operator system $\mathrm{C}^*(\Gamma)^{\mathrm{d}}_{(\Sigma\Sigma^{-1})}$ of the Fourier system is in fact an operator system, and by the Fejér–Riesz property we have $\mathrm{C}^*(\Gamma)_{(\Sigma\Sigma^{-1})} \cong \mathrm{SOS}(\Sigma)$, which implies the claim by Corollary 5.3.17. Conversely if the operator systems $\mathrm{C}^*(\Gamma)^{(\Sigma)}$ and $\mathrm{C}^*(\Gamma)^{\mathrm{d}}_{(\Sigma\Sigma^{-1})}$ are completely order isomorphic we obtain complete order isomorphisms

$$\mathrm{C}^*(\Gamma)_{(\Sigma\Sigma^{-1})} \cong \mathrm{C}^*(\Gamma)^{\mathrm{dd}}_{(\Sigma\Sigma^{-1})} \cong (\mathrm{C}^*(\Gamma)^{(\Sigma)})^{\mathrm{d}} \cong \mathrm{SOS}(\Sigma),$$

so the one-to-one ucp map $\iota : \mathrm{SOS}(\Sigma) \to \mathrm{C}^*(\Gamma)_{(\Sigma\Sigma^{-1})} \subseteq \mathrm{C}^*(\Gamma)$ must be a complete order embedding, showing (4)⇒(3). □

Corollary 5.3.19. *Let Γ_1 and Γ_2 be discrete groups and assume that Γ_1 or Γ_2 is amenable. Let $\Sigma_1 \subseteq \Gamma_1$ and $\Sigma_2 \subseteq \Gamma_2$ be finite subsets which each have the complete extension property. Then the following statements are equivalent:*

(1) The subset $\Sigma_1 \times \Sigma_2 \subseteq \Gamma_1 \times \Gamma_2$ has the complete $\Sigma_1 \times \Sigma_2$-extension property.

(2) The pair of Toeplitz systems $(\mathrm{C}^*(\Gamma_1)^{(\Sigma_1)}, \mathrm{C}^*(\Gamma_2)^{(\Sigma_2)})$ is $(\min, \max)$-nuclear.

Proof. Recall that by amenability, we have nuclearity of the group C*-algebra tensor product

$$\mathrm{C}^*(\Gamma_1 \times \Gamma_2) \cong \mathrm{C}^*(\Gamma_1) \otimes_{\max} \mathrm{C}^*(\Gamma_2) \cong \mathrm{C}^*(\Gamma_1) \otimes_{\min} \mathrm{C}^*(\Gamma_2). \tag{5.7}$$

Assume now (1), i.e. that $\Sigma_1 \times \Sigma_2$ has the complete $\Sigma_1 \times \Sigma_2$-extension property. Then, by Theorem 5.3.18, the canonical inclusion map $\iota : \mathrm{SOS}(\Sigma_1 \times \Sigma_2) \to \mathrm{C}^*(\Gamma_1 \times \Gamma_2)$ is a complete order embedding. By the assumption that Σ_i has the complete Σ_i-extension property it follows from Theorem 5.3.18 that Σ_i has the complete Fejér–Riesz property, i.e. the Fourier system $\mathrm{C}^*(\Gamma_i)_{(\Sigma_i\Sigma_i^{-1})}$ is completely order isomorphic to the sums-of-squares system $\mathrm{SOS}(\Sigma_i)$, for $i = 1, 2$. By Proposition 5.3.6 and (5.7), we obtain $(\min, \max)$-nuclearity of the pair of sums-of-squares systems:

$$\begin{aligned} &\mathrm{SOS}(\Sigma_1) \otimes_{\max} \mathrm{SOS}(\Sigma_2) \\ &= \mathrm{SOS}(\Sigma_1 \times \Sigma_2) \cong \mathrm{C}^*(\Gamma_1 \times \Gamma_2)_{(\Sigma_1 \times \Sigma_2)} \\ &= \mathrm{C}^*(\Gamma_1)_{(\Sigma_1)} \otimes_{\min} \mathrm{C}^*(\Gamma_2)_{\Sigma_2} = \mathrm{SOS}(\Sigma_1) \otimes_{\min} \mathrm{SOS}(\Sigma_2) \end{aligned}$$

Duality of the Toeplitz and sums of squares systems, Proposition 5.3.16, together with the complete order isomorphisms $(X \otimes_{\min} Y)^{\mathrm{d}} \cong X^{\mathrm{d}} \otimes_{\max} Y^{\mathrm{d}}$ and $(X \otimes_{\max} Y)^{\mathrm{d}} \cong X^{\mathrm{d}} \otimes_{\min} Y^{\mathrm{d}}$, for finite-dimensional operator systems X and Y, it follows that the pair of Toeplitz systems $(\mathrm{C}^*(\Gamma_1)^{(\Sigma_1)}, \mathrm{C}^*(\Gamma_2)^{(\Sigma_2)})$ is $(\min, \max)$-nuclear.

Conversely assume (2), i.e. the operator systems $\mathrm{C}^*(\Gamma_1)^{(\Sigma_1)} \otimes_{\max} \mathrm{C}^*(\Gamma_2)^{(\Sigma_2)}$ and $\mathrm{C}^*(\Gamma_1)^{(\Sigma_1)} \otimes_{\min} \mathrm{C}^*(\Gamma_2)^{(\Sigma_2)}$ are completely order isomorphic. Since Σ_1 and Σ_2 respectively have the complete Fejér–Riesz property and by injectivity of the minimal operator system tensor product, we have that the tensor product $\iota_1 \otimes \iota_2 : \mathrm{SOS}(\Sigma_1) \otimes_{\min} \mathrm{SOS}(\Sigma_2) \to \mathrm{C}^*(\Gamma_1) \otimes_{\min} \mathrm{C}^*(\Gamma_2)$ of the canonical inclusion maps is a complete order embedding. By the assumption that the pair of operator systems $(\mathrm{C}^*(\Gamma_1)^{(\Sigma_1)}, \mathrm{C}^*(\Gamma_2)^{(\Sigma_2)})$ is $(\min, \max)$-nuclear and by the duality $\mathrm{C}^*(\Gamma)^{(\Sigma)} \cong \mathrm{SOS}(\Sigma)^{\mathrm{d}}$ together with the duality of the minimal and maximal operator system tensor products for finite-dimensional operator systems, we obtain

$$\begin{aligned}
\mathrm{SOS}(\Sigma_1 \times \Sigma_2) &= \mathrm{SOS}(\Sigma_1) \otimes_{\max} \mathrm{SOS}(\Sigma_2) \\
&= \mathrm{SOS}(\Sigma_1) \otimes_{\min} \mathrm{SOS}(\Sigma_2) \\
&\cong \mathrm{C}^*(\Gamma_1)_{(\Sigma_1)} \otimes_{\min} \mathrm{C}^*(\Gamma_2)_{(\Sigma_2)} \\
&= \mathrm{C}^*(\Gamma_1 \times \Gamma_2)_{(\Sigma_1 \times \Sigma_2)}.
\end{aligned}$$

This shows that the set $\Sigma_1 \times \Sigma_2$ has the complete Fejér–Riesz property, whence the complete $\Sigma_1 \times \Sigma_2$-extension property, by Theorem 5.3.18. □

A scalar-valued version of Theorem 5.3.18 was shown by Rudin [127] and (according to [129]) independently by Calderón–Pepinsky [25]. From a result of Hilbert [69], they inferred that there are positive semi-definite trigonometric polynomials in two variables of respective degree 3 which cannot be expressed as a sum of squares. Equivalently, the subset $\Sigma := \{0, 1, 2, 3\}^2 \subseteq \mathbb{Z}^2$ does not have the (scalar-valued) Σ-extension property. This result was strengthened in [129] to the subset $\{0, 1, 2\}^2 \subseteq \mathbb{Z}^2$, which yields the following special case of [50, Theorem 6.5].

