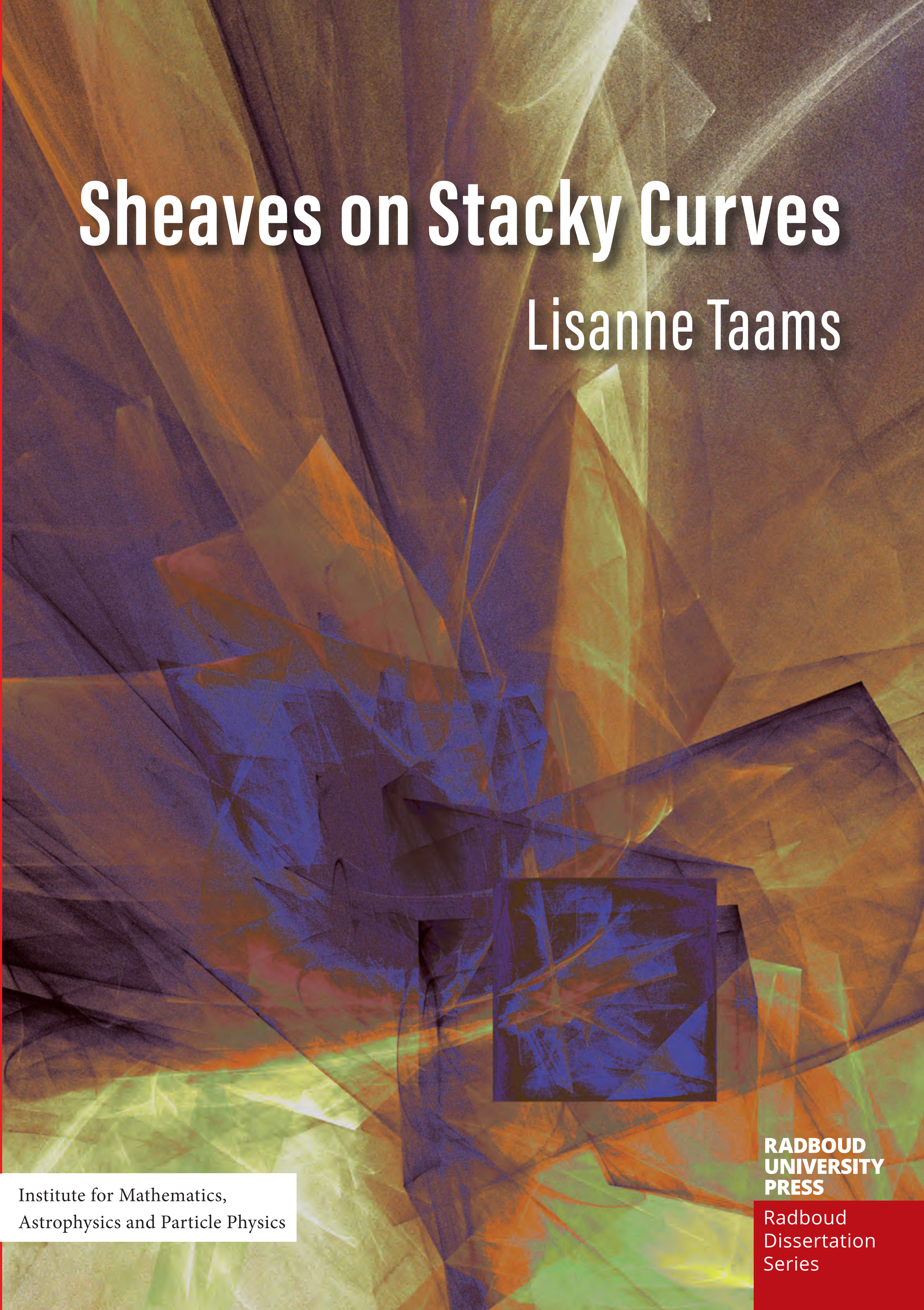

Sheaves on Stacky Curves

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Samenvatting

> Maer want een beclaghelicke verblintheyt,
> als duer Tschicsel veroirdent,
> t'verstant van velen alsoo verduystert ofte
> betoouert heeft, dat sy t'licht vande Sonne
> bouen dat der Sterren,
> ick meen de weerdicheyt deses Taels
> bouen al d'ander,
> niet en connen bemercken, tot groot
> achterdeel des Duytschen gheslachts
>
> *Beghinselen der Weeghconst*
> Simon Stevin

Dit proefschrift gaat over **algebraïsche meetkunde**. Meetkunde is de oudste vorm van wiskunde en gaat over het beschrijven en classificeren van vormen. Meer dan 2000 jaar geleden bestudeerden de eerste wiskundigen al vormen die je kan maken met passer en liniaal: lijnen, cirkels, driehoeken, enzovoort. Algebraïsche meetkunde gaat over vormen die je kan maken met algebraïsche formules, ook wel **polynomen** genoemd. Polynomen zijn formules met getallen en variabelen waarbij je mag optellen, vermenigvuldigen en machtsverheffen. Lijnen kan je bijvoorbeeld beschrijven met formules zoals $y = 2x + 3$. Ook de cirkel kan je beschrijven met zo'n formule. De stelling van Pythagoras vertelt ons namelijk dat een cirkel met straal r te beschrijven is met de formule $x^2+y^2 = r^2$. Zoals je ziet kunnen polynomen alles wat de passer en liniaal kunnen, maar polynomen kunnen daarnaast nog veel meer. Door meer variabelen te gebruiken kunnen vormen met meer dimensies beschreven worden, en door ingewikkeldere of meerdere polynomen tegelijk te gebruiken kunnen complexere vormen worden beschreven.

Een ander soort vorm die we in dit proefschrift behandelen is een “stack” of **stapel**. Een stapel is wat je krijgt als je een algebraïsche vorm **vouwt**. Soms kan je een gevouwen vorm voor een groot deel weer beschrijven met formules, maar formules kunnen geen vouwrandjes beschrijven. Een stapel kan op een technische manier de interactie tussen de formules en de vouwrandjes bevatten. Vouwen zorgt trouwens niet altijd voor vouw**randjes**. Als je bijvoorbeeld een feestmutsje vouwt krijg je bovenop een vouw**puntje**.

Er zijn veel te veel vormen om ze allemaal te kunnen beschrijven, dus we richten ons op een klein deel van de vormen. De basisvormen voor dit proefschrift zijn **krommen**, dat zijn 1-dimensionale vormen zoals lijnen en cirkels. De vernieuwing in het proefschrift ligt in de theorie van **stapelkrommen**: krommen met vouwpuntjes. In hoofdstuk 2 vind je een tabel waarin we de simpelste stapelkrommen classificeren.

De vormen waaraan we de meeste aandacht besteden zijn **schoven**. Schoven zijn meerdimensionale vormen die opgebouwd zijn uit stapelkrommen en **vectorruimtes**. Een vectorruimte is een vorm die oneindig recht is, zoals een lijn, vlak of de 3-dimensionale ruimte om ons heen. Een schoof maak je in twee stappen: je kiest eerst een stapelkromme als fundament en vervolgens plak je op elk punt van de kromme een vectorruimte. Schoven hebben twee belangrijke eigenschappen waarmee ze geclassificeerd kunnen worden. De eerste is de dimensie van de vectorruimtes die we gebruiken. De tweede is hoeveel de richting van de vectorruimtes veranderd als je over de kromme heen beweegt.

Zelfs als we deze twee eigenschappen weten zijn er nog oneindig veel verschillende schoven met dezelfde kromme als fundament. Het zijn er zelfs zoveel dat we ze niet in lijst kunnen zetten, zelfs al was de lijst oneindig lang. Om vat te krijgen op al deze schoven worden ze gerangschikt in een **moduliruimte**. De moduliruimte is een wereld waarin elke plek correspondeert met één schoof. Als je door de moduliruimte heen beweegt zie je dus één voor één alle schoven. Deze wereld is erg complex en heeft oneindig veel dimensies. Om je een idee te geven van hoe deze wereld eruitziet kan je in hoofdstuk 3 plaatjes vinden van de allersimpelste uithoekjes van deze wereld. Het belangrijkste doel van dit proefschrift is het geven van een soort routebeschrijving door de moduliruimte heen. Hierdoor begrijpen we beter welke soorten vormen er mogelijk zijn en hoe we ze kunnen beschrijven.

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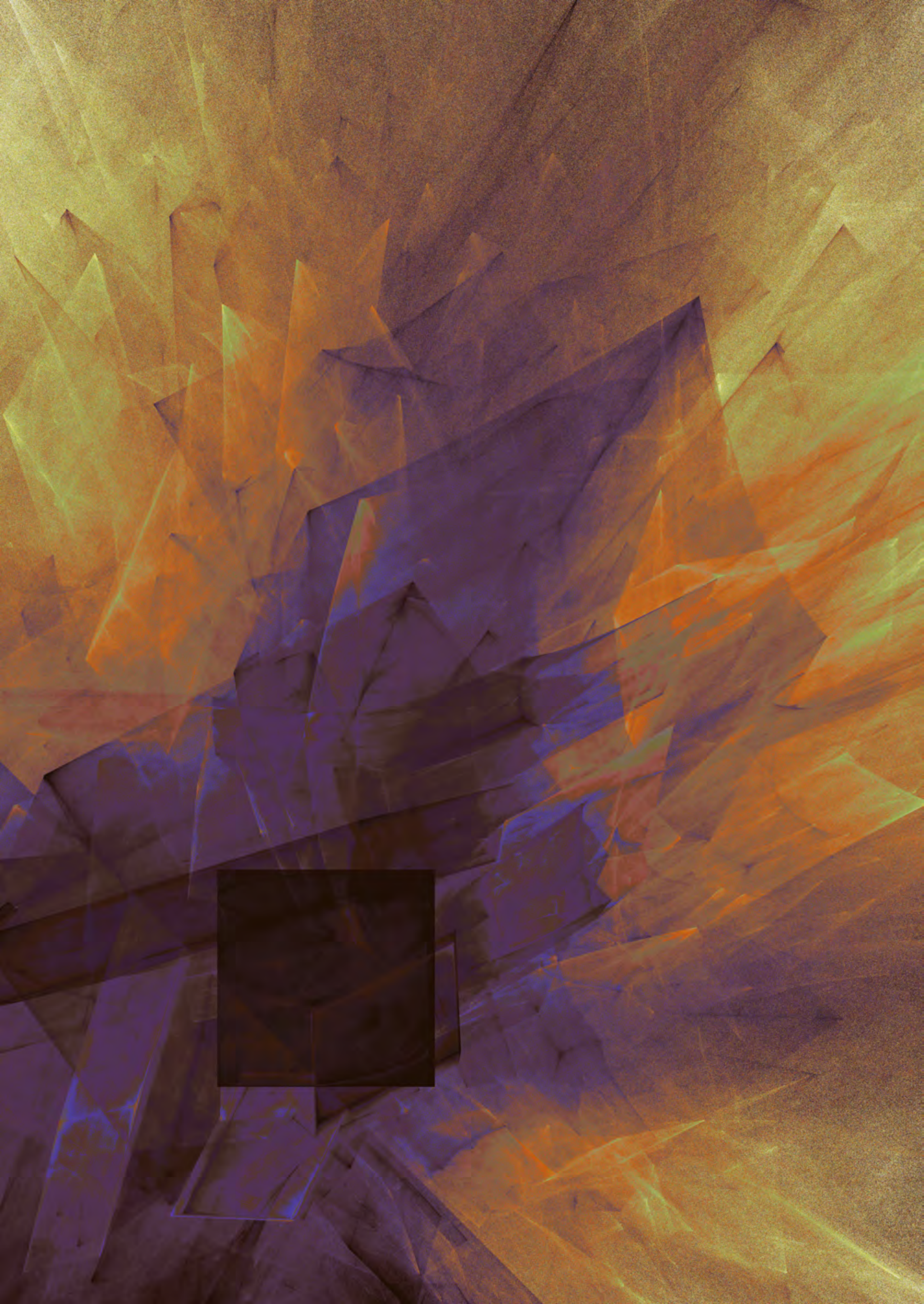


Introduction

en hoe verder hij ging
des te langer
was zijn terugweg

C.C.S. Crone

This thesis contributes to the field of **algebraic geometry** and more specifically to the study of **algebraic curves** and **vector bundles** on algebraic curves. Algebraic geometry is the study of **algebraic varieties**: topological spaces which are "locally" zero sets of polynomials, glued along rational maps. Historically the coefficients of the polynomials were complex numbers, in which case algebraic varieties are closely related to **complex manifolds**. We will take the more general perspective and allow the coefficients of our polynomials to lie in an arbitrary field. In addition to algebraic varieties we like to study basic algebraic structures over them such as vector bundles. A vector bundle is a way to associate to each point of an algebraic variety a vector space, such that the vector space varies "algebraically" as we move around the variety.

We will first give a historical overview of the study of algebraic curves and vector bundles on them. Then we will explain how this thesis fits into and builds on top of the existing theory.

Algebraic curves

One of the most fundamental problems in algebraic geometry is classification: to give a "list" of all the algebraic varieties. In general this problem is incredibly difficult; there are simply too many varieties and we have no idea what most of them look like. However, a lot of partial progress has been made by considering restricted classes of varieties. The most famous one is probably the class of smooth compact 1-dimensional varieties or **algebraic curves**.

If we assume our field is the complex numbers, then there is a correspondence between algebraic curves and **Riemann surfaces**, which are closed 1-dimensional complex manifolds. The underlying topological space of a Riemann surface is a topological surface and topological surfaces are classified by their **genus**: the number of "holes" in the surface. The classification problem for algebraic curves then becomes to describe all the ways to endow the genus g surface with a complex structure up to isomorphism.

The solution depends dramatically on the genus. For $g = 0$, the underlying surface is a sphere, which has a unique complex structure called the Riemann sphere or the complex projective line. For $g = 1$, the underlying surface is a torus and every complex structure can be obtained as a quotient $\mathbb{C}/L$, where L is a two dimensional lattice inside $\mathbb{C}$. This reduces the genus 1 classification to the algebraic problem of classifying lattices. For $g \geq 2$, the situation becomes more complicated. Riemann showed that a Riemann surface depends on exactly $3g - 3$ parameters, which he called **moduli** [Rie57].

Riemann's work implicitly assumes the existence of a space into which these parameters are arranged, which we now call a **moduli space**. More precisely, a moduli space is itself an algebraic variety, such that, in this example, the points precisely classify the Riemann surfaces of genus g up to isomorphism. The classification problem is now twofold: first show that a moduli space exists and then describe its geometry in as much detail as possible.

One of the most basic geometric problems is to identify the **connected components** of a moduli space. Two Riemann surfaces lie in the same connected component of a moduli space if and only if they can be continuously deformed into one another, so identifying the connected components corresponds to solving the simplified problem of classifying Riemann surfaces up to continuous deformations.

Moduli spaces as quotients

A common approach to showing the existence of moduli spaces is to present them as quotients. We start by giving an **overparameterization** X together with a group G which acts on X, such that the orbits of the action correspond precisely to the isomorphism classes of objects that need to be classified. The problem with this perspective is that orbit spaces are in general not themselves varieties.

This problem led to the development of geometric invariant theory by Mumford [Mum65]. When X is a **projective** variety (a very strong notion of compactness) and the group action is compatible with the projectivity, Mumford defines (up to some choices) two G-invariant open subsets $X^s \subset X^{ss} \subset X$, called the stable and semistable locus respectively, with particularly nice behavior. The orbit space X^s/G of the stable locus actually admits a natural structure of a variety. The orbit space X^{ss}/G of the semistable locus **almost** admits the structure of a variety: one first needs to identify orbits whose closures intersect. The resulting variety is called the GIT quotient, denoted by $X^{ss}/\!\!/G$, and it is a projective compactification of X^s/G. One major application of the theory was to define **stable vector bundles** and show the existence of a quasi-projective moduli space classifying stable bundles [Mum63].

Vector bundles

The classification of vector bundles on curves depends again on the genus of the underlying curve. Vector bundles on the projective line were classified by Dedekind and Weber: all vector bundles can be decomposed into line bundles and a line bundle is determined by its degree (a measure of twistedness) [DW82]. In genus 1 there is an explicit (but quite complicated) description of vector bundles due to Atiyah [Ati57].

When $g \geq 2$ we again end up in the realm of moduli theory. Equipped with the new notion of stable and semistable bundles, Narasimhan and Seshadri showed that over the complex numbers there is a correspondence between (twisted) irreducible unitary representations of the fundamental group of a Riemann surface and stable vector bundles on the Riemann surface [NS65]. Remarkably the moduli space of unitary representations is a priori only a real-analytic manifold, however this correspondence shows that it also admits the structure of a complex algebraic variety.

They also showed that the connected components of the moduli space are given by semistable bundles with fixed rank and degree. Moreover the dimension of each component is $n^2(g-1)+1$, where n is the rank of the bundles. Finally they gave an explicit description of the identifications between semistable bundles that are needed to form the GIT quotient, which is now known as S-equivalence.

The correspondence was later generalized by Mehta and Seshadri to the representations of the fundamental group of **punctured Riemann surfaces** [MS80]. On the other side of the correspondence they obtained vector bundles with added data around the punctures and called these **parabolic bundles**. They then carried out Mumford's GIT to show that there exists a projective moduli space of semistable parabolic bundles and gave an explicit description of the identification between semistable objects that is abstractly defined by GIT.