Corollary 5.3.20. *Let $n \geqslant 3$ and, denote by $\mathrm{T}_n \subseteq \mathrm{M}_n$ the operator system of $n \times n$ Toeplitz matrices. Then*

$$\mathrm{T}_n \otimes_{\min} \mathrm{T}_n \not\cong \mathrm{T}_n \otimes_{\max} \mathrm{T}_n.$$

Proof. By [127, 129], for $\Sigma_n = \{0, \ldots, n-1\}^2 \subseteq \mathbb{Z}^2$, the cone $\mathcal{Q}_1(\Sigma_n)$ is *strictly* contained in the cone $\mathrm{C}^*(\mathbb{Z}^2)^+ \cap \mathrm{S}[\Sigma_n \Sigma_n^{-1}]$, so the set Σ_n does not have the Fejér–Riesz property. The claim now follows from Theorem 5.3.18 and Corollary 5.3.19. □

5.4 Partially defined positive semi-definite functions

In this section we discuss partially defined positive semi-definite operator-valued functions on discrete groups and whether they admit positive semi-definite extensions to the whole group. Such functions are generally defined on a positivity domain $\Delta \subseteq \Gamma$, rather than on a difference $\Sigma\Sigma^{-1} \subseteq \Gamma$ as in the previous section. We also drop the finiteness assumption on the positivity domain. Our analysis follows a similar path as in the previous section by first identifying the operator system predual of the matrix-ordered vector space of positive semi-definite functions; this operator system is constructed using (the matrix-cones of) sums-of-squares systems as building blocks. In the case of the positivity domain consisting of five points in $\mathbb{Z}^2$ as in Figure 5.3 this allows us to realize the associated operator system as an amalgamated direct sum, giving us access to its maximal and minimal C*-cover and allowing us to infer that there must be positive semi-definite operator-valued functions on this positivity domain which do not admit positive semi-definite extensions to the whole group.

5.4.1 The operator system $\mathrm{P}(\Delta)$ and the complete factorization property

Let Γ be a discrete group and $\Delta \subseteq \Gamma$ a positivity domain, i.e. $\Delta^{-1} = \Delta \ni e$. We begin by defining the positive matrix-cones which we will use to define an operator system structure on the *-vector space $\mathrm{S}[\Delta] := \operatorname{span}\{\delta_s \mid s \in \Delta\}$. For a *-vector space V and a subset $\mathcal{P} \subseteq V_{\mathrm{h}}$, we denote by $\operatorname{cone}(\mathcal{P})$ the *cone generated* by $\mathcal{P}$, i.e. $\operatorname{cone}(\mathcal{P}) := [0, \infty) \cdot \mathcal{P} + \mathcal{P}$.

Definition 5.4.1. For all $n \in \mathbb{N}$, set

$$\mathcal{E}_n(\Delta) := \operatorname{cone}\left(\bigcup_{\substack{\Sigma \subseteq \Gamma \\ \Sigma\Sigma^{-1} \subseteq \Delta \\ |\Sigma| < \infty}} \mathcal{Q}_n(\Sigma) \right) \subseteq \mathrm{M}_n(\mathrm{S}[\Delta])_{\mathrm{h}}.$$

Note that, since $\mathcal{Q}_n(\Sigma)$ is a cone, for any finite subset $\Sigma \subseteq \Gamma$, it is equivalent to take the convex hull of the union of the cones $\mathcal{Q}_n(\Sigma)$ in the above definition. Since the cone generated by a union of cones is their sum, we obtain the following lemma:

Lemma 5.4.2. *For all $n \in \mathbb{N}$, the following holds:*

$$\mathcal{E}_n(\Delta) = \sum_{\substack{\Sigma \subseteq \Gamma \\ \Sigma\Sigma^{-1} \subseteq \Delta \\ |\Sigma| < \infty}} \mathcal{Q}_n(\Sigma)$$

The next proposition follows from Definition 5.4.1 and Lemma 5.3.3:

Proposition 5.4.3. *The family $(\mathcal{E}_n(\Delta))_{n \in \mathbb{N}}$ is a compatible family of convex proper cones respectively contained in $\mathrm{M}_n(\mathrm{C}^*(\Gamma))^+$. It is the smallest such family whose n-th level contains the set $\{(\delta_{st^{-1}})_{s,t \in \Sigma} \mid \Sigma \subseteq \Gamma, \Sigma\Sigma^{-1} \subseteq \Delta, |\Sigma| < \infty\}$. The element δ_e is a matrix order-unit for the family $(\mathcal{E}_n(\Delta))_{n \in \mathbb{N}}$. In particular, the triple $(\mathrm{S}[\Delta], (\mathcal{E}_n(\Delta))_{n \in \mathbb{N}}, \delta_e)$ is a matrix order-unit space and its canonical inclusion into $\mathrm{C}^*(\Gamma)$ is ucp.*

Definition 5.4.4. We denote the matrix order-unit space $(\mathrm{S}[\Delta], (\mathcal{E}_n(\Delta))_{n \in \mathbb{N}}, \delta_e)$ by $\mathrm{P}_0(\Delta)$ and its archimedeanization by $\mathrm{P}(\Delta) := \mathrm{Arch}(\mathrm{P}_0(\Delta))\ \mathrm{P}_0(\Delta)$.

Example 5.4.5. Consider the positivity domain

$$\Delta := \{(-1,0),(0,-1),(0,0),(0,1),(1,0)\} \subseteq \mathbb{Z}^2,$$

as in Figure 5.3.

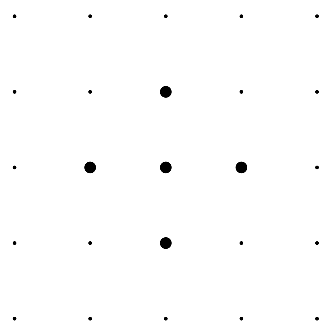

Figure 5.3: The positivity domain $\Delta := \{(-1,0),(0,-1),(0,0),(0,1),(1,0)\} \subseteq \mathbb{Z}^2$.

Set $\Sigma_{\mathrm{h}} := \{(0,0),(1,0)\}$, $\Sigma_{\mathrm{v}} := \{(0,0),(0,1)\}$ and $\mathrm{SOS}_{\mathrm{h}} := \mathrm{SOS}(\Sigma_{\mathrm{h}})$, $\mathrm{SOS}_{\mathrm{v}} := \mathrm{SOS}(\Sigma_{\mathrm{v}})$. We have $\mathrm{P}_0(\Delta) = \mathrm{Arch}(\mathrm{P}_0(\Delta)) = \mathrm{P}(\Delta)$ and the canonical ucp maps $\phi_{\mathrm{h}}, \phi_{\mathrm{v}} : \mathrm{SOS}_{\mathrm{h}}, \mathrm{SOS}_{\mathrm{v}} \to \mathrm{P}(\Delta)$ are complete order embeddings; indeed, we have $\mathcal{E}_n(\Delta) = \mathcal{Q}_n(\Sigma_{\mathrm{h}}) + \mathcal{Q}_n(\Sigma_{\mathrm{v}})$, which is a closed cone and hence δ_e is an archimedean matrix order-unit for $\mathrm{P}_0(\Delta)$. Moreover, $\mathrm{M}_n(\mathrm{S}[\{(-1,0),(0,0),(1,0)\}]) \cap \mathcal{E}_n(\Delta) = \mathcal{Q}_n(\Sigma_{\mathrm{h}})$, so ϕ_{h} is a complete order embedding, and analogously for ϕ_{v}.