Algebraic stacks

Ever since Deligne and Mumford's seminal paper [DM69] it is widely understood that moduli spaces are more naturally moduli **stacks**. Algebraic stacks are a natural generalization of varieties that allow for quotients by algebraic groups. When we quotient a topological space by a continuous group action we naturally obtain a topological groupoid, so a good way to visualize a stack is as a topological space together with automorphisms groups attached at certain points.

From the quotient perspective of moduli spaces, it is already clear why we would like to work with stacks, as the existence problem is solved almost tautologically. However from the classification perspective there is an added bonus: when we classify objects which have non-trivial automorphism groups, a moduli stack can naturally record these automorphisms as well.

In their paper Deligne and Mumford gave a treatment of the moduli stack $\mathcal{M}_g$ of algebraic curves. There are several ways to show that there exists a moduli space M_g of curves of genus g and, over the complex numbers, it was quickly shown to be connected using analytic techniques, which could not be generalized to arbitrary fields. Deligne and Mumford's breakthrough was to instead show that the moduli stack $\mathcal{M}_g$ over an arbitrary base field was connected, which then implied the result for the moduli space M_g as well.

The most important stack from the perspective of this thesis is the moduli stack of vector bundles on a given curve. For technical reasons is it is often convenient to slightly extend the class of vector bundles by taking kernels and cokernels of vector bundle morphisms to obtain the class of **coherent sheaves**. Fixing a curve C, there is a moduli stack $\mathsf{Coh}(C)$ of coherent sheaves, together with an open substack of vector bundles $\mathsf{Bun}(C)$.

The connected components are the substacks $\mathsf{Bun}_{n,d}(C) \subset \mathsf{Coh}_{n,d}(C)$, consisting of bundles with fixed rank n and degree d. The dimension of $\mathsf{Coh}_{n,d}(C)$ is $n^2(g-1)$, which is 1 less than the dimension of Narasimhan and Seshadri's variety. This is because vector bundles have positive dimensional automorphism groups and the dimension of a stack incorporates the dimension of the automorphism groups. In particular all vector bundles are invariant under scalar multiplication, which is the reason for the difference of 1. One nice consequence is that the stacky dimension formula also holds for $g = 0$ and $g = 1$.

Good moduli spaces

Another breakthrough in the theory of moduli spaces was a stack-theoretic treatment of Mumford's GIT. For an arbitrary stack we can ask if there is a variety (more precisely an **algebraic space**) that best approximates it. If such a variety exists and satisfies a list of technical properties, it is deemed a **good moduli space** for the stack [Alp13]. The motivating example is the fact that the GIT quotient $X^{ss}/\!\!/G$ is a good moduli space for the stack quotient $[X^{ss}/G]$. Alper, Halpern-Leistner and Heinloth have given two valuative criteria for the existence of good moduli spaces [AHH23], which can actually be checked in practice.

For example the moduli stack of semistable vector bundles admits a good moduli space, which is in fact the same moduli space constructed by Narasimhan and Seshadri [ABBLT22]. The same is true for the stack of semistable parabolic bundles and the corresponding variety constructed by Mehta and Seshadri. Note that in these cases the good moduli space only parameterizes objects up to S-equivalence, and in general the map from the stack to the good moduli space is a topological quotient.

One major advantage of this approach is that we no longer need any projectivity assumptions. However the theory also does not explain when the resulting good moduli space is projective. For this reason the modern approach to good moduli spaces has two parts, first check the existence criteria of [AHH23] and then give a separate proof of projectivity.

Cohomology and motives

Identifying connected components is only a first step for understanding a moduli stack. A very popular approach to understanding complicated varieties and stacks in algebraic geometry is via **cohomology theories**, such as singular cohomology or étale cohomology. Cohomology theories are functors from some geometric category (such as stacks over a fixed field) to a linear category (such as graded Abelian groups) satisfying some natural properties. Cohomology theories are useful as they take away some of the complexity, letting us isolate some specific geometric information.

The idea of **Voevodsky motives** is to build a universal functor M and a category of motives through which other cohomology theories factor, and which encodes the **Chow groups** of smooth varieties, another important invariant. In practice this means that if we can make computations in the category of motives, we automatically obtain computations in other cohomology theories and Chow groups. As an illustration, the geometric fact that projective space $\mathbb{P}^n$ can be naturally stratified by affine spaces

$$\mathbb{P}^n = \mathbb{A}^n \amalg \mathbb{A}^{n-1} \amalg \cdots \amalg \mathbb{A}^1 \amalg \mathtt{pt},$$

gives rise to a motivic formula

$$M(\mathbb{P}^n) \simeq \bigoplus_{i=0}^{n} \mathbb{Z}\{i\},$$

where $\mathbb{Z}\{1\}$ is a fundamental object called the **Tate** motive and $\mathbb{Z}\{i\} = \mathbb{Z}\{1\}^{\otimes i}$. This formula implies the classic computations of singular cohomology, étale cohomology and Chow groups:

$$\begin{aligned} H^*_{\mathsf{sing}}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) &\simeq \mathbb{Z}[x]/(x^{n+1}), \\ H^*_{\text{ét}}(\mathbb{P}^n_{\mathbb{F}_p}, \mathbb{Q}_\ell) &\simeq \mathbb{Q}_\ell[x]/(x^{n+1}), \\ \mathsf{CH}^*(\mathbb{P}^n) &\simeq \mathbb{Z}[H]/(H^{n+1}). \end{aligned}$$

Note that a priori the motivic formula only gives us these objects as Abelian groups, to compute the ring structure one needs to do some extra work.

There are very nice formulas for the motives of moduli stacks of vector bundles on a curve C, expressed in terms of the motive of symmetric powers of C and its Picard stack $\mathtt{Pic}_0(C)$ [HP22; HP21a]. Namely, for $n \geq 1$ we have

$$M\left(\mathsf{Bun}_{n,d}(C)\right) = M(\mathtt{Pic}_0(C)) \otimes \bigotimes_{i=1}^{n-1} \bigoplus_{j \geq 0} M(\mathsf{Sym}^j(C))\{ij\},$$

$$M\left(\mathsf{Coh}_{n,d}(C)\right) = M(\mathtt{Pic}_0(C)) \otimes \bigotimes_{i \geq 1} \bigoplus_{j \geq 0} M(\mathsf{Sym}^j(C))\{ij\}.$$

Geometrically the tensor product of motives corresponds to products of stacks and the direct sum corresponds to a disjoint union. This means that we can interpret the formulas as saying that the moduli space of vector bundles is "made up" of products of the Picard stack and symmetric powers of the curve.

Stacky curves

This thesis investigates coherent sheaves on **stacky curves**. Stacky curves are regular 1-dimensional stacks with finite stabilizer groups at finitely many points (Definition 1.1.1). Stacky curves naturally arise as quotients of curves by finite group actions. More generally they can be obtained by gluing together several such quotients. The underlying topological space of a stacky curve always admits the structure of an algebraic curve and is called the **coarse space** of the stacky curve. Therefore we like to intuitively think of a stacky curve as a curve together with stabilizer groups attached at finitely many points, which we call the stacky points.

When the field has positive characteristic it is often important to assume that the order of the stabilizer groups is not divisible by the characteristic. In this case we call the stacky curve **tame**. In the tame setting a stacky curve is completely determined by the coarse space and the order of the stabilizer groups. This is formalized in the following theorem, which was possibly well-known to a handful of experts; however, our proof addresses some subtleties over non-separably closed fields which had not been considered previously.

Theorem A (Proposition 1.1.27 and Theorem 1.1.31) Every tame regular stacky curve is a root stack over its coarse space and conversely every root stack over a regular algebraic curve is a regular stacky curve.

The above theorem classifies stacky curves as curves together with a set of weighted marked points; however the **spherical** stacky curves with genus $g < 1$ are particularly well behaved and we show that there is a particularly nice classification in terms of root systems. Note that stacky curves can have a fractional genus, this reflects the idea that a stacky point can be thought of as a "fractional" point.

Theorem B (Section 2.1) Let $\mathcal{C}$ be a smooth projective stacky curve of genus $g < 1$. Then there exists a natural irreducible root system in a quotient of the Grothendieck group $\mathsf{K}_0(\mathcal{C})$. This root system together with the residue fields of the stacky points uniquely determines the stacky curve up to isomorphism.

Sheaves on stacky curves

In his thesis [Nir09] Nironi proves that there exists a moduli stack of coherent sheaves on a projective Deligne-Mumford stack. Moreover, using the concept of generating sheaves from [OS03], he introduces a notion of semistability and shows that the substack of semistable sheaves admits a projective good moduli space by using GIT. This notion of semistability depends on the chosen generating sheaf and thus gives many different substacks and corresponding moduli spaces.

In this thesis we specialize to the case of a stacky curve $\mathcal{C}$ and define discrete invariants called **twisted degrees** that together with the rank determine the connected components of $\mathsf{Coh}(\mathcal{C})$ and $\mathsf{Bun}(\mathcal{C})$. The twisted degrees are a finite set of integers that encode the action of the stabilizer groups on the fibers at the stacky points.

Theorem C (Corollary 3.1.9 and Theorem 3.1.12) Let $\mathcal{C}$ be a smooth projective stacky curve. The stack $\mathsf{Coh}_{n,\underline{d}}(\mathcal{C})$ of coherent sheaves with fixed rank n and twisted degrees $\underline{d}$ is smooth and connected.

There are forgetful maps $\mathsf{Bun}_{n,\underline{d}}(\mathcal{C}) \to \mathsf{Bun}_{n,d}(C)$ from vector bundles on a stacky curve $\mathcal{C}$ to vector bundles on its coarse space C. These forgetful maps are fibrations by flag varieties (Example 3.2.3), which gives a motivic formula

$$M(\mathsf{Bun}_{n,\underline{d}}(\mathcal{C})) = M(\mathsf{Bun}_{n,d}(C)) \otimes M(\mathtt{Flag}_{\underline{d}}(k^{\oplus n})).$$

The corresponding map $\mathsf{Coh}_{n,\underline{d}}(\mathcal{C}) \to \mathsf{Coh}_{n,d}(C)$ is not a fibration and the motive seems to be much more complicated. Although we do not quite obtain a motivic formula, we can give a qualitative statement.

> **Theorem D** (Corollary 3.3.18) Let $k = \bar{k}$ be an algebraically closed field. The motive with rational coefficients $M(\mathsf{Coh}_{n,\underline{d}}(\mathcal{C}))$ lies in the thick tensor subcategory generated by $M(C)$.

Sheaves on stacky curves are closely related to parabolic bundles on the curve obtained by puncturing all the stacky points. This has been observed in several forms, first in [Bis97], via the existence of specific covers of the stacky curve. In fact there is a categorical equivalence between **quasi-parabolic** bundles and vector bundles on stacky curves [Bor07]. This categorical equivalence is upgraded to an equivalence of moduli stacks in [Nir09]. We analyze how these equivalences interact with the different notions of semistability and the discrete invariants, to give equivalences between stacks of semistable parabolic bundles and semistable vector bundles on stacky curves.

> **Theorem E** (Corollary 3.2.7) Every connected component of the moduli stack of semistable parabolic bundles on a smooth projective curve is isomorphic to a stack $\mathsf{Bun}^{\mathcal{E}\text{-ss}}_{n,\underline{d}}(\mathcal{C})$ of vector bundles on some smooth projective stacky curve $\mathcal{C}$, with rank n and twisted degrees $\underline{d}$, that are semistable with respect to some generating sheaf $\mathcal{E}$.

This shows that the moduli theory of parabolic bundles can be completely understood by considering vector bundles on stacky curves.

Even though Nironi has proven the existence of projective moduli spaces using GIT, these techniques are not **effective**: they do not explain in any way how to obtain actual embeddings of the moduli space into projective space. In joint work with C. Damiolini, V. Hoskins, S. Makarova we generalize the approach of [ABBLT22] from classical curves to stacky curves to give an effective proof of projectivity.

Theorem F (Corollary 4.5.4) The stack $\mathsf{Bun}_{n,\underline{d}}^{\mathcal{E}\text{-ss}}(\mathcal{C})$ admits a good moduli space B. Moreover there is a line bundle $L_{\mathcal{E}}$ on $\mathsf{Bun}_{n,\underline{d}}^{\mathcal{E}\text{-ss}}(\mathcal{C})$ and there are effective bounds for a power of $L_{\mathcal{E}}$ to define a finite map $B \to \mathbb{P}^N$, giving an effective proof of projectivity of B.

By the previous theorem, this is also an **effective** strengthening of the results of [MS80].

Structure of the thesis

Chapter 1 gives a basic treatment of the structure of stacky curves and their categories of sheaves. We show that every tame stacky curve is a root stack (see Theorem A) and thus a stacky curve is Zariski-locally a quotient of a curve by a finite group. This is a “well-known” result, certainly over the complex numbers, but we take some care to give these descriptions over an arbitrary (possibly non-separably closed) field. On the side of sheaves we give a computation of the Grothendieck group $\mathsf{K}_0(\mathcal{C})$ in terms of the geometry of the stacky curve $\mathcal{C}$. Finally we describe a relation between the category of (semistable) vector bundles on smooth projective stacky curves and the category of (semistable) parabolic bundles .

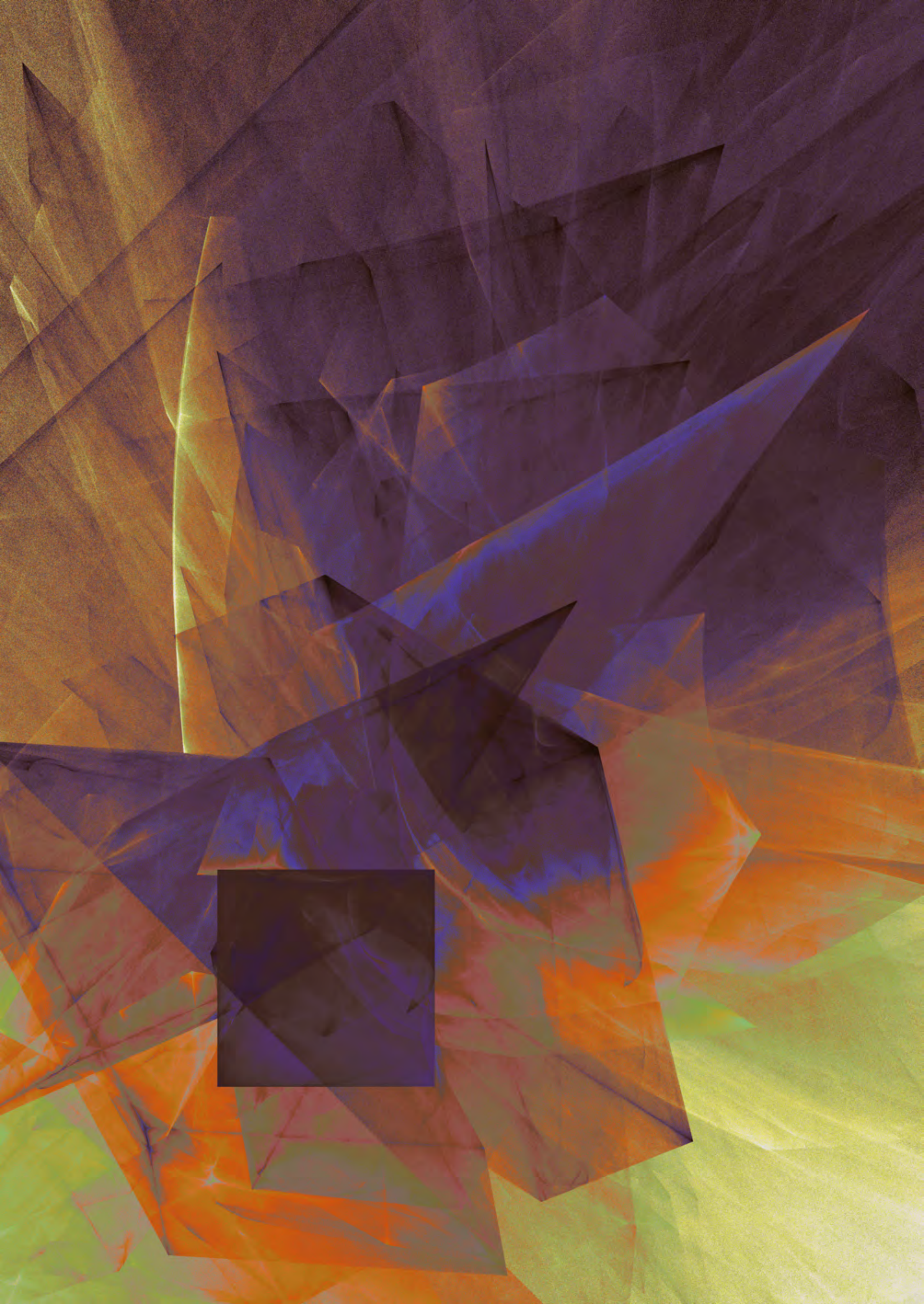
In Chapter 2 we focus on spherical stacky curves, which are the curves of genus < 1. These have the simplest behavior and we show that a quotient of their Grothendieck group contains a natural irreducible root system. This enables us to classify spherical stacky curves using Dynkin diagrams as described in Theorem B. We show that there is a close relation between spherical curves and finite subgroups of $\mathsf{PGL}_2(k)$, and we classify which bundles on $\mathbb{P}^1$ admit an equivariant structure for a particular finite group.