We now show that the operator system $\mathrm{P}(\Delta)$ is the direct sum of two copies of the Fourier system $\mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})} \subseteq \mathrm{C}^*(\mathbb{Z})$ amalgamated over the unit, i.e.

$$\mathrm{P}(\Delta) = \mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})} \oplus_{\mathbf{1}} \mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})}; \tag{5.8}$$

in fact, we have $\mathrm{P}(\Delta) = \mathrm{SOS}_{\mathrm{h}} \oplus_{\mathbf{1}} \mathrm{SOS}_{\mathrm{v}}$, which we show by checking that $\mathrm{P}(\Delta)$ satisfies the universal property of the amalgamated direct sum. To this end, assume that X is an operator system with ucp maps $\psi_{\mathrm{h}} : \mathrm{SOS}_{\mathrm{h}} \to X$ and $\psi_{\mathrm{v}} : \mathrm{SOS}_{\mathrm{v}} \to X$. Define a map $\psi : \mathrm{P}(\Delta) \to X$ by $\psi(\delta_{(0,0)}) := e_X$, $\psi(\delta_{(\pm1,0)}) := \psi_{\mathrm{h}}(\delta_{(\pm1,0)})$ and $\psi(\delta_{(0,\pm1)}) := \psi_{\mathrm{v}}(\delta_{(0,\pm1)})$. Clearly, ψ is the unique map such that $\psi \circ \phi_{\mathrm{v}} = \psi_{\mathrm{v}}$ and $\psi \circ \phi_{\mathrm{h}} = \psi_{\mathrm{h}}$, where ϕ_{h} and ϕ_{v} are the canonical maps $\mathrm{SOS}_{\mathrm{h}}, \mathrm{SOS}_{\mathrm{v}} \to \mathrm{P}(\Delta)$. If $x \in \mathcal{E}_n(\Delta)$, there are positive elements $x_{\mathrm{h}} \in \mathcal{Q}_n(\Sigma_{\mathrm{h}})$ and $x_{\mathrm{v}} \in \mathcal{Q}_n(\Sigma_{\mathrm{v}})$ such that $x = x_{\mathrm{h}} + x_{\mathrm{v}}$. Then $\psi^{(n)}(x) = \psi_{\mathrm{h}}^{(n)}(x_{\mathrm{h}}) + \psi_{\mathrm{v}}^{(n)}(x_{\mathrm{v}}) \in \mathrm{M}_n(X)^+$, so ψ is ucp, which completes the proof that $\mathrm{P}(\Delta)$ has the universal property of the amalgamated direct sum. By the operator-valued Fejér–Riesz lemma, we have $\mathrm{SOS}_{\mathrm{h}} \cong \mathrm{SOS}_{\mathrm{v}} \cong \mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})}$, which shows (5.8).

Realizing the operator system $\mathrm{P}(\Delta)$ as an amalgamated direct sum as above, allows us to compute its maximal and minimal C*-covers. We prepare for the former with the following lemma which is probably well-known. Recall that the universal unital C*-algebra $\mathrm{C}^*_{\mathrm{u}}(\mathrm{con})$ of a contraction is defined as the universal unital C*-algebra generated by a unit $\mathbf{1}$ and a contraction $x \neq \mathbf{1}$ such that for any contraction

$y \in \mathcal{B}(H)$ there is a canonical unital $*$-homomorphism

$$\begin{aligned} \Phi\colon \mathrm{C}^*(\mathrm{con}) &\to \mathrm{C}^*(\mathbf{I}^H, y) \\ x &\mapsto y. \end{aligned}$$

Lemma 5.4.6. *The maximal* C^**-cover of the Fourier system* $\mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})}$ *is canonically isomorphic to* $\mathrm{C}^*_{\mathrm{u}}(\mathrm{con})$.

Proof. For notational convenience let us denote by $\mathrm{C}^*(\mathbf{1}, x) := \mathrm{C}^*(\mathrm{con})$ the universal C*-algebra of a contraction, and by x its generator. We will also assume that $\mathrm{C}^*(\mathbf{1}, x)$ is concretely represented in some $\mathcal{B}(H)$ with $x \neq \mathbf{1} = \mathbf{I}^H$. By the universal property of $\mathrm{C}^*(1, x)$, there is a unital $*$-homomorphism

$$\begin{aligned} \Phi\colon \mathrm{C}^*(1, x) &\to \mathrm{C}^*(\mathbb{Z}) \\ x &\mapsto \delta_1. \end{aligned}$$

Its restriction $\Phi|_{\mathrm{span}\{x^*, \mathbf{1}, x\}} \subseteq \mathrm{C}^*(\mathbf{1}, x)$ gives a ucp map from $\mathrm{span}\{x^*, 1, x\}$ to the Fourier system $\mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})} := \mathrm{span}(\{\delta_1^*, \delta_0, \delta_1\}) \subseteq \mathrm{C}^*(\mathbb{Z})$.

Conversely, by the Sz.-Nagy dilation theorem the contraction x dilates to a unitary $u \in \mathcal{B}(K)$ for some Hilbert space K. Recall that $\mathrm{C}^*(\mathbb{Z})$ is universal with respect to unitaries, providing a unital $*$-homomorphism

$$\begin{aligned} \Psi\colon \mathrm{C}^*(\mathbb{Z}) &\to \mathcal{B}(K) \\ \delta_1 &\mapsto u. \end{aligned}$$

By further compressing to H we obtain a ucp map

$$\begin{aligned} \psi\colon \mathrm{C}^*(\mathbb{Z}) &\to \mathcal{B}(H) \\ \psi(a) &:= P_H \Psi(a)|_H, \end{aligned}$$

which satisfies

$$\psi(\delta_1) = P_H \Psi(\delta_1)|_H = P_H u|_H = x.$$

Thus the restriction of ψ to $\mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})}$ has $\Phi|_{\mathrm{span}\{x^*,1,x\}}$ as a unital completely contractive inverse. Therefore ψ is a unital complete order embedding of the operator system $\mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})}$ into $\mathrm{C}^*(\mathbf{1}, x)$, making the latter a C*-cover.

It remains to show that $\mathrm{C}^*(\mathbf{1}, x)$ has the universal property of the universal C*-cover $\mathrm{C}^*_{\mathrm{u}}(\mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})})$. Let

$$\phi\colon \mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})} \to \mathcal{B}(K)$$

be a ucp map. Then $\phi(\delta_1)$ is a contraction and by the universal property of $\mathrm{C}^*(1, x)$ there is a canonical $*$-homomorphism

$$\begin{aligned} \tilde{\phi}\colon \mathrm{C}^*(\mathbf{1}, x) &\to \mathcal{B}(K) \\ x &\mapsto \phi(\delta_1). \end{aligned}$$

By definition $\tilde{\phi}$ is an extension of the ucp map ϕ, as required. ☐

We are now in position to compute the maximal and minimal C^*-cover of the operator system $\mathrm{P}(\Delta)$ for the positivity domain from Example 5.4.5.

Proposition 5.4.7. *Consider the positivity domain*

$$\Delta = \{(-1,0),(0,-1),(0,0),(0,1),(1,0)\} \subseteq \mathbb{Z}^2,$$

as in Figure 5.3. The following identities hold:

$$\mathrm{C}^*_{\max}(\mathrm{P}(\Delta)) = \mathrm{C}^*_{\mathrm{u}}(\mathrm{con}_1, \mathrm{con}_2), \tag{5.9}$$

and

$$\mathrm{C}^*_{\min}(\mathrm{P}(\Delta)) = \mathrm{C}^*(\mathbb{F}_2). \tag{5.10}$$

Here $\mathrm{C}^*_{\mathrm{u}}(\mathrm{con}_1, \mathrm{con}_2)$ *is the universal unital* C^**-algebra of two contractions.*

Proof. Recall that by (5.8) we have

$$\mathrm{P}(\Delta) = \mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})} \oplus_{\mathbf{1}} \mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})}.$$

The identity for the maximal C^*-cover (5.9) follows by the compatibility of the coproduct with the free product of the universal C^*-algebra of the Fourier system $\mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})}$. That is, we have

$$\begin{aligned}\mathrm{C}^*_{\max}(\mathrm{P}(\Delta)) &= \mathrm{C}^*_{\max}\left(\mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})} \oplus_{\mathbf{1}} \mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})}\right)\\ &= \mathrm{C}^*_{\max}\left(\mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})}\right) *_{\mathbf{1}} \mathrm{C}^*_{\max}\left(\mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})}\right)\\ &= \mathrm{C}^*_{\mathrm{u}}(\mathrm{con}) *_{\mathbf{1}} \mathrm{C}^*_{\mathrm{u}}(\mathrm{con}),\end{aligned}$$

where we applied Lemma 5.4.6 in the last *-isomorphism. Now, since the universal unital C^*-algebra of two contractions equals $\mathrm{C}^*_{\mathrm{u}}(\mathrm{con}) *_{\mathbf{1}} \mathrm{C}^*_{\mathrm{u}}(\mathrm{con})$, we obtain (5.9).

For the identity for the minimal C^*-cover (5.10), note that, since the Fourier system

$$\mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})} = \mathrm{span}\{\delta_1^*, \delta_0, \delta_1\} \subseteq \mathrm{C}^*(\mathbb{Z})$$

contains the unitary generator of $\mathrm{C}^*(\mathbb{Z})$, we have $\mathrm{C}^*_{\min}(\mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})}) = \mathrm{C}^*(\mathbb{Z})$. As noticed in [51, Theorem 5.2], if $X \subseteq A$ and $Y \subseteq B$ are operator systems which respectively contain the unitary generators of the C^*-algebras A, B, then the amalgamated free product (over the unit) $X \oplus_{\mathbf{1}} Y$ contains the unitary generators of $A *_{\mathbf{1}} B$, and hence, by [81, Proposition 5.6], we have $\mathrm{C}^*_{\min}(X) = A$, $\mathrm{C}^*_{\min}(Y) = B$ and $\mathrm{C}^*_{\min}(X \oplus_{\mathbf{1}} Y) = A *_{\mathbf{1}} B$. In our situation, this gives

$$\begin{aligned}\mathrm{C}^*_{\min}(\mathrm{P}(\Delta)) &= \mathrm{C}^*_{\min}\left(\mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})} \oplus_{\mathbf{1}} \mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})}\right)\\ &= \mathrm{C}^*_{\min}\left(\mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})}\right) *_{\mathbf{1}} \mathrm{C}^*_{\min}\left(\mathrm{C}^*(\mathbb{Z})_{(\{-1,0,1\})}\right)\\ &= \mathrm{C}^*(\mathbb{Z}) *_{\mathbf{1}} \mathrm{C}^*(\mathbb{Z})\\ &= \mathrm{C}^*(\mathbb{F}_2),\end{aligned}$$

and the proof is complete. □

We point out that [51, Theorem 5.2] can be strengthened to operator systems which are hyperrigid in their minimal C^*-cover, see [47, Theorem 1], [38, Theorem 5.3.21], [26, Theorem 4.11]. Note that an operator system which contains the unitary generators of its ambient C^*-algebra is hyperrigid in it [26, Proposition 4.8]. In particular, the Fourier systems $\mathrm{C}^*(\mathbb{Z})_{(-1,0,1)}$, $\mathrm{C}^*(\mathbb{Z}^2)_{(\Delta)}$, for Δ as in Figure 5.3, are hyperrigid in $\mathrm{C}^*(\mathbb{Z})$, $\mathrm{C}^*(\mathbb{Z}^2)$ respectively; as a consequence of the above proposition, also the operator system $\mathrm{P}(\Delta)$ is hyperrigid in $\mathrm{C}^*(\mathbb{F}_2)$.

For the next definition recall the canonical map $\mathrm{P}_0(\Delta) \to \mathrm{C}^*(\Gamma)_{(\Delta)}$, $\delta_s \mapsto \delta_s$ which we denote by ι_0. It follows from the definition of the positive cones $\mathcal{E}_n(\Delta)$ of the matrix order-unit space $\mathrm{P}_0(\Delta)$ that ι_0 is ucp. In the diagram Figure 5.4 we furthermore include the quotient map $q : \mathrm{P}_0(\Delta) \to \mathrm{P}(\Delta)$ from the archimedeanization as well as the map $\iota : \mathrm{P}(\Delta)$ which is canonically induced by the universal property of the archimedeanization.

$$\begin{array}{ccc} \mathrm{C}^*(\Gamma) \supseteq \mathrm{C}^*(\Gamma)_{(\Delta)} & \xleftarrow{\ \iota_0\ } & \mathrm{P}_0(\Delta) \\ & \nwarrow^{\iota} & \big\downarrow q \\ & & \mathrm{P}(\Delta) \end{array}$$

Figure 5.4: The canonical maps ι_0 and ι.

Definition 5.4.8. We say that a positivity domain $\Delta \subseteq \Gamma$ has the *complete factorization property* if the canonical map $\iota : \mathrm{P}(\Delta) \to \mathrm{C}^*(\Gamma)$ is a complete order embedding.

Using the minimal C^*-cover of the operator system $\mathrm{P}(\Delta)$, for Δ as in Figure 5.3, we now show that Δ does not have the complete factorization property.

Corollary 5.4.9. *The positivity domain* $\Delta = \{(-1,0),(0,-1),(0,0),(0,1),(1,0)\} \subseteq \mathbb{Z}^2$*, as in Figure 5.3, does not have the complete factorization property.*

Proof. Observe that the minimal C^*-cover of the Fourier system $\mathrm{C}^*(\mathbb{Z}^2)_{(\Delta)}$ is $\mathrm{C}^*(\mathbb{Z}^2)$. Indeed, since $\mathrm{C}^*(\mathbb{Z}^2)$ is the C^*-algebra generated by the unitary generators of the Fourier system $\mathrm{C}^*(\mathbb{Z}^2)_{(\Delta)}$, it follows that $\mathrm{C}^*_{\mathrm{min}}(\mathrm{C}^*(\mathbb{Z}^2)_{(\Delta)}) = \mathrm{C}^*(\mathbb{Z}^2)$ by [81, Proposition 5.6].

Now, if the canonical isomorphism $\iota : \mathrm{P}(\Delta) \to \mathrm{C}^*(\mathbb{Z}^2)_{(\Delta)}$ was a complete order isomorphism, we would have

$$\mathrm{C}^*(\mathbb{F}_2) = \mathrm{C}^*_{\mathrm{min}}(\mathrm{P}(\Delta)) \cong \mathrm{C}^*_{\mathrm{min}}(\mathrm{C}^*(\mathbb{Z}^2)_{(\Delta)}) = \mathrm{C}^*(\mathbb{Z}^2),$$

which is absurd. The claim follows. □

5.4.2 The Fourier–Stieltjes matrix-ordered vector space and the complete extension property

Let Γ be a discrete group, $\Delta \subseteq \Gamma$ a positivity domain and H a Hilbert space.

Definition 5.4.10. A function $u : \Delta \to \mathcal{B}(H)$ is called *positive semi-definite* if the Toeplitz matrix $(u(st^{-1}))_{s,t\in\Sigma} \in \mathrm{M}_\Sigma(\mathcal{B}(H))$ is positive semi-definite, whenever $\Sigma \subseteq \Gamma$ is a finite subset with $\Sigma\Sigma^{-1} \subseteq \Delta$. We denote the cone of all positive semi-definite $\mathcal{B}(H)$-valued functions on Δ by $\mathrm{B}(\Delta, H)^+$ and its linear span by $\mathrm{B}(\Delta, H)$.

Proposition 5.4.11. *Let Γ be a discrete group and $\Delta \subseteq \Gamma$ a positivity domain.*

(i) If $u \in \mathrm{B}(\Delta, H)^+$ is a positive semi-definite function on Δ then there exists a unique completely positive map $\phi_u : \mathrm{P}_0(\Delta) \to \mathcal{B}(H)$ such that $\phi_u(\delta_s) = u(s)$, for all $s \in \Delta$.

(ii) If $\phi : \mathrm{P}_0(\Delta) \to \mathcal{B}(H)$ is a completely positive map then there exists a unique positive semi-definite function $u_\phi : \Delta \to \mathcal{B}(H)$ such that $u_\phi(s) = \phi(\delta_s)$, for all $s \in \Delta$.

Proof. (i) Let $u \in \mathrm{B}(\Delta, H)^+$. By the characterization of $\mathrm{M}_n(\mathrm{P}_0(\Delta))^+ = \mathcal{E}_n(\Delta)$ as the smallest compatible family of positive matrix-cones containing the set of matrices $\{(\delta_{st^{-1}})_{s,t\in\Sigma} \mid \Sigma \subseteq \Gamma, \Sigma\Sigma^{-1} \subseteq \Delta, |\Sigma| < \infty\}$ from Proposition 5.4.3 and by the same computation as in (5.6), the induced map $\phi_u : \mathrm{P}_0(\Delta) \to \mathcal{B}(H)$ is completely positive.