Chapter 3 considers the moduli theory of coherent sheaves on smooth projective stacky curves and several related moduli problems. We show that the stack of coherent sheaves on a stacky curve is smooth and show that the substacks with fixed invariants are irreducible, proving Theorem C. We also show that any stack of semistable parabolic bundles is isomorphic to a stack of semistable vector bundles on a stacky curve as in Theorem E. Finally we show that the stack of coherent sheaves on a stacky curve is stratified by the invariants of the torsion part and these strata admit the structure of vector bundle stacks and the stack of torsion sheaves is in turn stratified by (graded) Young diagrams. These stratifications are used to prove Theorem D.

Chapter 4 proves Theorem F and is based on joint work with C. Damiolini, V. Hoskins and S. Makarova. We apply the existence theorem of [AHH23] to obtain a proper good moduli space for the stack of $\mathcal{E}$-semistable vector bundles on a stacky curve for any generating sheaf $\mathcal{E}$. We then construct an explicit determinantal line bundle on the stack and give effective bounds for the global generation of a power of this line bundle. We show this defines a finite map from the good moduli space to projective space, showing that the good moduli space is projective.

Appendix A is a technical appendix where we construct a universal flattening stratification for cyclotomic stacks, which is applied in Chapter 3. This construction is a specific case of the conjectured construction in [OS03].

Appendix B collects some facts about Voevodsky motives, which will be used in Chapter 3.



CHAPTER 1

Fundamentals of stacky curves

Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.

the Elements
Euclides

In this chapter we will introduce and develop the basic theory of stacky curves. In the first section we cover the local and global structure results for stacky curves, which relate stacky curves to classical curves, proving Theorem A. In the second section we start our analysis of coherent sheaves on stacky curves and prove analogues of many of the classical results, like the existence of a torsion filtration and a description of invertible sheaves. In the third section we study projective stacky curves and give analogues of Serre-duality, the Riemann-Roch theorem and the Riemann-Hurwitz theorem. We also discuss Hilbert polynomials and stability for vector bundles on projective stacky curves. In the final section we will relate vector bundles on stacky curves to parabolic vector bundles on stacky curves and compare the notions of stability on both sides.

1.1 Structure results for stacky curves

In this section we will describe the basic geometry of stacky curves. The main results are two structure results for stacky curves: a local structure result describing stacky curves locally as finite quotients of classical curves and a global structure result describing stacky curves as a classical curve together with finite ramification data. The results in this chapter are certainly well-known; however, they are often stated in such high generality that it might obfuscate the simplicity of the case of curves. Consequently, we will restate these results in the case of curves and use the fact that we are on a curve to give simplified proofs. What is new is that we work over an arbitrary (potentially imperfect) base field. For this reason we will have to work with regular curves rather than smooth curves.

> **Definition 1.1.1** A **stacky curve** is a regular separated finite type geometrically connected Deligne-Mumford stack $\mathcal{C}$ of dimension 1 over a field k, such that there exists a (non-empty) scheme X and an open immersion $X \to \mathcal{C}$.

The condition that $\mathcal{C}$ contains an open subscheme excludes things like gerbes over curves and ensures that $\mathcal{C}$ has only finitely many stacky points. We will only consider regular stacky curves, which is why we include it in the definition. Note that by definition a curve is just a stacky curve that happens to be a scheme. When we want to emphasize that a curve is scheme, we will call it a **classical** curve.

We will now define the basic properties of points on a stacky curve. For a nice discussion on residual gerbes of points of an algebraic stack see [BL24, Appendix A].

> **Definition 1.1.2** Let $\mathcal{C}$ be a stacky curve and p be a closed point of $\mathcal{C}$. We define the **residual gerbe** of p to be the unique reduced closed substack of $\mathcal{C}$ supported on p and denote it by $\iota_p\colon \mathcal{G}_p \hookrightarrow \mathcal{C}$. There is a field $\kappa(p)$, called the **residue field** of p, with a map $\mathcal{G}_p \to \mathbf{Spec}(\kappa(p))$ that is initial among such maps. We say that p is a **stacky point** if $\mathcal{G}_p$ is a stack, i.e. is not the spectrum of a point. Let l be a finite extension of $\kappa(p)$ over which $\mathcal{G}_p$ splits, so $(\mathcal{G}_p)_l \simeq [\mathbf{Spec}(l)/G_{p,l}]$, where $G_{p,l}$ acts trivially on l. We say that p is a **tame** point if the order of $G_{p,l}$ is coprime to the order of k for any (or every) such l. We say that $\mathcal{C}$ is **tame** if all of its points are tame.

Note that, since stacky curves are locally Noetherian, this definition of the residual gerbe is equivalent to the more general definition of [Stacks, Definition 06MU] via [Stacks, Lemma 0H27].

We will refer to $G_{p,l}$ as the **stabilizer group** of a closed point p and denote it by G_p.

Strictly speaking this is not well defined, since we need to choose a field extension l, but since we are usually working étale locally this is not a big problem. Moreover we will see in Lemma 1.1.35 that for **tame** stacky curves the residual gerbe always splits over $\kappa(p)$ and G_p is a well defined group scheme defined over $\kappa(p)$ isomorphic to μ_e.

The motivating example of a stacky curve is the following.

> **Example 1.1.3** Let C be a curve over a field k and G be a finite subgroup of $\mathsf{Aut}(C)$; then the stack quotient $[C/G]$ is a stacky curve. The stacky points of $[C/G]$ correspond to the orbits of G with non-trivial inertia. Let p be a fixed point of the G-action and denote by $G_s(p)$ the stabilizer group and by $G_i(p) \subset G_s(p)$ the inertia group, i.e. the subgroup that acts trivially on the residue field $\kappa(p)$ of p (see [SGA1, Exposé V.2]). The residual gerbe $\mathcal{G}_{Gp}$ is isomorphic to $[\mathsf{Spec}\left(\kappa(p)^{G_s(p)/G_i(p)}\right)/G_i(p)]$.

In the next example we glue together two quotient curves to get a stacky curve that is not itself a quotient of a curve (see Proposition 1.3.5 for the proof of this claim).

> **Definition 1.1.4** The **football space** $\mathcal{F}(p,q)$, with weights $p,q \in \mathbb{N}_{\geq 1}$, is given by gluing the two stacky curves $U_0 = [\mathbb{A}^1_k/\mu_p]$ and $U_1 = [\mathbb{A}^1_k/\mu_q]$, where μ_p and μ_q act by multiplication and the gluing map $\mathsf{Spec}\left(k[x,x^{-1}]\right) \simeq [\mathbb{A}^1_k - \{0\}/\mu_p] \to [\mathbb{A}^1_k - \{0\}/\mu_q] \simeq \mathsf{Spec}\left(k[y,y^{-1}]\right)$ is defined by $y \to x^{-1}$.

The football space $\mathcal{F}(1,1)$ is simply the projective line $\mathbb{P}^1_k$ and topologically $\mathcal{F}(p,q)$ is just $\mathbb{P}^1_k$ where the points 0 and ∞ are stacky with residual gerbes $B\mu_p$ and $B\mu_q$ respectively. Over the complex numbers, we can think of this as a sphere with two pointy sides, i.e. an American football. When p and q are coprime, $\mathcal{F}(p,q)$ is isomorphic to the weighted projective stack $\mathcal{P}(p,q) := [\mathbb{A}^2_k - \{(0,0)\}/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts as $\lambda \cdot (x,y) = (\lambda^p x, \lambda^q y)$. When $\mathsf{gcd}(p,q) = e > 1$, there is a map $\mathcal{P}(p,q) \to \mathcal{F}(p/e,q/e)$, making $\mathcal{P}(p,q)$ into a μ_e-gerbe over $\mathcal{F}(p/e,q/e)$.

> **Definition 1.1.5** Let $\mathcal{C}$ be a stacky curve. A **coarse space morphism** for $\mathcal{C}$ is a morphism $\pi\colon \mathcal{C} \to C$ to an algebraic space satisfying the following properties.

- Any morphism $f\colon \mathcal{C} \to X$ to an algebraic space factors uniquely through π.
- The induced map $|\mathcal{C}(\Omega)| \to |C(\Omega)|$ is a bijection for algebraically closed fields Ω.

The algebraic space C is called the **coarse space** of $\mathcal{C}$.

By the factorization property, the coarse space morphism is unique up to unique isomorphism if it exists. To mirror the idea that the coarse space is a rough (coarse) approximation of the stacky curve we will write stacky curves with calligraphic letters and their coarse spaces with the same non-calligraphic letter. In the literature coarse spaces are sometimes called coarse **moduli** spaces, in analogy with the concept of fine/coarse moduli spaces. Since stacky curves are not (always) moduli spaces, we omit the word "moduli".

To show the existence of coarse spaces we can apply the Keel-Mori theorem [KM97]; see for example [Con05] for a proof.

Theorem 1.1.6 (Keel-Mori) Let $\mathcal{X}$ be an Artin stack that is locally of finite presentation over a field k, with finite inertia stack $I(\mathcal{X})$. Then there exists a coarse space morphism $\pi\colon \mathcal{X} \to X$ with the following additional properties.

(1) If $\mathcal{X}$ is separated, then so is X.

(2) The coarse space X is locally of finite type over k.

(3) The map π is proper and quasi-finite.

(4) For any flat map $X' \to X$ of algebraic spaces, the pullback $\pi'\colon \mathcal{X} \times_X X' \to X'$ is also a coarse space morphism.

Clearly stacky curves satisfy the conditions of the Keel-Mori theorem, so they always have a coarse space morphism. Using this fact we can give the local structure result for stacky curves we alluded to above.

Proposition 1.1.7 (Local form of stacky curves) Let $\mathcal{C}$ be a stacky curve with coarse space map $\pi\colon \mathcal{C} \to C$ and p be a closed point of C with stabilizer group

G_p. Then there exists an étale morphism $V \to C$ from a classical curve, with p in its image, and a (possibly disconnected) classical curve U with an action of G_p such that $\mathcal{C} \times_C V \simeq [U/G_p]$.

Proof. The existence of the schemes V, U and the action by G_p follows from the proof of [AV02, Lemma 2.2.3]. The quotient $U \to [U/G_p]$ is finite and smooth, so U is finite and smooth over $\mathcal{C}$. It follows that U is regular separated and 1-dimensional over k, so it is a (possibly disconnected) classical curve. ⬡

Proposition 1.1.8 Let $\mathcal{C}$ be a stacky curve with coarse space C, then C is a classical curve.

Proof. By Proposition 1.1.7, we know there exists a surjective étale cover by a (possibly disconnected) curve $f\colon V \to C$. It follows that C is regular and a fortiori normal. By Theorem 1.1.6 (1), we know that C is separated and since π is a homeomorphism, C is irreducible. Finally we have an open substack $X \to \mathcal{C}$ that is a 1-dimensional scheme. Now the coarse space of X, which is X, is an open subspace of C. Thus C is 1-dimensional as it contains an open 1-dimensional scheme. By [Knu71, Theorem V.4.4], a normal, separated, irreducible algebraic space over a field is a scheme in codimension 1. It follows C is a scheme and hence a curve. ⬡

Ramification theory and root stacks

We will now develop some basic ramification theory for stacky curves. This is based on [GS17], which gives a treatment for more general (smooth) DM-stacks. The goal is to understand the ramification of the coarse space map and see how it characterizes the curves.

Definition 1.1.9 Let $f\colon \mathcal{C} \to \mathcal{D}$ be a morphism of stacky curves. Let $p \in \mathcal{C}$ be a closed point with image $f(p) = q \in \mathcal{D}$. Take an étale cover by a scheme $V \to \mathcal{D}$ and then another étale cover by a scheme $U \to V \times_{\mathcal{D}} \mathcal{C}$. Then take a point $u \in U$ that maps to p and let v be its image in V. Then we define the **ramification index** $e_{p/q}$ to be the ramification index $e_{u/v}$ of u over v.

Lemma 1.1.10 The above definition is independent of the chosen covers.

Proof. Fix an étale cover $V \to \mathcal{D}$ and choose two different U and U'. Then $U \times_{V \times_{\mathcal{D}} \mathcal{C}} U'$ is also étale over $V \times_{\mathcal{D}} \mathcal{C}$, so we may assume there is an étale morphism $U' \to U$ commuting with the map to $V \times_{\mathcal{D}} \mathcal{C}$. Let v, u, u' be such that $u' \mapsto u \mapsto v$; then $e_{u'/v} = e_{u'/u} e_{u/v} = e_{u/v}$. Now pick two pairs of étale covers U, V and U', V'. Since $V \times_{\mathcal{D}} V'$ is étale over $\mathcal{D}$, we may assume that there is an étale morphism $V' \to V$. By the first point, we may replace U' by $U \times_{\mathcal{D}} U'$ so that we have a commutative diagram,

$$\begin{array}{ccc} U' & \longrightarrow & V' \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

where the vertical arrows are étale. Now pick u, v, u', v' appropriately; then we have $e_{u/v} = e_{u'/u} e_{u/v} = e_{u'/v'} e_{v'/v} = e_{u'/v'}$. ◯

Definition 1.1.11 For f as above, the **ramification locus** R_f is the set of closed points $p \in \mathcal{C}$ such that $e_{p/f(p)} > 1$. The **branch locus** is the image of R_f inside $\mathcal{D}$. We denote by $\underline{e_f}$ the set of ramification indices $e_{p/f(p)}$ for $p \in R_f$. A map f is called **unramified** if R_f is empty. We say that f is **tamely ramified at** p if the characteristic of k does not divide $e_{p/f(p)}$. The map f is **tamely ramified** if it is tamely ramified at every point. The pair $(R_f, \underline{e_f})$ is the **ramification data** of f.

The ramification data of tame quotients is particularly well behaved.

Lemma 1.1.12 Let G be a finite group acting faithfully on a curve C. Consider the coarse space morphism $\pi\colon [C/G] \to C/G$ from the stack quotient to the schematic quotient. Assume that the orders of the inertia groups $G_i(x)$ are not divisible by the characteristic of k for any closed point $x \in C$, i.e. $[C/G]$ is tame. Then for any closed point $y \in [C/G]$, with $z := \pi(y)$, we have that the ramification index $e_{y/z}$ is equal to the order of the inertia group $G_i(x)$ for a point $x \in C$ lying above y.

Proof. Since C/G is already a scheme, we can take the identity map as its étale cover. The map $C \to [C/G]$ is étale by the assumption on the orders of the inertia groups, so we may pick a point x in C that maps to y and compute $e_{x/z} = e_{y/z}$ using the map $C \to C/G$. Now the result is classical. ◯

Lemma 1.1.13 An unramified map $\mathcal{C} \to \mathcal{D}$ between tame stacky curves is representable.

Proof. Let $U \to \mathcal{D}$ be an étale cover of $\mathcal{D}$ by a scheme U, then $\mathcal{C} \times_{\mathcal{D}} U \to U$ is also unramified, so we may assume that $\mathcal{D}$ is a scheme. Let $[V/G] \to \mathcal{C}$ be as in the local form of Proposition 1.1.7 around any point. The map $[V/G] \to \mathcal{C} \to \mathcal{D}$ is unramified and factors through the coarse space V/G as $\mathcal{D}$ is a scheme. Since ramification indices are multiplicative in compositions, the map $[V/G] \to V/G$ is unramified. This means that G acts freely on V by Lemma 1.1.12, hence $[V/G] = V/G$. It follows that the coarse space map $\mathcal{C} \to C$ is étale locally an isomorphism, so $\mathcal{C}$ is a scheme. ◯

Lemma 1.1.14 Let $f\colon \mathcal{C} \to \mathcal{D}$ be an unramified map of tame stacky curves that induces an isomorphism of coarse spaces $C \simeq D$; then f is an isomorphism.

Proof. Since being an isomorphism can be checked étale locally, we can assume $\mathcal{D} = [V/G]$ and $D = V/G$ for a curve V and finite group G by Proposition 1.1.7. Since f is unramified, it is representable by Lemma 1.1.13, so $V' := \mathcal{C} \times_{\mathcal{D}} V$ is a scheme. As $V' \to \mathcal{C}$ is a finite étale morphism, V' is also a curve. By definition there are non-empty open subschemes $X \subset \mathcal{C}$ and $Y \subset \mathcal{D}$ and we can take their intersection $X \cap Y \subset C \simeq D$ in the coarse spaces. Now $X \cap Y$ is a non-empty open subscheme of both $\mathcal{C}$ and $\mathcal{D}$ and f restricts to an isomorphism on this open subscheme. It follows that the morphism $V' \to V$ between regular curves is birational and a bijection on points, hence an isomorphism. Consequently f is an isomorphism. ◯

Definition 1.1.15 Let $\mathcal{C}$ be a stacky curve. A **Weil divisor** D on $\mathcal{C}$ is a finite formal sum $\sum_Z n_Z Z$ of reduced closed substacks Z of $\mathcal{C}$ of codimension 1. If all the coefficients $n_Z \geq 0$, we say D is **effective**.

The reduced closed substacks of $\mathcal{C}$ of codimension 1 are in one to one correspondence with the reduced closed subschemes of the coarse space C, hence they are in one to one correspondence with the closed points of both $\mathcal{C}$ and C. When p is a stacky point, the associated closed substack is precisely the residual gerbe $\mathcal{G}_p$ of p. This is the motivation for the following definition.