(ii) Conversely if $\phi : \mathrm{P}_0(\Delta) \to \mathcal{B}(H)$ is a completely positive map, let $\Sigma \subseteq \Gamma$ be a finite subset with $\Sigma\Sigma^{-1} \subseteq \Delta$. Then we have

$$(u_\phi(st^{-1}))_{s,t\in\Sigma} = (\phi(\delta_{st^{-1}}))_{s,t\in\Sigma} = \phi^{(|\Sigma|)}((\delta_{st^{-1}})_{s,t\in\Sigma}) \in \mathcal{B}(H)^+,$$

so $u_\phi \in \mathrm{B}(\Delta, H)^+$.

The uniqueness statements follow since the assignments $u \mapsto \phi_u$ and $\phi \mapsto u_\phi$ are inverse to each other. □

Corollary 5.4.12. *The pair $(\mathrm{B}(\Delta), (\mathrm{B}(\Delta, \mathbb{C}^n)^+)_n)$ is the matrix-ordered vector space dual of the operator system $\mathrm{P}(\Delta)$.*

Proof. By Proposition 5.4.11 and identifying $\mathcal{CP}(\mathrm{P}(\Delta), \mathrm{M}_n)$ with $(\mathrm{M}_n(\mathrm{P}(\Delta))^*)^+$ as in [106, Chapter 6], the pair $(\mathrm{B}(\Delta), (\mathrm{B}(\Delta, \mathbb{C}^n)^+)_n)$ is the matrix order-unit space dual of the matrix-ordered vector space $\mathrm{P}_0(\Delta)$. Since the matrix-ordered vector space dual is preserved by archimedeanization, the claim follows. □

We refer to the matrix-ordered vector space $\mathrm{B}(\Delta)$ as the *Fourier–Stieltjes (matrix-ordered vector) space.* Moreover, the identification

$$\begin{aligned} \mathrm{B}(\Delta, H) &= \mathrm{span}(\mathrm{B}(\Delta, H)^+) \\ &\cong \mathrm{span}(\mathcal{CP}(\mathrm{P}(\Delta), \mathcal{B}(H))) \\ &= \mathcal{CB}(\mathrm{P}(\Delta), \mathcal{B}(H)) \end{aligned}$$

turns $\mathrm{B}(\Delta)$ into a (matrix-ordered) operator space.

5.4.3 The complete extension and the complete factorization property

Definition 5.4.13. We say that the positivity domain $\Delta \subseteq \Gamma$ has the *cp extension property* if, for all $n \in \mathbb{N}$, every positive semi-definite function $u : \Delta \to \mathrm{M}_n$ admits an extension to a function on Γ which is positive semi-definite.

Definition 5.4.14. We say that the positivity domain $\Delta \subseteq \Gamma$ has the *cb extension property* if, for all $n \in \mathbb{N}$, and every function $u \in \mathrm{B}(\Delta, \mathbb{C}^n)$ there is a function $\tilde{u} \in \mathrm{B}(\Gamma, \mathbb{C}^n)$ with $\tilde{u}(s) = u(s)$, for all $s \in \Delta$, and $\|\tilde{u}\|_{\mathrm{B}(\Gamma,\mathbb{C}^n)} = \|u\|_{\mathrm{B}(\Delta,\mathbb{C}^n)}$.

Remark 5.4.15. Note that the positivity domain Δ has the cp/cb extension property if and only if the restriction map $\rho : \mathrm{B}(\Gamma) \to \mathrm{B}(\Delta)$ is an **MVS**-/**OSp**-quotient map.

Theorem 5.4.16. *Let Γ be a discrete group and $\Delta \subseteq \Gamma$ a positivity domain. The following are equivalent:*

(1) The positivity domain Δ has the cp extension property.

(2) The positivity domain Δ has the cb extension property.

(3) The positivity domain Δ has the complete factorization property.

Proof. Similarly as in the proof of Theorem 5.3.18, note that the dual map of the canonical inclusion map $\iota : \mathrm{P}(\Delta) \to \mathrm{C}^*(\Gamma)$ is the restriction map $\rho : \mathrm{B}(\Gamma) \to \mathrm{B}(\Delta)$, where $\mathrm{B}(\Gamma)$ and $\mathrm{B}(\Delta)$ are, at the same time, the matrix-ordered vector space and operator space duals of $\mathrm{C}^*(\Gamma)$ and $\mathrm{P}(\Delta)$ respectively. Recall from Remark 5.4.15 that the positivity domain Δ has the cp/cb extension property if and only if the restriction map ρ is an **MVS**-/**OSp**-quotient map. By Proposition 2.2.7, both of these statements are equivalent to the statement that the inclusion map ι is a complete order embedding, i.e. that Δ has the complete factorization property. □

Corollary 5.4.17. *The positivity domain $\Delta \subseteq \mathbb{Z}^2$, as in Figure 5.3, given by $\Delta = \{(-1,0),(0,-1),(0,0),(0,1),(1,0)\}$ does not have the cp extension property; i.e. there exists a positive integer $n \in \mathbb{N}$ and a positive semi-definite function $u : \Delta \to \mathrm{M}_n$ which does not admit any extension to a positive semi-definite function on $\mathbb{Z}^2$.*

Proof. By Corollary 5.4.9, the positivity domain Δ does not have the complete factorization property which is equivalent to the claim by Theorem 5.3.18. □

Appendix

Research data management

This research was conducted according to the institute research data management policy of IMAPP, Radboud University Nijmegen. No data was produced or analyzed.

Summary

Geometry can be studied in many different ways. Mathematicians study, for example, symmetries, the curvature of surfaces, or the numbers of holes. Another fundamental geometric concept which plays a major role in this thesis is *distance*: the distance between two points on a curved surface is defined as the length of the shortest path connecting them (Figure A.1). The mathematical field of *noncommutative geometry* is concerned with developing geometric foundations of spaces which may be hard to visualize but which naturally arise e.g. from dynamical systems, as singular spaces, or from quantum physics. One can also make sense of the concept of distance in such *noncommutative spaces*, borrowing ideas from the field of *optimal transport* and using *spectra* to compute it. I will now explain the concepts of *states*, *distance in optimal transport* and *spectra*, before discussing one the main themes of this thesis which is trying to understand distance in noncommutative geometry in a certain approximate way.

Points such as the position of a ship at sea are described by coordinates. The speed of the ship and the direction in which it is traveling can also be considered coordinates, and together we can say that the *state* of the ship at a fixed point in time is described by these coordinates – position, speed, and direction. All possible states of the ship then form the *state space* which can be considered as a geometric object: in this state space, a path between two states is described by how the individual coordinates – position, speed, and direction – change from one state to another. I.e. a path in state space consists of a path that the ship travels together with specifications as to how much the ship accelerates or decelerates along this path and in which direction it steers. In general, states of a classical physical system can be understood as points described by coordinates in the state space belonging to this system.

In quantum physics, however, things are more complicated. The state of a quantum system (here referred to as *quantum state*) cannot be specified by exact coordinates. It is rather like a distribution of possibilities that indicates, for instance, how likely it is that certain positions, velocities, and directions (of a quantum particle, for instance) will be measured. In other words, a quantum state is like a weighting of possible measurement results. A suitable image to illustrate a quantum state is a distribution of sand (say of total mass one ton) over a region: each point in this region represents a possible measurement result, and the mass of sand above a point represents the probability that this result will actually be measured.

In the field of *optimal transport*, mathematicians ask how one sand distribution can be transformed into another while minimizing the cost for performing this trans-

port. The lowest possible transport cost from one sand distribution to another is then a measure of the distance between them. Similarly, the distance between two quantum states in the state space of a quantum system can be thought of as a kind of minimum cost (such as energy) required to transform the system from one state to another. This concept of distance differs from the classical concept of distance between points. Nevertheless, it allows to consider quantum state space as a geometric object.

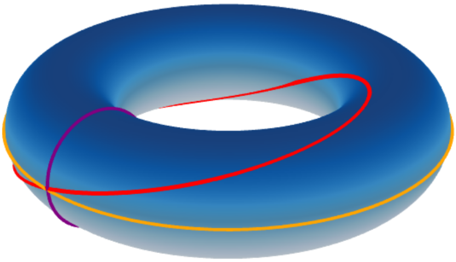

Figure A.1: The torus with some shortest paths.

A fundamental difficulty in quantum physics is that states and thus their distance cannot be measured directly. In noncommutative geometry this issue is addressed by using *spectra* to measure distances: think of a tone produced by a musical instrument which can be decomposed into fundamental frequencies, i.e. a tone corresponds to a sound spectrum. Indeed, such spectra are closely linked to geometry: larger musical instruments produce lower tones, for instance. Extending this, in the mathematical field of *spectral geometry* one tries to infer as much information as possible about the underlying geometry of a shape from its fundamental modes of vibration. Mark Kac put this theme in a nutshell by asking:

"Can one hear the shape of a drum?"

In fact, in noncommutative geometry there is a formula which allows to compute distance from spectra and this formula generalizes to a formula for the distance between quantum states.

Approximating the geometry of flat tori and quantum groups using partial spectral data

In this thesis, I address the issue that in practice, often only part of the spectral data is available: for example, the human ear can only perceive sounds up to a certain maximum frequency, and similar limitations apply to every physical measuring device. Geometrically, such partial spectral data correspond to deformations of the underlying body, which can be recognized, for example, by changes in the above-mentioned distance function (Figure A.2). The question arises as to how similar these deformed bodies still are to the original ones, and in particular whether it is possible to approximate the distances between quantum states using partial spectral data.

Chapter 3 and Chapter 4 are case studies for this question. There I examine several examples of geometric objects and quantum systems (with certain useful properties) for which both, the complete associated spectrum and the actual distance function are well-known. I first consider the example of a *flat torus*. A *torus* is the surface of a doughnut (Figure A.1), but you can best imagine a *flat* torus as being obtained from a sheet of paper as follows: glue two opposite sides together to obtain a cylinder. If you now glue the two opposite circular ends of the cylinder together, you get a torus. (In 3d this can only be done by wrinkling the paper, but a flat torus can be embedded in 4d without distortion and is then – just like the sheet of paper – indeed flat.) For instance, the intersection point of the purple, red, and orange lines in Figure A.1 then corresponds to a corner of the horizontal square in Figure A.2 (actually all four corners, since they are glued together). The function which specifies the distance from this point to every other point on the flat torus surface is plotted on the right-hand side of Figure A.2. On the left-hand side and in the middle of Figure A.2, distance functions are shown that were computed from a portion of the spectral data. In Chapter 3, I prove that these distance functions become increas-

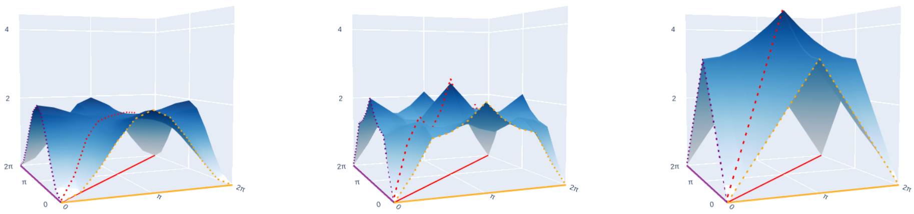

Figure A.2: The distance function on the flat torus (right) and two of its deformed versions (left and center) which were computed from part of the associated spectral data. The horizontal plane corresponds to the surface of the flat torus and the purple, red and orange paths to the ones highlighted in Figure A.1.

ingly similar to the original function, meaning that the deformed versions of the flat torus become increasingly similar to the actual flat torus, the more spectral data is taken into account. Furthermore, in Chapter 4, I show that a similar result holds for a certain class of quantum systems, so-called *compact quantum groups*.

Relationship between two types of compressions of quantum systems

Only having part of the spectral data available is like a compression of a quantum system. In Chapter 5, I deal with two types of such compressions. One is like a blurred version of an image, while the other is like disregarding high frequencies in a sound. The underlying spectral data for both types of compressions are the same, but the resulting mathematical structures have quite different properties. Through a connection to a problem that has played an important role in many areas of mathematics since the 1960s, I relate these structures to each other mathematically.

Samenvatting

Meetkunde kan op veel verschillende manieren worden bestudeerd. Wiskundigen bestuderen bijvoorbeeld symmetrieën, de kromming van oppervlakken of het aantal gaten. Een ander fundamenteel meetkundig concept dat een belangrijke rol speelt in dit proefschrift is *afstand*: de afstand tussen twee punten op een gekromd oppervlak wordt gedefinieerd als de lengte van de kortste weg die ze met elkaar verbindt (Figuur A.1). Het wiskundige gebied van de *niet-commutatieve meetkunde* houdt zich bezig met het ontwikkelen van geometrische grondslagen van ruimten die misschien moeilijk voor te stellen zijn, maar die van nature voorkomen, bijvoorbeeld in dynamische systemen, als singuliere ruimten, of in de kwantumfysica. Men kan ook betekenis geven aan het concept van afstand in dergelijke *niet-commutatieve ruimten* door ideeën uit het gebied van *optimaal transport*, en *spectra* gebruiken om deze te berekenen. Ik zal nu de concepten van *toestanden*, *afstand in optimaal transport* en *spectra* uitleggen, alvorens een van de hoofdthema's van dit proefschrift te bespreken: het begrijpen van afstand in niet-commutatieve meetkunde op een bepaalde benaderende manier.

Punten zoals de positie van een schip op zee worden beschreven door coördinaten. De snelheid van het schip en de richting waarin het vaart, kunnen ook worden beschouwd als coördinaten, en samen kunnen we zeggen dat de *toestand* van het schip op een vast tijdstip wordt beschreven door deze coördinaten: positie, snelheid en richting. Alle mogelijke toestanden van het schip vormen dan de *toestandsruimte*, die kan worden beschouwd als een geometrisch object: in deze toestandsruimte wordt een pad tussen twee toestanden beschreven door hoe de afzonderlijke coördinaten – positie, snelheid en richting – veranderen van de ene toestand naar de andere. Dat wil zeggen, een pad in de toestandsruimte bestaat uit een pad dat het schip aflegt samen met specificaties over hoeveel het schip versnelt of vertraagt langs dit pad en in welke richting het stuurt. In het algemeen kunnen toestanden van een klassiek fysisch systeem worden opgevat als punten die worden beschreven door coördinaten in de toestandsruimte die bij dit systeem hoort.

In de kwantumfysica liggen de zaken echter ingewikkelder. De toestand van een kwantumsysteem (die we hier *kwantumtoestand* noemen) kan niet worden gespecificeerd door exacte coördinaten. Het is eerder een verdeling van mogelijkheden die aangeeft hoe waarschijnlijk het is dat bepaalde posities, snelheden en richtingen (van een kwantumdeeltje bijvoorbeeld) worden gemeten. Anders gezegd is een kwantumtoestand zoals een weging van mogelijke meetresultaten. Een geschikt beeld om een kwantumtoestand te illustreren is een verdeling van zand (laten we zeggen met een totale massa van één ton) over een gebied: elk punt in dit gebied vertegenwoordigt een mogelijk meetresultaat, en de massa zand boven een punt vertegenwoordigt de waarschijnlijkheid dat dit resultaat daadwerkelijk wordt gemeten.

Binnen de optimale transporttheorie vragen wiskundigen zich af hoe de ene zandverdeling kan worden omgezet in een andere verdeling, terwijl de transportkosten worden geminimaliseerd. De laagst mogelijke transportkosten van de ene zandverdeling naar de andere zijn dan een maat voor de afstand tussen beide. Op dezelfde manier kan de afstand tussen twee kwantumtoestanden in de toestandsruimte van een kwantumsysteem worden beschouwd als een soort minimale kosten (zoals energie) die nodig zijn om het onderliggende systeem van de ene toestand naar de andere te transformeren. Dit concept van afstand verschilt van het klassieke concept van afstand tussen punten. Niettemin maakt het het mogelijk om de kwantumtoestandsruimte als een geometrisch object te beschouwen.