Definition 1.1.16 Let p be a stacky point of order e_p on a stacky curve. We define $\frac{1}{e_p}p$ to be the Weil divisor $\mathcal{G}_p$. This lets us write a Weil divisor as a formal sum of closed points with coefficients in $\mathbb{Q}$, namely we define $\sum_p \frac{n_p}{e_p} p \coloneqq \sum_{\mathcal{G}_p} n_p \mathcal{G}_p$.

Definition 1.1.17 Let $\mathcal{C}$ be a stacky curve. An **effective Cartier divisor** D on $\mathcal{C}$ is a non-zero map $D\colon \mathcal{C} \to [\mathbb{A}^1/\mathbb{G}_m]$ i.e. a line bundle $\mathcal{L}$ on $\mathcal{C}$ together with a non-zero section s of $\mathcal{L}$.

Note that one can similarly define a possibly non-effective Cartier divisor to be a map to $[\mathbb{P}^1/\mathbb{G}_m]$. This definition is more familiar than it might look on first glance, namely the isomorphism classes of maps into $[\mathbb{A}^1/\mathbb{G}_m]$ are nothing more than elements of $H^0(\mathcal{O}/\mathcal{O}^\times)$. Similarly maps into $[\mathbb{P}^1/\mathbb{G}_m]$ are parameterized by $H^0(\mathcal{K}^\times/\mathcal{O}^\times)$, where $\mathcal{K}$ is the sheaf of meromorphic functions.

Definition 1.1.18 For a closed substack $Z \subset \mathcal{C}$, we define the **ideal sheaf**

$$\mathcal{O}_{\mathcal{C}}(-Z) \subset \mathcal{O}_{\mathcal{C}}$$

on étale covers of $\mathcal{C}$ as follows. Let $f\colon U \to \mathcal{C}$ be an étale cover; then

$$\mathcal{O}_{\mathcal{C}}(-Z)|_U = \mathcal{O}_U(-Z \times_{\mathcal{C}} U) \subset \mathcal{O}_U.$$

Given an effective Weil divisor D, we can associate an ideal sheaf

$$\mathcal{O}_{\mathcal{C}}(-D) \coloneqq \bigotimes_p \mathcal{O}\left(-\frac{1}{e_p}p\right)^{\otimes n_p} \subset \mathcal{O}_{\mathcal{C}}$$

and the corresponding effective Cartier divisor $(\mathcal{O}_{\mathcal{C}}(D), s_D)$, where $\mathcal{O}_{\mathcal{C}}(D) := \mathcal{H}om(\mathcal{O}_{\mathcal{C}}(-D), \mathcal{O}_{\mathcal{C}})$, and s_D corresponds to the inclusion map $\mathcal{O}_{\mathcal{C}}(-D) \to \mathcal{O}_{\mathcal{C}}$. This process can be inverted by sending $(\mathcal{L}, s)$ to $\sum_p \frac{v_p(s)}{e_p} p$, where $v_p(s)$ is defined by considering the inclusion $\iota_p \colon \mathcal{G}_p \to \mathcal{C}$ and setting $v_p(s)$ to be one less than the length of $\iota_p^{-1}\mathcal{L}$ considered as an $\iota_p^{-1}\mathcal{O}_{\mathcal{C}}$-module via $\iota_p^{-1}s \colon \iota_p^{-1}\mathcal{O}_{\mathcal{C}} \to i^{-1}\mathcal{L}$. To see that these two operations are inverse to each other, we can pass to an étale cover, where it follows from the case of classical curves.

> **Definition 1.1.19** Let $f \colon \mathcal{C} \to \mathcal{D}$ be a non-constant map of stacky curves and D be an effective Cartier divisor on $\mathcal{D}$. We define the **pullback** f^*D of D to be the composition $\mathcal{C} \to \mathcal{D} \to \left[\mathbb{A}^1/\mathbb{G}_m\right]$.

The following proposition expresses the pullback of a divisor in terms of Weil divisors and ramification data.

> **Proposition 1.1.20** Let $f \colon \mathcal{C} \mapsto \mathcal{D}$ be a tamely ramified map of stacky curves and let q be a closed point of $\mathcal{D}$, with pre-images $\{p_i\} = f^{-1}(q)$. We have $f^*\mathcal{G}_q = \sum_{p_i} e_{p_i/q}\mathcal{G}_{p_i}$.

Proof. We first show the case where $\mathcal{C} = C$ is a scheme and f is étale. We then have $f^*\mathcal{G}_q := (\mathcal{O}(\mathcal{G}_q \times_{\mathcal{D}} C), s_{\mathcal{G}_q \times_{\mathcal{D}} C}) = \sum_i p_i$.

For the general case, we let $u \colon U \to \mathcal{D}$ be an étale neighborhood of q such that q has a unique preimage $\tilde{q}$ and let $V \to U \times_{\mathcal{D}} \mathcal{C}$ be an étale cover, so we have the following diagram.

$$\begin{tikzcd} V \arrow[d, "v"] & \\ U \times_{\mathcal{D}} \mathcal{C} \arrow[r, "g"] \arrow[d, "w"] & U \arrow[d, "u"] \\ \mathcal{C} \arrow[r, "f"] & \mathcal{D} \end{tikzcd}$$

We can now verify the equality by passing to the cover V, i.e. we have to show

$$v^*w^*f^*\mathcal{G}_q = v^*w^*\sum_{p_i} e_{p_i/q}\mathcal{G}_{p_i}.$$

Note that $v^*w^*f^*\mathcal{G}_q = v^*g^*u^*\mathcal{G}_q = (v\circ g)^*\tilde{q}$. Let r_{ij} be the preimages of the p_i under $(v \circ w)$, then by the first case $v^*w^*\sum_{p_i} e_{p_i/q}\mathcal{G}_{p_i} = \sum_{r_{ij}} e_{p_i/q} r_{ij}$. Note that the r_{ij} are exactly the preimages of $\tilde{q}$ under $(v \circ g)$ and $e_{p_i/q} = e_{r_{ij}/\tilde{q}}$. So we have reduced to the case of classical curves, which is [Liu02, Chapter 7, Exercise 2.3(b)].

We now go over the construction of root stacks, which should be viewed as "degree 1 covers" with specified ramification data. We will prove that all stacky curves are actually root stacks over their coarse space in Theorem 1.1.31. For a more general treatment of root stacks see [Cad07].

Definition 1.1.21 Let $\mathcal{C}$ be a stacky curve, p be a closed point and $e > 1$ be a natural number not divisible by the characteristic of k. Consider the Cartier divisor $(\mathcal{O}(\mathcal{G}_p), s_p)$ associated to p. The **root stack** $\sqrt[e]{p/\mathcal{C}}$ is defined as the fiber product of the diagram

$$\begin{tikzcd} \sqrt[e]{p/\mathcal{C}} \arrow[r] \arrow[d, "\rho"] & \left[\mathbb{A}^1_k/\mathbb{G}_m\right] \arrow[d, "\theta_e"] \\ \mathcal{C} \arrow[r, "(\mathcal{O}(\mathcal{G}_p), s_p)"] & \left[\mathbb{A}^1_k/\mathbb{G}_m\right], \end{tikzcd}$$

where the right arrow is induced by the e-th power maps on $\mathbb{A}^1$ and $\mathbb{G}_m$ and the bottom arrow is induced by p. The top map $\sqrt[e]{p/\mathcal{C}} \to \left[\mathbb{A}^1/\mathbb{G}_m\right]$ defines an effective Cartier divisor (T_p, s_p), which is called the tautological divisor. We refer to T_p as the tautological line bundle. The left arrow $\rho\colon \sqrt[e]{p/\mathcal{C}} \to \mathcal{C}$ is called the root morphism.

For a finite set of points $\underline{p} = (p_1, \dots p_n)$ and multiplicities $\underline{e} = (e_1, \dots e_n)$ we define the **iterated root stack**

$$\sqrt[\underline{e}]{\underline{p}/\mathcal{C}} \coloneqq \sqrt[e_1]{p_1/\mathcal{C}} \times_{\mathcal{C}} \sqrt[e_2]{p_2/\mathcal{C}} \times_{\mathcal{C}} \cdots \times_{\mathcal{C}} \sqrt[e_n]{p_n/\mathcal{C}},$$

which comes with tautological Cartier divisors $(\mathrm{T}_{p_i}, s_{p_i})$ for each i and an iterated root morphism $\sqrt[\underline{e}]{\underline{p}/\mathcal{C}} \to \mathcal{C}$.

Technically the root construction also allows us to root in non-reduced divisors;

however this might result in singular stacks, so we avoid it. We do note that rooting in $f \cdot D$ with degree fe is the same as rooting in D with degree e.

Since root stacks commute with pullback by construction, the following lemma explains the local structure of root stacks.

Lemma 1.1.22 Let $C = \mathsf{Spec}(A)$ be an affine curve and let $x \in A$ correspond to a point $p = (x)$. We have

$$\sqrt[e]{p/C} \simeq [\mathsf{Spec}(A[t]/(t^e - x))/\mu_e],$$

where μ_e acts by multiplication on the variable t.

Proof. Since $\mathcal{O}_C(p) \simeq \mathcal{O}_C$, the morphism $C \xrightarrow{p} [\mathbb{A}^1/\mathbb{G}_m]$ factors as $C \xrightarrow{x} \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$. First consider the diagram of Cartesian squares.

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "\theta_e"] \\
\mathbb{A}^1 \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d] \\
\mathsf{Spec}(k) \arrow[r] & B\mathbb{G}_m
\end{tikzcd}$$

We claim that $X := \mathbb{A}^1 \times_{[\mathbb{A}^1/\mathbb{G}_m]} [\mathbb{A}^1/\mathbb{G}_m] \simeq [\mathbb{A}^1/\mu_e]$. Indeed

$$X \simeq \mathsf{Spec}(k) \times_{B\mathbb{G}_m} [\mathbb{A}^1/\mathbb{G}_m] \simeq [\mathbb{A}^1/(\mathbf{ker}\,\theta_e : \mathbb{G}_m \to \mathbb{G}_m)] = [\mathbb{A}^1/\mu_e].$$

Now consider another commutative diagram of Cartesian squares (pictorially represented in Figure 1.1).

$$\begin{tikzcd}
\mathsf{Spec}(A[t]/(t^e - x)) \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] & \\
\sqrt[e]{p/C} \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mu_e]} \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d] \\
C \arrow[r] & \mathbb{A}^1 \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]}
\end{tikzcd}$$

The action of μ_e on $\mathbb{A}^1$ pulls back to an action on **Spec**$(A[t]/(t^e - x))$ and the result follows. ⬡

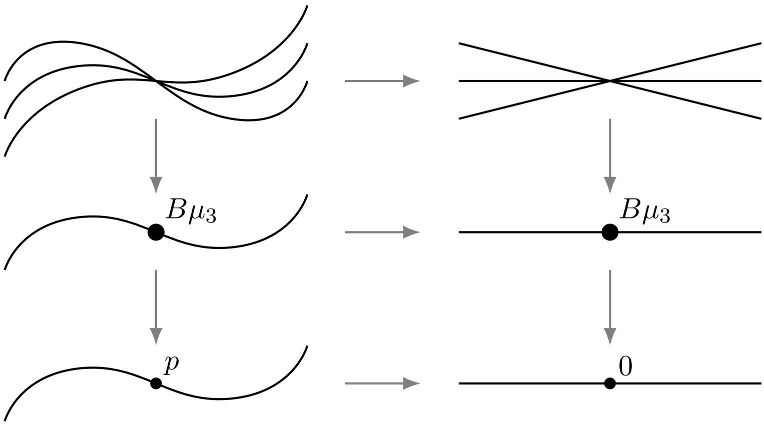


Figure 1.1: The local structure of a root stack with $e = 3$. The horizontal maps can be thought of as projections to the tangent space of the point p. We can see that the root stack sits in between the curve and a ramified cover of the curve. Note that the nodes are branch points, not singular points.

Remark 1.1.23 In the case that we are rooting in a non-stacky point this example shows that the Weil divisor associated to (T_p, s_p) is supported on the single closed point lying above p and has stabilizer μ_e. We abuse notation and the point lying above p will also be called p, so that the corresponding Weil divisor is denoted by $\frac{1}{e}p$. By construction we have $\pi^*(\mathcal{O}_C(p)) = \mathcal{O}_{\mathcal{C}}(\frac{1}{e}p)^e$, which motivates the "root" terminology.

Lemma 1.1.24 The root morphism $\rho\colon \sqrt[e]{p/\mathcal{C}} \to \mathcal{C}$ is an isomorphism away from the rooted point.

Proof. Away from the rooted point the section s_p does not vanish, therefore the restriction $\mathcal{C} - \{p\} \to [\mathbb{A}^1_k/\mathbb{G}_m]$ factors through the open substack **Spec**$(k) = [\mathbb{A}^1_k - \{0\}/\mathbb{G}_m] \subset [\mathbb{A}^1_k/\mathbb{G}_m]$ and the restricted map $\theta_e\colon [\mathbb{A}^1_k - \{0\}/\mathbb{G}_m] \to [\mathbb{A}^1_k - \{0\}/\mathbb{G}_m]$ is the identity. ⬡

For completeness we will prove two lemmas on the regularity and smoothness properties of branched coverings.

Lemma 1.1.25 Let A be a regular local ring with maximal ideal $\mathfrak{m}$ and $k = A/\mathfrak{m}$. Let $s \in A - \{0\}$ such that $A/(s)$ is regular and e be a positive integer that is invertible in A. Then $B := A[t]/(t^e - s)$ is regular.

Proof. We split up the proof into two cases. First assume $s \notin \mathfrak{m}$; then we claim that $A \to B$ is étale. Indeed $\Omega_{B/A} = \langle dt | et^{e-1}dt = 0\rangle$ and $et^{e-1} \in B^\times$ by assumption. Hence $\Omega_{B/A} = 0$.

Now assume that $s \in \mathfrak{m}$. We see that $\mathfrak{m} + (t)$ is the unique maximal ideal of B and we compute

$$\begin{aligned}\texttt{dim}_k \frac{\mathfrak{m} + (t)}{(\mathfrak{m} + (t))^2} &= \texttt{dim}_k \frac{\mathfrak{m} \oplus tA \oplus \cdots \oplus t^{e-1}A}{(\mathfrak{m}^2 + (s)) \oplus t\mathfrak{m} \oplus t^2 A \oplus \cdots \oplus t^{e-1}A} \\ &= \texttt{dim}_k \frac{\mathfrak{m}}{\mathfrak{m}^2 + (s)} + \texttt{dim}_k A/\mathfrak{m} < \texttt{dim}_k \frac{\mathfrak{m}}{\mathfrak{m}^2} + 1.\end{aligned}$$

The final inequality follows as $s \in \mathfrak{m}$, but $s \notin \mathfrak{m}^2$, because $A/(s)$ was assumed to be regular. It follows that we must have $\texttt{dim}_k \frac{\mathfrak{m}+(t)}{(\mathfrak{m}+(t))^2} = \texttt{dim}_k \frac{\mathfrak{m}}{\mathfrak{m}^2}$, so B is regular. ◯

Lemma 1.1.26 Let A be a smooth k-algebra, $s \in A$ be an irreducible element and $e \geq 2$ an integer invertible in k. Let $B = A[t]/(t^e - s)$. Then B is smooth over k if and only if $A/(s)$ is smooth over k.

Proof. First notice that B_t is smooth, since it is étale over A_s. Any prime ideal of B containing s also contains t so they are in bijection with the prime ideals of $B/(t, s) = A/(s)$. Let $\mathfrak{p} \subset B$ be such a prime and let $\mathfrak{q}$ be the corresponding prime in $A/(s)$. We may assume that A has a standard smooth presentation $A \simeq k[x_1, \ldots x_n]/(f_1, \ldots, f_c)$, and write $B = k[x_1, \ldots x_n, t]/(f_1, \ldots, f_c, h)$, where $h = t^e - s$.

If $A/(s)$ is smooth, then by [Stacks, Lemma 00TE], for any $\mathfrak{q}$, we can rename variables so that

$$\det \begin{bmatrix} \frac{\partial f_i}{\partial x_j} & \frac{\partial s}{\partial x_j} \end{bmatrix}_{1\leq i\leq c, 1\leq j\leq c+1}$$

does not map to an element of $\mathfrak{q}$. It then follows that

$$\det \begin{bmatrix} \frac{\partial f_i}{\partial x_j} & \frac{\partial t^e - s}{\partial x_j} \end{bmatrix}_{1\leq i\leq c, 1\leq j\leq c+1}$$

does not map to $\mathfrak{p}$, so B is smooth at $\mathfrak{p}$ for all $\mathfrak{p}$. (Note that $\frac{\partial t^e - s}{\partial x_j} = \frac{\partial s}{\partial x_j}$, so the determinant does not have any t-terms.)