Een fundamenteel probleem in de kwantumfysica is dat toestanden en dus ook hun afstand niet direct gemeten kunnen worden. In de niet-commutatieve meetkunde wordt dit probleem opgelost door spectra te gebruiken om afstanden te meten: denk aan een toon die door een muziekinstrument wordt geproduceerd en die kan worden opgesplitst in fundamentele frequenties. Dat wil zeggen, een toon komt overeen met een geluidsspectrum. Dergelijke spectra zijn inderdaad nauw verbonden met meetkunde: grotere muziekinstrumenten produceren bijvoorbeeld lagere tonen. In het verlengde hiervan probeert men in het wiskundige gebied van de spectrale meetkunde uit de fundamentele trillingsmodi van een geometrisch lichaam zoveel mogelijk informatie af te leiden over de onderliggende meetkunde. Mark Kac heeft de centrale vraag hierover treffend samengevat:

“Kan men de vorm van een trommel horen?”

Feitelijk bestaat er in de niet-commutatieve meetkunde een formule waarmee de afstand tussen spectra kan worden berekend, en deze formule kan worden gegeneraliseerd tot een formule voor de afstand tussen kwantumtoestanden.

Benadering van de geometrie van vlakke tori en kwantumgroepen met behulp van gedeeltelijke spectrale gegevens

In dit proefschrift behandel ik het probleem dat in de praktijk mogelijk slechts een deel van de spectrale gegevens beschikbaar is: het menselijk oor alleen geluiden tot een bepaalde maximale frequentie waarnemen en elk fysisch meetinstrument heeft soortgelijke beperkingen. Meetkundig gezien komen dergelijke gedeeltelijke spectrale gegevens overeen met vervormingen van het onderliggende lichaam, wat bijvoorbeeld te zien is aan veranderingen in de bovengenoemde afstandsfunctie (Figuur A.2). De vraag rijst in hoeverre deze vervormde lichamen nog lijken op de oorspronkelijke en in het bijzonder of het mogelijk is om de afstanden tussen kwantumtoestanden bij benadering te berekenen aan de hand van gedeeltelijke spectrale gegevens.

Hoofdstuk 3 en Hoofdstuk 4 zijn casestudy’s voor deze vraag. Daarin onderzoek ik verschillende voorbeelden van geometrische objecten en kwantumsystemen (met bepaalde nuttige eigenschappen) waarvan zowel het volledige bijbehorende spectrum als de werkelijke afstandsfunctie bekend zijn. Ik bekijk eerst het voorbeeld van een *vlakke torus*. Een *torus* is het oppervlak van een donut (Figuur A.1), maar je kunt je een *vlakke* torus het beste voorstellen als volgt verkregen uit een vel papier: lijm twee tegenoverliggende zijden aan elkaar om een cilinder te verkrijgen. Als je nu de twee tegenoverliggende cirkelvormige einden van de cilinder aan elkaar plakt, krijg je een torus. (In 3D kan dit alleen worden gedaan door het papier te kreuken, maar een platte torus kan zonder vervorming in 4D worden ingebed en is dan – net als het vel papier – inderdaad vlak.) Het snijpunt van de paarse, rode en oranje lijnen in Figuur A.1 komt dan bijvoorbeeld overeen met een hoek van het horizontale vierkant in Figuur A.2 (eigenlijk alle vier de hoeken, omdat ze aan elkaar zijn geplakt). De functie die de afstand van dit punt tot elk ander punt op het vlakke torusoppervlak aangeeft, is aan de rechterkant van Figuur A.2 uitgezet. Aan de linkerkant en in het midden van Figuur A.2 worden afstandsfuncties weergegeven die zijn berekend op basis van een deel van de spectrale gegevens. In Hoofdstuk 3 bewijs ik dat deze afstandsfuncties steeds meer gaan lijken op de oorspronkelijke functie, wat betekent dat de vervormde versies van de vlakke torus steeds meer gaan lijken op de werkelijke vlakke torus, naarmate er meer spectrale gegevens in aanmerking worden genomen. Verder laat ik in Hoofdstuk 4 zien dat een soortgelijk resultaat geldt voor een bepaalde klasse van kwantumsystemen, de zogenaamde *compacte kwantumgroepen*.

Relatie tussen twee soorten compressies van kwantumsystemen

Als je maar een deel van de spectrale gegevens hebt, is dat net als een compressie van een kwantumsysteem. In Hoofdstuk 5 behandel ik twee soorten van zulke compressies. De ene is te vergelijken met een wazige versie van een afbeelding, terwijl de andere te vergelijken is met het negeren van hoge frequenties in een geluid. De onderliggende spectrale gegevens voor beide soorten compressies zijn hetzelfde, maar de resulterende wiskundige structuren hebben verschillende eigenschappen. Door een verband te leggen met een probleem dat sinds de jaren zestig een belangrijke rol heeft gespeeld op veel gebieden van de wiskunde, breng ik deze structuren wiskundig met elkaar in verbinding.

Zusammenfassung

Geometrie kann man auf viele verschiedene Arten untersuchen. Mathematiker:innen erforschen beispielsweise Symmetrien, die Krümmung von Oberflächen oder die Anzahl an Löchern von Körpern. Ein weiteres grundlegendes geometrisches Konzept, das eine wichtige Rolle in dieser Dissertation spielt, ist *Abstand*. Der Abstand zwischen zwei Punkten auf einer gekrümmten Oberfläche etwa ist als die Länge des kürzesten Weges, der sie verbindet, definiert (Abbildung A.1). Auf dem mathematischen Gebiet der *nichtkommutativen Geometrie* geht es darum, Grundlagen für die Analyse bestimmter Arten von geometrischen Räumen zu entwickeln, die man sich zwar nur schwierig vorstellen kann, die aber natürlicherweise z.B. bei dynamischen Systemen, als singuläre Räume oder in der Quantenphysik vorkommen. Basierend auf Ideen aus dem *Optimaltransport* kann man einen Abstandsbegriff für solche *nichtkommutativen Räume* definieren. Um Abstände konkret zu berechnen, werden *Spektren* benutzt. Im folgenden erkläre ich die Konzepte von *Zuständen*, des *Abstandsbegriffs aus dem Optimaltransport* und von *Spektren*. Danach erläutere ich eines der Hauptthemen dieser Arbeit, bei dem es darum geht, Abstände in der nichtkommutativen Geometrie auf eine bestimmte Art zu approximieren.

Punkte wie z.B. die Position eines Schiffs auf dem Meer werden durch Koordinaten beschrieben. Auch die Geschwindigkeit des Schiffs und die Richtung, in die es fährt, kann man als Koordinaten auffassen und zusammen kann man sagen, dass der *Zustand* des Schiffs zu einem festen Zeitpunkt durch diese Koordinaten – Position, Geschwindigkeit und Richtung – beschrieben wird. Alle möglichen Zustände des Schiffs bilden dann den *Zustandsraum*, den man sich als geometrisches Objekt vorstellen kann. In diesem Zustandsraum wird ein Weg zwischen zwei Zuständen dadurch beschrieben, wie sich die einzelnen Koordinaten – Position, Geschwindigkeit und Richtung – von einem zu einem anderen Zustand verändern. D.h. ein Weg im Zustandsraum besteht aus einem Weg, den das Schiff fährt, zusammen mit Vorgaben, wie stark das Schiff auf diesem Weg beschleunigt oder abbremst und in welche Richtung es lenkt. Ganz allgemein lassen sich Zustände eines klassischen physikalischen Systems als durch Koordinaten beschriebene Punkte im zu diesem System gehörigen Zustandsraum verstehen.

In der Quantenphysik sind die Dinge jedoch komplizierter. Der Zustand eines Quantensystems (im folgenden *Quantenzustand*) kann nicht durch genaue Koordinaten angegeben werden, sondern ist eher wie eine Verteilung von Möglichkeiten, die z.B. besagt, wie wahrscheinlich es ist, dass bestimmte Positionen, Geschwindigkeiten und Richtungen (eines Quantenteilchens) gemessen werden. Anders gesagt ist ein

Quantenzustand wie eine Gewichtung möglicher Messergebnisse. Ein geeignetes Bild, um einen Quantenzustand anschaulicher zu machen, ist eine Verteilung von Sand (etwa der Gesamtmasse einer Tonne) auf einem Gebiet: Jeder Punkt auf diesem Gebiet stellt ein mögliches Messergebnis dar und die über diesem Punkt liegende Masse an Sand steht für die Wahrscheinlichkeit, dass dieses Ergebnis auch tatsächlich gemessen wird.

Im mathematischen Gebiet das *Optimaltransports* beschäftigt man sich unter anderem damit, wie eine Sandverteilung in eine andere überführt werden kann, wobei die dabei entstehenden Transportkosten minimiert werden sollen. Die günstigstmöglichen Transportkosten von einer Sandverteilung zu einer anderen sind dann ein Maß für den Abstand zwischen diesen. Analog dazu kann man sich den Abstand zwischen zwei Quantenzuständen im Zustandsraum eines Quantensystems als eine Art Mindestkosten (wie Energie) vorstellen, um das zugrunde liegende System von einem Zustand in den anderen zu überführen. Dieser Abstandsbegriff unterscheidet sich vom klassischen Abstandsbegriff zwischen Punkten. Dennoch gibt er dem Quantenzustandsraum geometrischen Gehalt.

Eine fundamentale Schwierigkeit in der Quantenphysik liegt darin, dass man Zustände und damit auch deren Abstand nicht direkt messen kann. In der nichtkommutativen Geometrie benutzt man stattdessen *Spektren*, um Abstände zu messen. Der Ton etwa, den ein Musikinstrument erzeugt, kann in Grundfrequenzen zerlegt werden, d.h. ein Ton entspricht einem Klangspektrum. Spektren sind eng mit Geometrie verbunden: Größere Musikinstrumente erzeugen beispielsweise tiefere Töne. Darauf aufbauend versucht man im mathematischen Gebiet der *Spektralgeometrie* so viele Informationen wie möglich über die zugrundeliegende Geometrie eines Objekts aus seinen möglichen Eigenschwingungen zu gewinnen. Mark Kac hat die dabei zentrale Frage auf den Punkt gebracht:

“*Kann man die Form einer Trommel hören?*”

In der nichtkommutativen Geometrie gibt es tatsächlich eine Formel, anhand der man Abstände aus Spektren errechnen kann und die sich zu einer Abstandsformel für Quantenzustände verallgemeinern lässt.

Annäherung der Geometrie von flachen Tori und Quantengruppen durch partielle Spektraldaten

In dieser Arbeit beschäftige ich mich mit dem Problem, dass in der Praxis möglicherweise nur ein Teil der Spektraldaten verfügbar ist: Beispielsweise kann das menschliche Ohr nur Töne bis zu einer gewissen maximalen Frequenz wahrnehmen und ähnliche Einschränkungen hat jedes physikalische Messgerät. Geometrisch entsprechen solche partiellen Spektraldaten Verformungen des zugrundeliegenden Körpers, was man beispielsweise an Veränderungen der oben genannten Abstandsfunktion erkennen kann (Abbildung A.2). Es stellt sich die Frage, wie ähnlich diese verformten Körper noch dem ursprünglichen sind, und insbesondere, ob es möglich ist die Abstände zwischen Quantenzuständen anhand partieller Spektraldaten näherungsweise zu berechnen.

Kapitel 3 und Kapitel 4 sind Fallstudien dazu. Dabei schaue ich mir Beispiele geometrischer Objekte und Quantensysteme (mit gewissen nützlichen Eigenschaften) an, bei denen sowohl das zugehörige Spektrum als auch die tatsächliche Abstandsfunktion bekannt sind. Zunächst beschäftige ich mich mit dem Beispiel des *flachen Torus*. Ein *Torus* ist die Oberfläche eines Donuts (Abbildung A.1), aber einen *flachen* Torus kann man sich wie folgt wie ein Blatt Papier vorstellen: Man klebt zunächst zwei gegenüberliegende Seiten zusammen, um einen Zylinder zu bekommen. Danach klebt man die gegenüberliegenden kreisförmigen Enden zusammen und erhält einen Torus. (In 3D geht das nur, indem man das Blatt Papier zerknittert, aber der flache Torus kann ohne Verformung in 4D eingebettet werden und ist dann – genau wie das Blatt Papier – tatsächlich flach.) Der Schnittpunkt der violetten, roten und orangfarbenen Linie in Abbildung A.1 beispielsweise entspricht dann der Ecke des horizontalen Quadrats in Abbildung A.2 (eigentlich allen vier Ecken zugleich, da sie zusammengeklebt werden). Die Funktion, die den Abstand jedes Punktes auf dem flachen Torus zu diesem Schnittpunkt angibt, ist rechts in Abbildung A.2 dargestellt. Links und in der Mitte in Abbildung A.2 sind Abstandsfunktionen abgebildet, die aus nur einem Teil der Spektraldaten errechnet wurden. In Kapitel 3 beweise ich, dass diese Abstandsfunktionen der ursprünglichen Funktion immer ähnlicher werden, dass die verformten Versionen des flachen Torus dem eigentlichen flachen Torus also immer ähnlicher werden, je mehr Spektraldaten berücksichtigt werden. Darüber hinaus zeige ich in Kapitel 4, dass ein ähnliches Ergebnis für eine bestimmte Klasse von Quantensystemen, sogenannte *kompakte Quantengruppen*, gilt.

Beziehung von zwei Arten von Komprimierungen von Quantensystemen

Dass nur ein Teil der Spektraldaten verfügbar ist, ist wie eine Komprimierung eines Quantensystems. In Kapitel 5 beschäftige ich mich zwei Arten solcher Komprimierungen. Eine ist wie eine verschwommene Version eines Bildes, während die andere wie die Vernachlässigung hoher Frequenzen in einem Ton ist. Die zugrunde liegenden Spektraldaten für beide Arten von Komprimierungen sind die gleichen, aber die entstehenden mathematischen Strukturen haben unterschiedliche Eigenschaften. Über eine Verbindung zu einem Problem, das in vielen Bereichen der Mathematik seit den sechziger Jahren eine wichtige Rolle spielt, setze ich diese Strukturen mathematisch in Beziehung zueinander.

Curriculum vitae

Malte Leimbach (born in Detmold, Germany, on August 28, 1995) studied Mathematics at Freie Universität Berlin, obtaining the degrees of Bachelor of Science in 2018 and Master of Science in 2021. He spent the fall semester 2016/17 at Universidad de Granada, Spain, and the fall semester 2019/20 at Université Paris Didérot (Paris VII), France. His B.Sc. thesis, supervised by Dirk Werner, was entitled *"Der Satz von Choquet"* and his M.Sc. thesis, supervised by Sylvie Paycha and Alfonso Garmendia, was entitled *"A Groupoid Perspective on Pseudodifferential Operators. The Non-commutative Residue and the Tangent Groupoid"*. During his studies he held a scholarship with Stiftung der Deutschen Wirtschaft.

In 2021, he began is Ph.D. studies at Radboud Universiteit Nijmegen, the Netherlands, in the project *"Scale-dependent approach to noncommutative geometry: Applications to quantum physics"*, supervised by Walter van Suijlekom. In spring 2024, he did a two-months research visit at UC Berkeley. During his Ph.D. studies he participated in conferences as a contributing or invited speaker including the *NSeaG* workshops in Bonn in May 2023, and in Odense in August 2024, the *IWOTA*'s in Helsinki in August 2023, in Canterbury in 2024, and in Enschede in 2025, *Noncommutative Geometry and Applications* in Cortona in June 2024, and *Noncommutativity behind the dunes* in Delft in October 2024. He was invited to speak and co-lead a research session on noncommutative geometry at the workshop on *operator systems and applications* in Banff in 2025. Furthermore he was invited to give seminar talks at the *Analysis seminars* in Potsdam in January 2023, in Newcastle and Glasgow in November 2024, and in an online *Quantum groups seminar* in December 2024.

He co-organized the mathematics Ph.D. colloquium at Radboud Universiteit Nijmegen in the academic year 2022/23 and represented the mathematics Ph.D. students at the advisory board of the mathematics department. He competed for the Ph.D. award for scientific communication of the local *IMAPP* institute. He participated in training courses on teaching, scientific integrity, and diversity and inclusion.

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