On the other hand assume that $A/(s)$ is not smooth. Then, again by [Stacks, Lemma 00TE], there is a prime $\mathfrak{q}$ such that for every relabeling of the x_i, the determinant

$$\det \begin{bmatrix} \frac{\partial f_i}{\partial x_j} & \frac{\partial s}{\partial x_j} \end{bmatrix}_{1\leq i\leq c, 1\leq j\leq c+1}$$

maps to an element of $\mathfrak{q}$. It follows that if we want a relabeling on the level of B we need to include t. Now consider

$$\det \begin{bmatrix} \frac{\partial f_i}{\partial x_j} & \frac{\partial t^e - s}{\partial x_j} \\ \frac{\partial f_i}{\partial t} & \frac{\partial t^e - s}{\partial t} \end{bmatrix}_{1\leq i,j\leq c} = et^{e-1} \det \begin{bmatrix} \frac{\partial f_i}{\partial x_j} \end{bmatrix}_{1\leq i,j\leq c},$$

where we use $\frac{\partial f_i}{\partial t} = 0$ and $\frac{\partial t^e - s}{\partial t} = et^{e-1}$. So we see also for relabelings containing t, the determinant lands in $\mathfrak{q}$. It follows that B is not smooth at $\mathfrak{q}$. ◯

Proposition 1.1.27 Let $\mathcal{C}$ be a stacky curve, p be a closed point and $e > 1$ be a natural number that is not divisible by the characteristic of k. The root stack $\sqrt[e]{p/\mathcal{C}}$ is a stacky curve. Moreover, $\sqrt[e]{p/\mathcal{C}}$ is smooth over k if and only if $\mathcal{C}$ and $\mathcal{G}_p$ are smooth over k.

Proof. The only non-trivial facts are that $\sqrt[e]{p/\mathcal{C}}$ is a DM-stack and that $\sqrt[e]{p/\mathcal{C}}$ is regular. By Proposition 1.1.7 and Lemma 1.1.22, we can cover $\mathcal{C}$ by affine curves $\mathsf{Spec}(A) \to \mathcal{C}$ such that $\mathsf{Spec}(A) \times_{\mathcal{C}} \sqrt[e]{p/\mathcal{C}} \simeq [\mathsf{Spec}(B)/\mu_e]$, where $B = A[t]/(t^e - s)$ and $s \in A$ is a section corresponding to a reduced point. Since s is assumed to be reduced, B is regular by Lemma 1.1.25 and it follows that $\sqrt[e]{p/\mathcal{C}}$ is a regular DM stack. The smoothness statement is immediate from Lemma 1.1.26. ◯

This proposition shows that root stacks naturally give rise to regular, but non-smooth stacky curves, since over an imperfect base we can have closed points of a smooth curve that are not smooth themselves.

Example 1.1.28 Let $k = \mathbb{F}_p(t)$ and consider the curve $\mathbb{A}^1_k = \mathsf{Spec}(k[x])$, with the point $(-x^p - t)$. Then $\sqrt[e]{p/\mathbb{A}^1}$ is the curve

$$[\mathsf{Spec}(k[x,y]/(x^p + y^e + t))/\mu_e],$$

so it is singular at the point $y = 0, x = t^{1/p}$ by [Zar47, Example 3].

Proposition 1.1.29 Let C be a curve and let $\underline{p}$ be a set of closed points together with a set of multiplicities $\underline{e}$ and consider the root stack $\mathcal{X} := \sqrt[\underline{e}]{\underline{p}/C}$. The root morphism $\mathcal{X} \to C$ is the coarse space morphism.

Proof. Let $\pi\colon \mathcal{X} \to X$ be the coarse space morphism. By the universal property of the coarse space, $\mathcal{X} \to C$ factors through a map $X \to C$. We can check that this is an isomorphism Zariski-locally. Take an affine open $\mathsf{Spec}(A) = U \subset C$ containing a single $p \in \underline{p}$. By Lemma 1.1.22, we have $\mathcal{X} \times_C U = [\mathsf{Spec}(A[t]/(t^e - s))/\mu_e]$ and

$$X \times_C U = \mathsf{Spec}(A[t]/(t^e - s)^{\mu_e}) = \mathsf{Spec}(A) = U.$$

Since C can be covered by affine opens of this type, we conclude that $X \to C$ is an isomorphism. ◯

Proposition 1.1.30 Let $\mathcal{C}$ be a stacky curve, let p be a closed point on $\mathcal{C}$ and let e be a natural number that is not divisible by the characteristic of k. The root morphism $\mathcal{X} = \sqrt[e]{p/\mathcal{C}} \to \mathcal{C}$ is ramified above p with degree e and it is universal (terminal) with respect to this property.

Proof. The ramification at p can be computed using Lemmas 1.1.12 and 1.1.22. Let $f\colon \mathcal{X} \to \mathcal{C}$ be a map of stacky curves and q be a point of $\mathcal{X}$ ramified with degree

e above $p \in \mathcal{C}$; then $(\mathcal{O}(\frac{1}{e_q}q), s_q)$ defines a map to $[\mathbb{A}^1_k/\mathbb{G}_m]$ and by Proposition 1.1.20 $f^*(\mathcal{O}_{\mathcal{C}}(\frac{1}{e_p}p), s_p) = (\mathcal{O}_{\mathcal{X}}(\frac{1}{e_q}q)^{\otimes e_q}, s_q^{e_q})$. Hence f factors through $\mathcal{X} \to \mathcal{C}$ by the universal property of the fiber product. ⬠

Theorem 1.1.31 Let $\mathcal{C}$ be a tame stacky curve with coarse space $\pi\colon \mathcal{C} \to C$ and let R_π be the ramification locus. Identifying the ramification locus with the branch locus, we have that $\mathcal{C}$ is canonically isomorphic to $\sqrt[e_\pi]{R_\pi/C}$.

Proof. Note that by the tameness assumption the integers $e \in \underline{e_\pi}$ are not divisible by the characteristic of k. By the universal property of root stacks it follows that π factors via a map $\mathcal{C} \to \sqrt[e_\pi]{R_\pi/C}$. This map is unramified and induces an isomorphism of coarse spaces. By Lemma 1.1.14 it is an isomorphism. ⬠

One immediate consequence of this important structure result is a strengthening of Proposition 1.1.7 for tame stacky curves.

Corollary 1.1.32 Let $\mathcal{C}$ be a tame stacky curve with coarse space map $\pi\colon \mathcal{C} \to C$ and let p be a point of $\mathcal{C}$ with automorphism group of order e. Then there exists an open neighbourhood $V \subset C$ containing $\pi(p)$ and a curve U with a μ_e-action such that $V \simeq U/\mu_e$, fitting into a Cartesian square.

$$\begin{tikzcd} {[U/\mu_e]} \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] \\ V \arrow[r, hook] & C \end{tikzcd}$$

We also obtain a somewhat mysterious characterization of fixed points of finite group actions on curves in positive characteristic.

Corollary 1.1.33 Let G be a finite group of order not divisible by the characteristic of k acting on a smooth curve C. Then for any fixed point x, the residue field $\kappa(x)$ is separable over k.

Remark 1.1.34 In [VZ22], the authors define a **separably rooted** smooth stacky curve to be a smooth stacky curve such that the residual gerbes of stacky points admit an l-point for a separable extension l/k. By Theorem 1.1.31 and Proposition 1.1.27, it follows that all smooth tame stacky curves are separably rooted.

The root stack description also defines a canonical isomorphism from the residual gerbe of a stacky point to $B\mu_e$.

Lemma 1.1.35 Consider the following commutative diagram.

$$\begin{tikzcd} \mathcal{G}_p \arrow[r] \arrow[d] & \sqrt[e]{p/C} \arrow[r] \arrow[d] & \left[\mathbb{A}^1/\mathbb{G}_m\right] \arrow[r] \arrow[d] & B\mathbb{G}_m \arrow[d] \\ \mathsf{Spec}(\kappa(p)) \arrow[r, "p"] & C \arrow[r] & \left[\mathbb{A}^1/\mathbb{G}_m\right] \arrow[r] & B\mathbb{G}_m \end{tikzcd}$$

The outer square is a 2-Cartesian diagram. As a consequence, the residual gerbe $\mathcal{G}_p$ is naturally isomorphic to $B\mu_e$, where $B\mu_e$ is considered as the kernel of the map $\theta_e \colon B\mathbb{G}_m \to B\mathbb{G}_m$ i.e. the map $B\mu_e \to B\mathbb{G}_m$ is induced by the inclusion $\mu_e \to \mathbb{G}_m$.

Proof. By the universal property of the 2-fiber product, we get a morphism $\mathcal{G}_p \to (B\mu_e)_{\kappa(p)} = B\mathbb{G}_m \times_{B\mathbb{G}_m} \mathsf{Spec}(\kappa(p))$. On the other hand, the inclusion $B\mu_e \to B\mathbb{G}_m$ factors through $\left[\mathbb{A}^1/\mathbb{G}_m\right]$, so again by the universal property of 2-fiber products we get a morphism $(B\mu_e)_{\kappa(p)} \to \sqrt[e]{p/C}$. The image of this morphism is precisely p and since $B\mu_e$ is reduced it follows that it factors through $\mathcal{G}_p$. Summarizing we get a factorisation $\mathcal{G}_p \to (B\mu_e)_{\kappa(p)} \to \mathcal{G}_p \to \sqrt[e]{p/C} \to (B\mathbb{G}_m)_{\kappa(p)}$, showing that the natural morphism $\mathcal{G}_p \to B\mu_e$ is an isomorphism. ◇

We end this section with a technical definition that will be used when we want to reduce to the case of a stacky curve with a single stacky point.

Definition 1.1.36 Let $\mathcal{C}$ be a stacky curve. A **coarsening** $f \colon \mathcal{C} \to \mathcal{C}'$ is a map to a stacky curve $\mathcal{C}'$ inducing an isomorphism on coarse spaces.

Proposition 1.1.37 Let $\pi\colon \mathcal{C} \to \mathcal{C}'$ be a coarsening of tame stacky curves. Then $\mathcal{C}$ is canonically isomorphic to $\sqrt[e_\pi]{R_\pi/\mathcal{C}'}$.

Proof. This follows immediately by applying Theorem 1.1.31 to $\mathcal{C}$ and $\mathcal{C}'$. ◯

Example 1.1.38 let $\mathcal{C}$ be a stacky curve with coarse space $\pi\colon \mathcal{C} \to C$ and ramification divisor $R_\pi = \sum_{i=1}^n e_i p_i$. Set $\mathcal{C}_0 = C$ and $\mathcal{C}_i = \sqrt[e_i]{p_i/\mathcal{C}_{i-1}}$. Then $\mathcal{C}_n = \mathcal{C}$ and the maps $r_i\colon \mathcal{C}_i \to \mathcal{C}_{i-1}$ are all coarsenings such that $\pi = r_1 \circ \cdots \circ r_{n-1} \circ r_n$.

1.2 Sheaves on stacky curves

In this section we will develop the basic theory of coherent sheaves on stacky curves. We start by giving technical results relating sheaves on a stacky curve to sheaves on its coarse space. We then describe the discrete data of coherent sheaves on a stacky curve and explain some of their computational properties. We classify the invertible bundles in terms of invertible sheaves on the coarse space, and we describe torsion sheaves in terms of cyclic quiver representations. We then compute the Grothendieck group of a stacky curve by showing that a coherent sheaf has a torsion filtration and that a locally free sheaf has a filtration by invertible sheaves. We end with a computation of the canonical sheaf of a stacky curve.

The functors π_* and π^*

We begin by giving an equivalent characterization of the tameness condition for a stacky curve in terms of coherent sheaves.

Proposition 1.2.1 ([AOV08, Theorem 3.2]) Let $\mathcal{C}$ be a stacky curve with coarse space map $\pi\colon \mathcal{C} \to C$; then $\mathcal{C}$ is tame if and only if the pushforward on the categories of quasi-coherent sheaves $\pi_*\colon \mathfrak{QCoh}(\mathcal{C}) \to \mathfrak{QCoh}(C)$ is exact.

Note that the forward implication, which is most relevant for us, is already in [AV02, Lemma 2.3.4].

Proposition 1.2.2 ([AV02, Lemma 2.3.4]) Let $\mathcal{C}$ be a tame stacky curve with coarse space morphism $\pi\colon \mathcal{C} \to C$. The functor π_* restricts to a functor of coherent sheaves $\mathfrak{Coh}(\mathcal{C}) \to \mathfrak{Coh}(C)$ and to a functor of vector bundles $\mathfrak{Vect}(\mathcal{C}) \to \mathfrak{Vect}(C)$.

Proposition 1.2.3 Let $\mathcal{C}$ be a tame stacky curve. The functor $\pi^*\colon \mathfrak{Coh}(C) \to \mathfrak{Coh}(\mathcal{C})$ is exact.

Proof. The map $\theta_e\colon [\mathbb{A}^1_k/\mathbb{G}_m] \to [\mathbb{A}^1_k/\mathbb{G}_m]$ is faithfully flat, so by Theorem 1.1.31, the map π is also faithfully flat. ⬡

The formal properties of the pushforward π_* are essential for our applications to coherent sheaves, so from this point onward all our stacky curves will be assumed to be tame unless stated otherwise.

Proposition 1.2.4 Let $\pi\colon \mathcal{C} \to C$ be a stacky curve and $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{C}$. Then the following statements hold.

1. The natural map $\mathcal{O}_C \to \pi_*\mathcal{O}_{\mathcal{C}}$ is an isomorphism.
2. The natural map $\mathbf{Hom}_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, \pi^*\pi_*\mathcal{F}) \to \mathbf{Hom}_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, \mathcal{F})$ is an isomorphism.
3. There is a natural isomorphism $\mathbf{Hom}_C(\mathcal{O}_C, \pi_*\mathcal{F}) \to \mathbf{Hom}_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, \mathcal{F})$ and as a consequence $H^i(\mathcal{C}, \mathcal{F}) \simeq H^i(C, \pi_*\mathcal{F})$.

Proof. For the first part let $U \to C$ be étale; then $U \times_C \mathcal{C} \to U$ is a coarse space morphism by Theorem 1.1.6, so any morphism $U \times_C \mathcal{C} \to \mathbb{A}^1$ factors uniquely through a morphism $U \to \mathbb{A}^1$. For the second part, the inverse is given by sending a section $s\colon \mathcal{O}_{\mathcal{C}} \to \mathcal{F}$ to the composition

$$\mathcal{O}_{\mathcal{C}} \to \pi^*\mathcal{O}_C \to \pi^*\pi_*\mathcal{O}_{\mathcal{C}} \to \pi^*\pi_*\mathcal{F}.$$

For the final part, we can compose a series of natural isomorphisms.

$$\mathbf{Hom}_C(\mathcal{O}_C, \pi_*\mathcal{F}) \to \mathbf{Hom}_C(\pi_*\mathcal{O}_{\mathcal{C}}, \pi_*\mathcal{F}) \to \mathbf{Hom}_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, \pi^*\pi_*\mathcal{F}) \to \mathbf{Hom}_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, \mathcal{F})$$

By [Nir09, Lemma 1.10], the functor π_* sends injective sheaves to flasque sheaves, so we may apply [Stacks, Lemma 015M] to conclude that

$$\mathsf{R}\,\mathsf{Hom}_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}},\,\_\,) = \mathsf{R}\,\mathsf{Hom}_{C}(\mathcal{O}_C, \mathsf{R}\pi_*(\,\_\,)) = \mathsf{R}\,\mathsf{Hom}_{C}(\mathcal{O}_C, \pi_*(\,\_\,)).$$

◯

The optimistic interpretation of this theorem is that it is easy to compute sheaf cohomology on stacky curves; in fact it is just as easy as computing sheaf cohomology on classical curves. The pessimistic interpretation is that sheaf cohomology does not help us understand anything about the stacky structure of either the curve or the sheaves. However, the above theorem is very specific to the structure sheaf $\mathcal{O}_{\mathcal{C}}$, so there is no analogue for Ext groups. In other words Ext groups do see the stacky structure. Therefore, we will phrase our results in terms of Ext groups whenever possible.

Using the local form for stacky curves, we can make the functors π_* and π^* very concrete.

Lemma 1.2.5 Let V be a curve together with the action of a finite group G, such that $[V/G]$ is a stacky curve. View a coherent sheaf on $[V/G]$ as a G-equivariant sheaf $\mathcal{F}$ on V. Then $\pi_*\mathcal{F} = \mathcal{F}^G$ is the G-invariant part of $\mathcal{F}$. If F is a coherent sheaf on V/G then π^*F is the pullback to V together with the trivial G-action.

Proof. This follows from the definitions. ◯

Using the local form for stacky curves we get the following corollary.

Corollary 1.2.6 Let $\pi\colon \mathcal{C} \to C$ be a stacky curve and let F be a coherent sheaf on C. Then the canonical morphism $F \to \pi_*\pi^*F$ is an isomorphism.

Discrete Invariants

Classically, coherent sheaves on curves contain two pieces of discrete data: the rank and the degree. These discrete data uniquely determine a connected component of the moduli stack of coherent sheaves. This reflects the fact that the Hilbert polynomial of a sheaf on a curve is given by a linear polynomial, whose coefficients are determined by the rank and degree, and the Hilbert polynomial uniquely identifies a connected component.

For stacky curves, the situation is more subtle. Even though our Hilbert polynomials are still linear, they no longer identify a unique connected component of the moduli space. To remedy this we we have to introduce more discrete invariants. It turns out that for different applications it is convenient to consider different (but equivalent) ways to package these discrete invariants.

Definition 1.2.7 Let $\mathcal{C}$ be a tame stacky curve and $\mathcal{F}$ be a coherent sheaf on $\mathcal{C}$. Let p be a stacky point with multiplicity e_p and $\iota_p \colon \mathcal{G}_p \simeq B\mu_{e_p} \to \mathcal{C}$ be the inclusion of the residual gerbe at p, where the isomorphism is the canonical one from Lemma 1.1.35. The coherent sheaf $\iota_p^*\mathcal{F}$ on $B\mu_{e_p}$ corresponds to a $\mathbb{Z}/e_p\mathbb{Z}$-graded vector space, so $\iota_p^*\mathcal{F} \simeq \bigoplus_{i\in\mathbb{Z}/e_p\mathbb{Z}} k(i)^{m_{p,i}}$, where $k(i)$ is the vector space k in grade i. The numbers $m_{p,i} = m_{p,i}(\mathcal{F})$ are called the **multiplicities** of $\mathcal{F}$ at p. We take the convention that $0 \leq i \leq e_p - 1$ and define the **multiplicity vector** of $\mathcal{F}$ at p by

$$m_p(\mathcal{F}) = m_p \coloneqq (m_{p,0}, \cdots, m_{p,e_p-1}).$$

Finally the collection of all the multiplicity vectors m_p for every stacky point p is called the **multiplicities** of $\mathcal{F}$ denoted by $\underline{m} = \underline{m}(\mathcal{F})$.

We define the **twisted degrees** of $\mathcal{F}$ to be $d_{p,i} = d_{p,i}(\mathcal{F}) =:= \mathbf{deg}\, \pi_*\mathcal{F} \otimes \mathcal{O}_{\mathcal{C}}(\frac{i}{e_p}p)$. We write $d_p(\mathcal{F}) = d_p \coloneqq (d_{p,0}, \ldots, d_{p,e_p-1})$ for the twisted degrees at p and finally $\underline{d} = \underline{d}(\mathcal{F})$ for the collection of all twisted degrees.

Example 1.2.8 The tautological line bundle $\mathrm{T}_p = \mathcal{O}_{\mathcal{C}}(\frac{1}{e}p)$ on $\sqrt[e]{p/C}$ has multiplicity vector $m_p = (0, 1, 0, \ldots, 0)$.

Proof. The pullback of the tautological line bundle corresponds to the composition

$$B\mu_e \to \sqrt[e]{p/C} \to \left[\mathbb{A}^1/\mathbb{G}_m\right] \to B\mathbb{G}_m,$$

which is the inclusion map by Lemma 1.1.35. ◇

Since pullback commutes with taking tensor products and $k(i) \otimes k(1) = k(i+1)$, we see that tensoring with the tautological sheaf at p acts as a (cyclical) shift operator on the multiplicities at p.

Example 1.2.9 Let $\mathcal{C} := \sqrt[e]{p/C}$ and let F be a coherent sheaf on C, then $\pi^* F$ has multiplicity vector $m_p = (n, 0, \ldots, 0)$, where n is the rank of $F\mid_p$.

Proof. We have a commutative diagram.

$$\begin{tikzcd}
\mathcal{G}_p \arrow[r, "i"] \arrow[d, "\phi"] & \mathcal{C} \arrow[d, "\pi"] \\
\mathsf{Spec}(k) \arrow[r, "\bar{i}"] & C
\end{tikzcd}$$

So we have $i_* \pi^* F = \phi^* \bar{i}^* F$, so $i_* \pi^* F$ lies completely in grade 0. ⬠

The above example actually classifies the coherent sheaves with "trivial" multiplicities.

Proposition 1.2.10 Let $\mathcal{C}$ be a stacky curve and $\mathcal{F}$ be a coherent sheaf on $\mathcal{C}$ such that $m_p = (n, 0, \ldots, 0)$ for every stacky point p. Then the canonical morphism $\pi^* \pi_* \mathcal{F} \to \mathcal{F}$ is an isomorphism.

Proof. Consider the local form of Corollary 1.1.32 around a stacky point p,

$$\begin{tikzcd}
{[V/\mu_e]} \arrow[r, "f"] \arrow[d, "\pi'"] & \mathcal{C} \arrow[d, "\pi"] \\
V/\mu_e \arrow[r, "g"] & C
\end{tikzcd}$$

where μ_e is the stabilizer of p. By [Nir09, Proposition 1.5] we have $f^* \pi^* \pi_* \mathcal{F} = \pi'^* g^* \pi_* \mathcal{F} = \pi'^* \pi'_* f^* \mathcal{F}$, so we can check that the canonical isomorphism is an isomorphism locally. View $\mathcal{F}$ as a μ_e-equivariant sheaf on V, so that $\mathcal{F} \simeq \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} \mathcal{F}_i$

decomposes into graded pieces. Then $\pi^*\pi_*\mathcal{F} = \mathcal{F}_0$, so we have to show that $\mathcal{F}_i = 0$ for $i \neq 0$. We have a Cartesian square.

$$\begin{tikzcd} \mathsf{Spec}(k) \arrow[r, "p"] \arrow[d] & V \arrow[d] \\ B\mu_e \arrow[r, "i"] & {[V/G]} \end{tikzcd}$$

Showing that $i^*\mathcal{F}$ is the same as the fiber of $\mathcal{F}$ at p together with the μ_e action on this fiber. Since $i^*\mathcal{F}$ is a trivial representation it follows that $\mathcal{F}_i = 0$ for $i \neq 0$. ⬠

Corollary 1.2.11 Let $\mathcal{C}$ be a stacky curve and let $\mathcal{L}$ be a line bundle on $\mathcal{C}$. For each stacky point p, let e_p be the order and let a_p be the unique number such that $m_{p,a_p}(\mathcal{L}) \neq 0$. We have $\mathcal{L} \simeq \pi^* L \otimes \bigotimes_p \mathcal{O}(\frac{1}{e_p}p)^{\otimes a_p}$ for a unique (up to isomorphism) line bundle L on C.

Proof. We can apply the Proposition 1.2.10 to $\mathcal{L} \otimes \bigotimes_p \mathcal{O}(\frac{1}{e_p}p)^{\otimes -a_p}$ to see

$$\pi^*\pi_*\left(\mathcal{L} \otimes \bigotimes_p \mathcal{O}\left(\frac{1}{e_p}p\right)^{\otimes -a_p}\right) = \mathcal{L} \otimes \bigotimes_p \mathcal{O}\left(\frac{1}{e_p}p\right)^{\otimes -a_p}.$$

Now set $L := \pi_*\left(\mathcal{L} \otimes \bigotimes_p \mathcal{O}(\frac{1}{e_p}p)^{\otimes -a_p}\right)$ to get

$$\pi^* L \otimes \bigotimes_p \mathcal{O}\left(\frac{1}{e_p}p\right)^{\otimes a_p} = \mathcal{L}.$$

Finally, if $\pi^* L \otimes \bigotimes_p \mathcal{O}(\frac{1}{e_p}p)^{\otimes a_p} \simeq \pi^* L' \otimes \bigotimes_p \mathcal{O}(\frac{1}{e_p}p)^{\otimes a_p}$, then $\pi^* L \simeq \pi^* L'$, so $L = \pi_*\pi^* L \simeq \pi_*\pi^* L' = L'$. ⬠

Corollary 1.2.12 Let $\pi\colon \mathcal{C} \to C$ be a stacky curve with stacky points p_i of order e_i for $1 \leq i \leq n$. Denote by $\mathtt{Pic}_{\mathcal{C}}$ the Picard group of $\mathcal{C}$, i.e. the set of line bundles over $\mathcal{C}$ up to isomorphism. We have an isomorphism of abelian groups

$$\mathtt{Pic}_C[x_1, \dots, x_n]/(e_1x_1 - \mathcal{O}_C(p_1), \dots, e_nx_n - \mathcal{O}_C(p_n)) \simeq \mathtt{Pic}_{\mathcal{C}},$$

given by $L \mapsto \pi^* L$ and $x_i \mapsto \mathcal{O}_{\mathcal{C}}(\frac{1}{e_i} p_i)$.

For completeness we also rephrase Corollary 1.2.11 in terms of Weil divisors.

Corollary 1.2.13 Let $\pi \colon \mathcal{C} \to C$ be a stacky curve and $p \in \mathcal{C}$ be a stacky point of order e. For $m \in \mathbb{Z}$ we have $\pi^*(mp) = \frac{em}{e} p$ and $\pi_*(\frac{m}{e} p) = \lfloor \frac{m}{e} \rfloor p$, where $\lfloor x \rfloor$ is the floor of x, i.e. the largest integer n such that $n \leq x$.

Proof. Consider the division with remainder $m = ae + b$. Then $\mathcal{O}_{\mathcal{C}}(\frac{m}{e} p) = \pi^* \mathcal{O}_C(ap) \otimes \mathcal{O}_{\mathcal{C}}(\frac{b}{e} p)$ and it follows that $\pi_* \mathcal{O}_{\mathcal{C}}(\frac{m}{e} p) = \mathcal{O}_C(ap) \otimes \pi_* \mathcal{O}_{\mathcal{C}}(\frac{b}{e} p) = \mathcal{O}_C(ap)$. ⬠

Another consequence is that we can compute the twisted degrees of line bundles.

Corollary 1.2.14 Let $\mathcal{C} := \sqrt[e]{p/C}$. Let $\mathcal{L} = \pi^* L \otimes \mathcal{O}_{\mathcal{C}}(\frac{i}{e} p)$, with $0 \leq i \leq e-1$ and $d = \deg L$. Then $\mathcal{L}$ has twisted degrees

$$d_{p,j} = \begin{cases} d & j < e - i \\ d + 1 & j \geq e - i \end{cases} .$$

Proposition 1.2.15 Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be a short exact sequence of locally free sheaves on a stacky curve then $\underline{m}(\mathcal{E}) + \underline{m}(\mathcal{G}) = \underline{m}(\mathcal{F})$.

Proof. This is immediate as the pullback functor to the residual gerbe $\mathcal{G}_p$ is exact on locally free sheaves. ⬠

The above proposition is false for general coherent sheaves. Consider for example a short exact sequence of the form $0 \to \mathcal{O}(-\mathcal{G}_p) \to \mathcal{O}_{\mathcal{C}} \to \mathcal{T} \to 0$. Then pulling back to $\mathcal{G}_p$ we get the short exact sequence $k(e-1) \to k(0) \to \iota_p^* \mathcal{T} \to 0$. The first arrow must be the zero map, so we get $m_p(\mathcal{T}) = m_p(\mathcal{O}_{\mathcal{C}})$.

Proposition 1.2.16 Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be a short exact sequence of coherent sheaves on a stacky curve; then $\underline{d}(\mathcal{E}) + \underline{d}(\mathcal{G}) = \underline{d}(\mathcal{F})$.

Proof. This is immediate as tensoring with $\mathcal{O}_{\mathcal{C}}(\frac{i}{e_p}p)$ is exact, π_* is exact and **deg** is additive in short exact sequences of coherent sheaves on C. ◇

Locally free sheaves

Having classified line bundles on stacky curves, we now show that every torsion-free sheaf is a vector bundle i.e. locally free, and that vector bundles are iterated extensions of line bundles, as in the case of classical curves. Note that "locally free" should always be interpreted in the étale topology. For a stacky point p there is no Zariski neighborhood U of p such that $\mathcal{O}_{\mathcal{C}}|_U \simeq \mathcal{O}_{\mathcal{C}}(\frac{1}{e}p)|_U$, since they are not isomorphic after pulling back to $\mathcal{G}_p$.

> **Definition 1.2.17** Let $\mathcal{C}$ be a stacky curve and $\mathcal{E}$ be a coherent sheaf on $\mathcal{C}$. We define the **torsion subsheaf** $\mathcal{E}_{\text{tor}} \subset \mathcal{E}$ to be the maximal subsheaf of $\mathcal{E}$ that is torsion. We say that $\mathcal{E}$ is **torsion-free** if $\mathcal{E}_{\text{tor}} = 0$.

Classically torsion-free sheaves on curves are locally free, and the same is true for stacky curves.

> **Lemma 1.2.18** Let $\mathcal{C}$ be a stacky curve and $\mathcal{E}$ be a torsion-free sheaf on $\mathcal{C}$; then $\mathcal{E}$ is locally free.

Proof. By Proposition 1.1.7 there is an étale cover $f\colon U \to \mathcal{C}$ of $\mathcal{C}$ by a classical curve. Then $f^*\mathcal{E}$ is a torsion-free sheaf on a (possibly disconnected) classical (regular) curve U, thus locally free. It follows that $\mathcal{E}$ is locally free. ◇

> **Corollary 1.2.19** Let $\mathcal{C}$ be a stacky curve and $\mathcal{E}$ be a coherent sheaf on $\mathcal{C}$. We have a short exact sequence
>
> $$0 \to \mathcal{E}_{\text{tor}} \to \mathcal{E} \to \mathcal{F} \to 0,$$
>
> where $\mathcal{E}_{\text{tor}}$ is the torsion subsheaf of $\mathcal{E}$ and $\mathcal{F}$ is locally free.

Proof. Let $q\colon \mathcal{E} \to \mathcal{E}/\mathcal{E}_{\text{tor}} =: \mathcal{F}$ be the quotient map and let $\mathcal{F}_{\text{tor}}$ be the torsion subsheaf of $\mathcal{F}$, then $q^{-1}(\mathcal{F}_{\text{tor}}) + \mathcal{E}_{\text{tor}}$ is torsion, so by maximality of $\mathcal{E}_{\text{tor}}$ we have that $q^{-1}(\mathcal{F}_{\text{tor}}) \subset \mathcal{E}_{\text{tor}}$, so $\mathcal{F}_{\text{tor}} = 0$. ⬠

Lemma 1.2.20 Let $\mathcal{F}$ be a locally free sheaf of rank r on a stacky curve $\mathcal{C}$. There exists a sequence of surjective maps

$$\mathcal{F} = \mathcal{E}_0 \twoheadrightarrow \mathcal{E}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{E}_r = 0,$$

such that $\mathcal{E}_i$ is locally free and $\mathcal{L}_i := \texttt{ker}\,(\mathcal{E}_i \to \mathcal{E}_{i+1})$ is an invertible sheaf. Moreover $\underline{m}(\mathcal{F}) = \sum_{i=1}^{r} \underline{m}(\mathcal{L}_i)$.

Proof. Let $D \gg 0$ be a positive divisor of large degree on the coarse space C; then $\pi_*\mathcal{F}(D)$ admits a non-zero section, so by Proposition 1.2.4 we get a non-zero section $\mathcal{O}_{\mathcal{C}} \to \mathcal{F} \otimes \pi^*\mathcal{O}_C(D)$. This gives rise to a subsheaf $\pi^*\mathcal{O}_C(-D) \to \mathcal{F}$. Let $\mathcal{T}$ be the torsion sheaf of $\mathcal{F}/\pi^*\mathcal{O}_C(-D)$ and take the saturation

$$\mathcal{L}_0 = \overline{\pi^*\mathcal{O}_{\mathcal{C}}(-D)} := \texttt{ker}\,(\mathcal{F} \to (\mathcal{F}/\pi^*\mathcal{O}_C(-D))/\mathcal{T})$$

and set $\mathcal{E}_1 := (\mathcal{F}/\pi^*\mathcal{O}_C(-D))/\mathcal{T}$. The saturation of an invertible sheaf is again an invertible sheaf and $\mathcal{E}_1$ is locally free by construction. The vector bundle $\mathcal{E}_1$ has rank $r-1$, so iteratively applying this construction finishes the proof. ⬠

We can now relate the twisted degrees and multiplicities for locally free sheaves.

Corollary 1.2.21 Let $\mathcal{F}$ be a vector bundle; then $m_{p,i} = d_{p,i} - d_{p,i-1}$ for $1 \leq i < e_p$ and $m_{p,0} = \texttt{rank}\,\mathcal{F} - \sum_{i=1}^{e-1} m_{p,i}$.

This corollary shows that for a vector bundle $\mathcal{F}$, we can recover $(\underline{d}(F), \texttt{rank}(\mathcal{F}))$ from $(\underline{m}(\mathcal{F}), \textbf{deg}\,\pi_*\mathcal{F})$ and visa versa. Twisted degrees have better computational behavior with respect to short exact sequences of coherent sheaves, so we will usually prefer them for general arguments. On the other hand the multiplicities are more geometric, so they are usually what we think about for intuition.

Torsion sheaves

Now that we have a basic understanding of vector bundles, we move on to torsion sheaves. We start by giving a very explicit description of torsion sheaves in terms of quiver representations.

Definition 1.2.22 A **k-quiver representation of the cyclic quiver with e vertices** is a $\mathbb{Z}/e\mathbb{Z}$-graded k-vector space V together with a degree 1 endomorphism u. More explicitly, it is a collection of k-vector spaces V_i and linear maps $u_i\colon V_i \to V_{i+1}$ indexed by $i \in \mathbb{Z}/e\mathbb{Z}$. See Figure 1.2 for a pictorial interpretation. A morphism of quiver representations $(V_i, u_i) \to (W_i, w_i)$ is a collection of linear maps $\phi_i\colon V_i \to W_i$, such that $\phi_i \circ u_i = w_i \circ \phi_i$.

A quiver representation is said to be **nilpotent** if u is nilpotent.

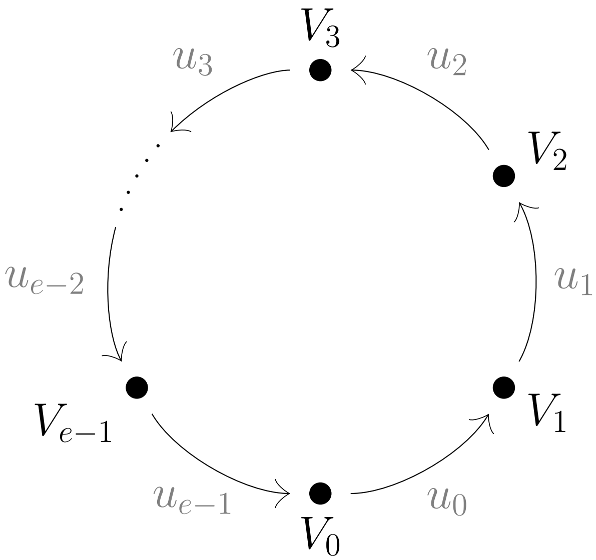


Figure 1.2: A quiver representation of the cyclic quiver

A form of the following proposition was stated in the language of parabolic torsion sheaves in [Hei04, Lemma 3.6].

Proposition 1.2.23 Let $\mathcal{C}$ be a stacky curve and p a stacky point of order e. There is an equivalence of categories between the category of torsion sheaves supported on p and the category of nilpotent $\kappa(p)$-quiver representations of the cyclic quiver with e vertices.

Proof. Take a local form $[V/\mu_e]$ around the point p, such that the μ_e-action fixes a unique point $q \in V$. Now the category of torsion sheaves on $\mathcal{C}$ supported on p is equivalent to the category of μ_e-equivariant torsion sheaves on V supported on q.

Let $R := \mathcal{O}_{V,q}$ be the local ring at q with maximal ideal $\mathfrak{m}$; then there is an induced μ_e-action on $\mathsf{Spec}(R)$, which induces a $\mathbb{Z}/e\mathbb{Z}$-grading $R = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} R_i$. Since the μ_e-action fixes $\mathfrak{m}$, it is a homogeneous ideal of R for this grading. It follows that there is a homogeneous uniformizer $u \in \mathfrak{m}$, which using the conventions of Lemma 1.1.35 has degree 1. Now the category of μ_e-equivariant torsion sheaves supported on q is naturally equivalent to the category of $\mathbb{Z}/e\mathbb{Z}$-graded torsion modules over R.

Next we notice that the torsion modules over R are precisely the R-module M such that $u^n M = 0$ for some n. This means that the category of torsion R-modules is equivalent to the category of pairs (M, n), where M is an $R/\mathfrak{m}^n$-module such that $u^{n-1}M \neq 0$, together with the pair $(0, -\infty)$, and the morphisms are morphisms of R-modules after extending scalars. Moreover $R/\mathfrak{m}^n$ inherits the grading of R and this equivalence respects gradings. Denote by $\hat{R}$ the completion of R in $\mathfrak{m}$. Since $R/\mathfrak{m}^n = \hat{R}/\mathfrak{m}^n$, it follows that the category of graded torsion modules over R is equivalent to the category of graded torsion modules over $\hat{R}$. Note that $\hat{R}$ has a natural $\mathbb{Z}/e\mathbb{Z}$-grading, since we complete in a homogeneous ideal.

Finally, by the Cohen structure theorem, we know that $\hat{R} \simeq \kappa(p)[[X]]$, where we can choose X to map to u. Then the induced grading on $\kappa(p)[[X]]$ is the one where X^i is homogeneous of degree i. A $\kappa(p)[[X]]$-module is torsion if and only if it is finite dimensional as a $\kappa(p)$-vector space. It follows that the category of graded torsion $\kappa(p)[[X]]$-modules is equivalent to the category pairs (V, u), where V is a $\mathbb{Z}/e\mathbb{Z}$-graded $\kappa(p)$-vector space and $u \colon V \to V$ is a degree 1 map. ◯

If we view non-stacky points as stacky points of order 1, we recover the fact that a torsion sheaf on a curve supported on single point corresponds to nilpotent representation of the Jordan quiver.

Example 1.2.24 Let $\mathcal{C}$ be a stacky curve and p be a stacky point of order e. For $1 \leq i \leq e$ define torsion sheaves $\mathcal{T}_i$ via the exact sequences

$$0 \to \mathcal{O}_{\mathcal{C}}(-\frac{i}{e}p) \to \mathcal{O}_{\mathcal{C}} \to \mathcal{T}_i \to 0.$$

On the level of $\kappa(p)[[X]]$-modules this exact sequence becomes

$$0 \to X^i\kappa(p)[[X]] \to \kappa(p)[[X]] \to \kappa(p)[[X]]/\langle X^i \rangle \to 0.$$

We can now see that $\mathcal{T}_i$ corresponds to the quiver representation

$$V_0 = V_1 = \cdots V_{i-1} = \kappa(q) \text{ and } V_i = \cdots = V_{e-1} = 0$$

with the identity maps if $i < e$. For $\mathcal{T}_e$, the map $V_{e-1} \to V_0$ is the zero map.

Remark 1.2.25 Chasing through all the definitions, we can see that for a torsion sheaf supported on a stacky point p corresponding to the quiver representation $(\sum_{i \in \mathbb{Z}/e\mathbb{Z}} V_i, u)$, we have $m_{p,i} = d_{p,i} = \texttt{dim}\, V_i$.

Lemma 1.2.26 Let $\mathcal{C}$ be a stacky curve with a stacky point p of order e. The irreducible torsion sheaves supported on p are all isomorphic to some $\mathcal{T}_i$, fitting into the exact sequence

$$0 \to \mathcal{O}(-\frac{i+1}{e}p) \to \mathcal{O}(-\frac{i}{e}p) \to \mathcal{T}_i \to 0,$$

for some $0 \leq i \leq e-1$.

Proof. Let $\mathcal{T}$ be an irreducible torsion sheaf supported on p and consider the associated quiver representation (V, u). Since u is nilpotent it must send some nonzero vector $v_i \in V_i \subset V$ to 0. Then we have a subrepresentation $\mathcal{T}_i = (k \cdot v_i, 0)$, which by irreducibility must be an isomorphism. Such a quiver representation corresponds to the module $u^i\kappa(p)[[u]]/u^{i+1}\kappa(p)[[u]]$, which fits into the exact sequence as stated. ⬡

The Grothendieck Group

We will now combine the results of the previous sections to give a description of the Grothendieck group $\mathsf{K}_0(\mathcal{C})$ of coherent sheaves on a tame stacky curve.

Proposition 1.2.27 Let $\mathcal{C}$ be a tame stacky curve with stacky points $\underline{p}$. The maps $\mathtt{det}_C \circ \pi_*$, $\mathtt{rank}$ and $\underline{m}_{p,i}$ define an injection of Abelian groups

$$\mathsf{K}_0(\mathcal{C}) \hookrightarrow \mathtt{Pic}_C \oplus \mathbb{Z} \oplus \bigoplus_{p \in \underline{p}} \mathsf{K}_0(\mathcal{G}_p).$$

This induces a non-canonical isomorphism

$$\mathsf{K}_0(\mathcal{C}) \simeq \mathtt{Pic}_C \oplus \mathbb{Z} \oplus \bigoplus_{p \in \underline{p}} \mathbb{Z}^{e_p - 1}.$$

Proof. Since $\mathsf{K}_0(\mathcal{C})$ is generated by the classes of vector bundles, we get natural maps $\iota_p^* \colon \mathsf{K}_0(\mathcal{C}) \to \mathsf{K}_0(\mathcal{G}_p) \simeq \mathbb{Z}^{e_p}$ of for each $p \in \underline{p}$. Note that these maps applied to a vector bundle are precisely the multiplicity vectors. The natural maps $\mathtt{rank}_p \colon \mathsf{K}_0(\mathcal{G}_p) \to \mathsf{K}_0(\mathsf{Spec}(\kappa(p))) \simeq \mathbb{Z}$ simply add the multiplicities together, which for a vector bundle is nothing more than the rank. Clearly the maps ι_p^* are surjective and the image of $\oplus \iota_p^* \colon \mathsf{K}_0(\mathcal{C}) \to \bigoplus_{p \in \underline{p}} \mathsf{K}_0(\mathcal{G}_p)$ is the sublattice where all the $\mathtt{rank}_p$ agree. This sublattice can then be identified with $\mathbb{Z} \oplus \bigoplus_{p \in \underline{p}} \mathbb{Z}^{e_p - 1}$. The kernel of $\oplus \iota_p^*$ is generated by classes of the form

$$\begin{aligned}[]
[\pi^* L_1 \otimes \bigotimes \mathcal{O}_{\mathcal{C}}(\frac{i}{e_p} p)] - [\pi^* L_2 \otimes \bigotimes \mathcal{O}_{\mathcal{C}}(\frac{i}{e_p})] &= [\pi^* L_1] - [\pi^* L_2] \\
&= [\pi^*(L_1 \otimes L_2^\vee)] - [\mathcal{O}_{\mathcal{C}}].
\end{aligned}$$

It follows that we have a natural exact sequence

$$0 \to \mathtt{Pic}_C \to \mathsf{K}_0(\mathcal{C}) \to \bigoplus_{p \in \underline{p}} \mathsf{K}_0(\mathcal{G}_p),$$

where $\mathtt{Pic}_C \to \mathsf{K}_0(\mathcal{C})$ is given by $L \mapsto [\pi^* L] - [\mathcal{O}_{\mathcal{C}}]$. Finally the map $\mathsf{K}_0(\mathcal{C}) \to \mathtt{Pic}_C$ given by $\mathtt{det} \circ \pi_*$ is a retraction of $\mathtt{Pic}_C \to \mathsf{K}_0(\mathcal{C})$, so the result follows. ⬠

We now define the determinant on the level of Grothendieck groups.

Definition 1.2.28 We define the determinant $\mathtt{det} = \mathtt{det}_{\mathcal{C}}$ to be the composition

$$\mathsf{K}_0(\mathcal{C}) \to \mathtt{Pic}_C \oplus \bigoplus_{p \in \underline{p}} \mathsf{K}_0(\mathcal{G}_p) \to \mathtt{Pic}_{\mathcal{C}},$$

where the first map is the projection and the second map is given by

$$(L, \underline{m}) \mapsto \pi^* L \otimes \bigotimes_{\substack{p \in \underline{p} \\ 0 < i < e_p}} \mathcal{O}_{\mathcal{C}}(\frac{i}{e_p} p)^{\otimes m_{p,i}}.$$

Note that this map is indeed the unique group homomorphism $\mathsf{K}_0(\mathcal{C}) \to \mathtt{Pic}_{\mathcal{C}}$ which sends the class of a line bundle $[\mathcal{L}] \mapsto \mathcal{L}$.

Definition 1.2.29 Consider the composition

$$\begin{aligned} \mathsf{K}_0(\mathcal{C}) &\xrightarrow{\mathsf{det}} \mathtt{Pic}_{\mathcal{C}} \\ &\xrightarrow{\sim} \mathtt{Pic}_C[x_1, \dots, x_n]/(e_1 x_1 - \mathcal{O}_C(p_1), \dots, e_n x_n - \mathcal{O}_C(p_n)) \\ &\to \mathbb{Z}[d_1/e_1, \dots, d_n/e_n] \subset \mathbb{Q}, \end{aligned}$$

where the last arrow is induced by the degree map on $\mathtt{Pic}_C$ and d_i is the degree of the residue field of p_i. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{C}$. We define the degree $\mathbf{deg}\, \mathcal{F}$ to be the image under this composition.

Note that we allow fractional degrees, but the denominators of the fractions are bounded in terms of the orders of the stacky points. This definition is chosen so that the pullback from the coarse space $\pi^* \colon \mathsf{K}_0(C) \to \mathsf{K}_0(\mathcal{C})$ is degree preserving and in fact it is uniquely defined by this property.

The rank of a vector bundle and its pushforward to the coarse space agree. The same is not true for the degree, but the difference can be expressed in terms of the multiplicities.

Proposition 1.2.30 Let $\mathcal{E}$ be a locally free sheaf with multiplicities $\underline{m}$. We have $\mathbf{deg}\, \mathcal{E} = \mathbf{deg}(\pi_* \mathcal{E}) + \sum_p \frac{1}{e_p} \sum_{i=0}^{e_p - 1} i m_{p,i}$.

Proof. Both sides of the equation are additive in short exact sequences, so we can reduce to the case of invertible sheaves by Lemma 1.2.20. The case of line bundles follows from Corollary 1.2.11. ∎

The cotangent sheaf

We end this section with a discussion on the cotangent sheaf of a stacky curve. We will start from a very abstract definition and then show that it can be very concretely described. The abstract definition is not necessary for any of our results, so it should only be viewed as motivation for the concrete description which we will actually use.

Definition 1.2.31 Following [Ill71], let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of DM-stacks, we define the **cotangent sheaf** $\Omega_{\mathcal{X}/\mathcal{Y}}$ on the étale site of $\mathcal{X}$ as follows. Let $\mathcal{I}$ be the kernel of the multiplication morphism $\mathcal{O}_{\mathcal{X}} \otimes_{f^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$, then $\Omega_{\mathcal{X}/\mathcal{Y}} := \mathcal{I}/\mathcal{I}^2$.

We have two canonical exact sequences.

Lemma 1.2.32 ([Ill71, (1.1.2.12)] and [Ill71, (1.1.2.13)]) Let $\mathcal{X} \xrightarrow{f} \mathcal{Y} \to \mathcal{Z}$ be morphisms of DM-stacks. We have a short exact sequence

$$f^*\Omega_{\mathcal{Y}/\mathcal{Z}} \to \Omega_{\mathcal{X}/\mathcal{Z}} \to \Omega_{\mathcal{X}/\mathcal{Y}} \to 0.$$

If $\mathcal{O}_{\mathcal{X}}$ is a locally free $f^{-1}\mathcal{O}_{\mathcal{Y}}$-module then we can extend the sequence to

$$0 \to f^*\Omega_{\mathcal{Y}/\mathcal{Z}} \to \Omega_{\mathcal{X}/\mathcal{Z}} \to \Omega_{\mathcal{X}/\mathcal{Y}} \to 0.$$

Lemma 1.2.33 ([Ill71, (1.1.6.2)]) Let $i\colon \mathcal{Y} \to \mathcal{X}$ be a closed immersion of DM-stacks with ideal sheaf $\mathcal{J}$. We have a canonical short exact sequence

$$\mathcal{J}/\mathcal{J}^2 \to i^*\Omega_{\mathcal{X}} \to \Omega_{\mathcal{Y}} \to 0.$$

Using the canonical short exact sequences we can compute the cotangent sheaf of a stacky curve.

Theorem 1.2.34 Let $\pi\colon \mathcal{C} \to C$ be a **smooth** tame stacky curve with stacky points $\underline{p}$. We have

$$\Omega_{\mathcal{C}} \simeq \pi^*\Omega_C \otimes \bigotimes_{p\in\underline{p}} \mathcal{O}(\frac{1}{e_p}p)^{\otimes e_p-1}.$$

Proof. Let $u\colon U \to \mathcal{C}$ be an étale atlas for $\mathcal{C}$; then U is a smooth (possibly disconnected) curve and Ω_U is a line bundle. From Lemma 1.2.32, we get an exact sequence $0 \to u^*\Omega_{\mathcal{C}} \to \Omega_U \to \Omega_{U/\mathcal{C}} = 0$, so $\Omega_{\mathcal{C}}$ is a line bundle.

Now apply Lemma 1.2.32 to the coarse space map $\pi\colon \mathcal{C} \to C$ to get a short exact sequence

$$\pi^*\Omega_C \to \Omega_{\mathcal{C}} \to \Omega_{\mathcal{C}/C} \to 0.$$

The sequence extends to the left since $\pi^*\Omega_C \to \Omega_{\mathcal{C}}$ is a map of line bundles that is generically an isomorphism, hence injective. Since $\Omega_{\mathcal{C}/C}$ is supported on the stacky points, it follows from Corollary 1.2.11 that $\Omega_{\mathcal{C}} = \pi^*\Omega_C \otimes \bigotimes_{p\in\underline{p}} \mathcal{O}_{\mathcal{C}}(\frac{1}{e_p}p)^{\otimes n_p}$ for some non-negative integers n_p.

To compute n_p, we can take a local form around p as follows.

$$\begin{tikzcd}
V \arrow[d, "\phi"] & \\
{[V/\mu_{e_p}]} \arrow[r, "g"] \arrow[d, "\pi'"] & \mathcal{C} \arrow[d, "\pi"] \\
V/\mu_{e_p} \arrow[r, "f"] & C
\end{tikzcd}$$

Let p also denote the preimage of p under g and let q be the unique point in V sitting above p. Then pulling back along g we get

$$\begin{aligned}
\Omega_{[V/\mu_{e_p}]} = g^*\Omega_{\mathcal{C}} &= g^*\pi^*\Omega_C \otimes \mathcal{O}_{[V/\mu_{e_p}]}(\frac{1}{e_p}p)^{\otimes n_p} \\
&= \pi'^*\Omega_{V/\mu_{e_p}} \otimes \mathcal{O}_{[V/\mu_{e_p}]}(\frac{1}{e_p}p)^{\otimes n_p}.
\end{aligned}$$

Pulling back once more along ϕ we see

$$\Omega_V = \phi^*\Omega_{[V/\mu_{e_p}]} = (\phi\circ\pi')^*\Omega_{V/\mu_{e_p}} \otimes \mathcal{O}_V(q)^{\otimes n_p}.$$

Now it follows from the ramification theory of classical curves that $n_p = e_p - 1$. ◯

To get a similar result for non-smooth curves one should work with the canonical sheaf instead, but we will not develop the theory of canonical sheaves for DM-stacks here.

1.3 Projective stacky curves

In this section we develop a theory of projective stacky curves analogous to the theory of classical projective curves. The main difference from the classical theory is that the polarization of a stacky curve is not given by a line bundle, but by a higher rank vector bundle called a generating sheaf, introduced in [OS03]. This generating sheaf is also used to define a notion of (semi)stability for vector bundles. Classically stability does not depend on the chosen polarization, this is very far from the case for stacky curves and different generating sheaves give many different notions of stability.

> **Definition 1.3.1** A **projective stacky curve** is a smooth tame stacky curve with a projective coarse space.

Note that we require projective stacky curves to be smooth and tame. This is probably not necessary for all the results in this section, but we will use these properties freely throughout.

Warning: The definition of a projective stack is more subtle, but for stacky curves this naive definition is good enough. See [Kre09] for the higher-dimensional case. One of the points is that for curves the existence of a generating sheaf is automatic.

Note that a proper smooth tame stacky curve is automatically projective. Indeed by Theorem 1.1.6, a stacky curve is proper if and only if its coarse space is, and a proper curve is projective.

> **Definition 1.3.2** Let $\mathcal{C}$ be a projective stacky curve. We define the Euler characteristic $\chi_{\mathcal{C}} := -\operatorname{\mathbf{deg}} \omega_{\mathcal{C}}$. We then define the **genus** $g_{\mathcal{C}}$ via $2 - 2g_{\mathcal{C}} = \chi_{\mathcal{C}}$.

Since the canonical bundle can have rational degree, the Euler characteristic and genus are not integers in general. This means, for example, that there is no cohomological description like $h^1(\mathcal{O}_C) = g_C$. One big motivation for this definition is that it satisfies an analogue of the Riemann-Hurwitz theorem. The following is an immediate corollary of Theorem 1.2.34.

Corollary 1.3.3 Let $\pi\colon \mathcal{C} \to C$ be a projective stacky curve with stacky points $\underline{p}$. We have

$$\chi_{\mathcal{C}} = \chi_C - \sum_{p \in \underline{p}} \frac{e_p - 1}{e_p} [\kappa(p) : k]$$

and

$$g_{\mathcal{C}} = g_C + \frac{1}{2} \sum_{p \in \underline{p}} \frac{e_p - 1}{e_p} [\kappa(p) : k].$$

Theorem 1.3.4 (Riemann-Hurwitz) Let $f\colon \mathcal{C} \to \mathcal{D}$ be a map of projective stacky curves tamely ramified at the points p_i with ramification index e_i. We have

$$f^* \omega_{\mathcal{D}} = \omega_{\mathcal{C}} \otimes \bigotimes_i \mathcal{O}(\mathcal{G}_{p_i})^{e_i - 1}.$$

Consequently,

$$\chi_{\mathcal{C}} = (\operatorname{\mathbf{deg}} f) \cdot \chi_{\mathcal{D}} - \sum_i (e_i - 1) \operatorname{\mathbf{deg}}(\mathcal{G}_{p_i}).$$

Proof. Let $\pi_C\colon \mathcal{C} \to C$ and $\pi_D\colon \mathcal{D} \to D$ be the coarse space morphisms and let $g\colon C \to D$ be the map induced by $\pi_D \circ f$. The result follows from an easy computation using Theorem 1.2.34 for $\pi_{\mathcal{C}}$ and $\pi_{\mathcal{D}}$ and the classical Riemann-Hurwitz theorem for g. ⬡

We give a short proof of the following well-known result to highlight the usefulness of the genus.

Proposition 1.3.5 Let $m \neq n$ by natural numbers not divisible by the characteristic of k; then the football space $\mathcal{F}(m,n)$ is not the quotient of a classical curve by a finite group.

Proof. Assume there is a classical curve C with an action of a finite group G such that $[C/G] \simeq \mathcal{F}(m,n)$. Then $C/G \simeq \mathbb{P}^1_k$, so C is projective. The map $C \to \mathcal{F}(m,n)$ is unramified, so we can apply Riemann-Hurwitz to see

$$\chi_C = |G|\chi_{\mathcal{F}(m,n)} = |G|(2 - (\frac{m-1}{m} + \frac{n-1}{n})) = |G|\frac{m+n}{mn}.$$

Since the right hand side is positive it follows that $\chi_C = 2$. Now write d for the greatest common divisor of m and n so that $m = da$ and $n = db$ for positive integers a and b. Since G contains subgroups of order m and n, namely the stabilizers of $0, \infty \in \mathcal{F}(m,n)$, we must have that dab divides $|G|$. Write $|G| = xdab$ so the equation $2 = |G|\frac{m+n}{mn}$ becomes $2 = x(a+b)$, which implies that $a = b = 1$, but this contradicts $m \neq n$. ⬠

We move on to proving Serre duality.

Theorem 1.3.6 (Serre Duality) Let $\mathcal{E}$ be a coherent sheaf on a projective stacky curve $\mathcal{C}$. For $i = 0, 1$, we have a natural isomorphism

$$\mathsf{Ext}^i(\mathcal{E}, \omega_{\mathcal{C}}) \simeq \mathsf{Ext}^{1-i}(\mathcal{O}_{\mathcal{C}}, \mathcal{E})^\vee.$$

Proof. Using Corollary 1.2.19 and Lemma 1.2.20 we can reduce to the case that $\mathcal{E}$ is a line bundle $\mathcal{L} \simeq \pi^* L \otimes \bigotimes_p \mathcal{O}(\frac{i_p}{e_p}p)$. Now we apply Serre duality on C to get

$$\begin{aligned}\mathsf{Ext}^i_{\mathcal{O}_{\mathcal{C}}}(\mathcal{L}, \omega_{\mathcal{C}}) \simeq \mathsf{Ext}^i_{\mathcal{O}_C}\left(\mathcal{O}_C, L^\vee \otimes \omega_C\right) &\simeq \mathsf{Ext}^{1-i}_{\mathcal{O}_C}(\mathcal{O}_C, L)^\vee \\ &\simeq \mathsf{Ext}^{1-i}_{\mathcal{O}_{\mathcal{C}}}(\mathcal{O}_{\mathcal{C}}, \mathcal{L})^\vee.\end{aligned}$$

The first isomorphism follows as

$$\pi_*(\mathcal{L}^\vee \otimes \omega_{\mathcal{C}}) = L^\vee \otimes \omega_C \otimes \pi_* \bigotimes_p \mathcal{O}(\frac{e_p - 1 - i_p}{e_p} p) = L^\vee \otimes \omega_C,$$

where we use the computation of $\omega_{\mathcal{C}}$ of Theorem 1.2.34. ⬠

Remark 1.3.7 Even though in general we have $\pi_*(\mathcal{F}^\vee) \neq (\pi_*\mathcal{F})^\vee$, the above proof shows that the Serre duals $\mathsf{SD}_{\mathcal{C}}(\mathcal{F}) := \mathcal{H}om_{\mathcal{C}}(\mathcal{F}, \omega_{\mathcal{C}})$ and $\mathsf{SD}_C(F) := \mathcal{H}om_C(F, \omega_C)$ do commute with π_*, i.e. $\pi_* \circ S_{\mathcal{C}} = S_C \circ \pi_*$.

We now state the naive Riemann-Roch theorem for a projective stacky curve [VZ22, Remark 5.5.12]. The reason for the terminology "naive" is that it does not involve any stacky structure of the line bundles nor the curve itself.

Proposition 1.3.8 (Naive Riemann-Roch) Let $\mathcal{C}$ be a projective stacky curve, with coarse space $\pi\colon \mathcal{C} \to C$. Let $\mathcal{L}$ be a line bundle on $\mathcal{C}$. Then

$$h^0(\mathcal{L}) - h^0(\mathcal{L}^\vee \otimes \omega_{\mathcal{C}}) = \mathbf{deg}\, \pi_*\mathcal{L} + 1 - g_C.$$

Proof. By the remark above, we have $h^0(\mathcal{L}) - h^0(\mathcal{L}^\vee \otimes \omega_{\mathcal{C}}) = h^0(\pi_*\mathcal{L}) - h^0((\pi_*\mathcal{L})^\vee \otimes \omega_C) = \mathbf{deg}\, \pi_*\mathcal{L} + 1 - g_C$. ⬠

Generating sheaves

We will now spend some time defining generating sheaves, which will serve as a polarization of a projective curve. Generating sheaves were first introduced in [OS03] in order to embed Quot schemes for tame DM-stacks into Quot schemes over their coarse spaces.

Definition 1.3.9 Following [OS03], let $\pi\colon \mathcal{C} \to C$ be a stacky curve and $\mathcal{E}$ be a locally free sheaf on $\mathcal{C}$. We define the functor $F_{\mathcal{E}}\colon \mathbf{Coh}\, \mathcal{C} \to \mathbf{Coh}\, C$ as

$$F_{\mathcal{E}}(\mathcal{F}) := \pi_*\mathcal{H}om(\mathcal{E}, \mathcal{F}) = \pi_*(\mathcal{F} \otimes \mathcal{E}^\vee)$$

and in the other direction $G_{\mathcal{E}}\colon \mathbf{Coh}\, C \to \mathbf{Coh}\, \mathcal{C}$ by

$$G_{\mathcal{E}}(F) := \pi^*(F) \otimes \mathcal{E}.$$

The identity map $\pi_*(\mathcal{H}om(\mathcal{E}, \mathcal{F})) \to \pi_*(\mathcal{H}om(\mathcal{E}, \mathcal{F}))$ has a left adjoint

$$\pi^*\pi_*(\mathcal{H}om(\mathcal{E}, \mathcal{F})) \to \mathcal{H}om(\mathcal{E}, \mathcal{F}),$$

which has a left adjoint

$$\pi^*\pi_*(\mathcal{H}om(\mathcal{E},\mathcal{F}))\otimes\mathcal{E}\to\mathcal{F}.$$

We denote this left adjoint of the left adjoint by $\theta_{\mathcal{E}}(\mathcal{F})\colon G_{\mathcal{E}}\circ F_{\mathcal{E}}(\mathcal{F})\to\mathcal{F}$.

Definition 1.3.10 Let $\mathcal{E}$ be a locally free sheaf on a stacky curve $\mathcal{C}$. If $\theta_{\mathcal{E}}(\mathcal{F})$ is surjective, then $\mathcal{E}$ is called a **generator** for $\mathcal{F}$. If $\mathcal{E}$ is a generator for all coherent sheaves $\mathcal{F}$ on $\mathcal{C}$, then $\mathcal{E}$ is a **generating sheaf** for $\mathcal{C}$.

It is not so obvious how to verify if a sheaf is generating directly, but the following local condition is easy to check in practice.

Theorem 1.3.11 (Local condition for generation) Let $\mathcal{C}$ be a stacky curve with stacky points $\underline{p}$ and $\mathcal{E}$ be a locally free sheaf. Then $\mathcal{E}$ is a generating sheaf if and only if $m_{p,j}>0$ for every $p\in\underline{p}$ and $0\le j\le e_p-1$. In other words the graded vector spaces $\iota_p^*\mathcal{E}$ for $\iota_p\colon\mathcal{G}_p\hookrightarrow\mathcal{C}$ are supported in all grades for all $p\in\underline{p}$.

Proof. The condition is certainly necessary: to generate $\mathcal{O}_{\mathcal{C}}(\frac{i}{e}p)$ we must have $m_{p,i}>0$. We will now prove that the condition is sufficient. As the surjectivity of $\theta_{\mathcal{E}}(\mathcal{F})$ can be checked locally, we may assume that $\mathcal{C}$ has a single stacky point p of order e.

Let $0\to\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3\to 0$ be a short exact sequence of coherent sheaves. We get a commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_1 \arrow[r] & \mathcal{F}_2 \arrow[r] & \mathcal{F}_3 \arrow[r] & 0 \\
0 \arrow[r] & G_{\mathcal{E}}\circ F_{\mathcal{E}}(\mathcal{F}_1) \arrow[r] \arrow[u, "\theta_{\mathcal{F}_1}"] & G_{\mathcal{E}}\circ F_{\mathcal{E}}(\mathcal{F}_2) \arrow[r] \arrow[u, "\theta_{\mathcal{F}_2}"] & G_{\mathcal{E}}\circ F_{\mathcal{E}}(\mathcal{F}_3) \arrow[r] \arrow[u, "\theta_{\mathcal{F}_3}"] & 0
\end{tikzcd}$$

By the snake lemma, we have that if $\theta_{\mathcal{E}}(\mathcal{F}_1)$ and $\theta_{\mathcal{E}}(\mathcal{F}_3)$ are surjective, so is $\theta_{\mathcal{E}}(\mathcal{F}_3)$, and when $\theta_{\mathcal{E}}(\mathcal{F}_2)$ is surjective, so is $\theta_{\mathcal{E}}(\mathcal{F}_3)$. By Lemma 1.2.20, Corollary 1.2.19 and the fact that every torsion sheaf admits a surjection by a vector bundle we only have to verify that $\mathcal{E}$ generates line bundles $\mathcal{L} \simeq \pi^* L \otimes \mathcal{O}_{\mathcal{C}}(\frac{j}{e}p)$. We can rewrite $\theta_{\mathcal{L}}$ as

$$\pi^*\pi_*\mathcal{H}om\left(\mathcal{E} \otimes \mathcal{O}\left(\frac{-j}{e}p\right), \mathcal{O}_{\mathcal{C}}\right) \otimes \pi^* L \otimes \mathcal{E} \to \pi^* L \otimes \mathcal{O}\left(\frac{j}{e}p\right).$$

Tensoring both sides by $\mathcal{L}^\vee$ and setting $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}(\frac{-j}{e}p)$, we get the morphism

$$\theta_{\mathcal{L}} \otimes \mathcal{L}^\vee : \pi^*\pi_*(\mathcal{H}om(\mathcal{E}', \mathcal{O}_{\mathcal{C}})) \otimes \mathcal{E}' \to \mathcal{O}_{\mathcal{C}},$$

which is precisely $\theta_{\mathcal{E}'}(\mathcal{O}_{\mathcal{C}})$. Since tensoring with a line bundle cyclically permutes the multiplicities $\mathcal{E}'$ also satisfies the local condition of generation. This means that we have reduced to the case $\mathcal{L} = \mathcal{O}_{\mathcal{C}}$.

Now we apply Lemma 1.2.20 to $\mathcal{E}'$ and get a chain of surjective maps $\mathcal{E}' = \mathcal{E}_0 \to \mathcal{E}_1 \to \ldots \to \mathcal{E}_r$. From the local condition for $\mathcal{E}'$ it follows that there exists some index i and a line bundle $\mathcal{J} = \mathbf{ker}(\mathcal{E}_i \to \mathcal{E}_{i+1})$ with $m_p(\mathcal{J}) = (1, 0, \ldots, 0)$, i.e. $\mathcal{J} \simeq \pi^* J$ for some line bundle J on C. Now we have a commutative diagram.

$$\begin{tikzcd}
\pi^*\pi_*\mathcal{J}^\vee \otimes \mathcal{J} = \mathcal{O}_{\mathcal{C}} \arrow[ddddr, bend left] & \\
\pi^*\pi_*\mathcal{E}_i^\vee \otimes \mathcal{J} \arrow[u, two heads] \arrow[d, hook] \arrow[dddr, bend left] & \\
\pi^*\pi_*\mathcal{E}_i^\vee \otimes \mathcal{E}_i \arrow[ddr, bend left] & \\
\pi^*\pi_*\mathcal{E}_i^\vee \otimes \mathcal{E}_0 \arrow[u, two heads] \arrow[d, hook] \arrow[dr] & \\
\pi^*\pi_*\mathcal{E}_0^\vee \otimes \mathcal{E}_0 \arrow[r] & \mathcal{O}_{\mathcal{C}}
\end{tikzcd}$$

The top diagonal arrow is an isomorphism and it follows that all of the other diagonal arrows below are surjective. ⬠

The general case of the above theorem is [OS08, Theorem 5.2]. However there it is claimed that for a stacky point $\zeta\colon \mathsf{Spec}(k) \to C$ with stabilizer G_ζ we have $\mathsf{Spec}(k) \times_C \mathcal{C} = BG_\zeta$. This is of course not true, since π is ramified above ζ. We do have $(\mathsf{Spec}(k) \times_C \mathcal{C})_{\mathsf{red}} = BG_\zeta$, which is enough to make their proofs work.

Definition 1.3.12 Let $\mathcal{C}$ be a stacky curve with stacky points $\underline{p}$, then

$$\mathcal{E}_{\mathsf{fav}} := \bigotimes_{p\in\underline{p}} \bigoplus_{j=0}^{e_p-1} \mathcal{O}(\frac{j}{e_p}p) \oplus \bigotimes_{p\in\underline{p}} \bigoplus_{j=0}^{e_p-1} \mathcal{O}(\frac{-j}{e_p}p)$$

is a generating sheaf, which we will call the **standard** (or favorite) generating sheaf for $\mathcal{C}$.

From the local condition of generation it is immediate that the standard generating sheaf is indeed a generating sheaf. The standard generating sheaf is definitely not very canonical, however it plays a very special role from a computational perspective. We will see that our formulas massively simplify whenever we apply them to the standard generating sheaf.

We now give a notion of degree that is relative to a locally free sheaf, which will usually be a generating sheaf. It is this degree that will show up in the stacky Riemann-Roch theorem.

Definition 1.3.13 Let $\mathcal{C}$ be a projective curve, $\mathcal{E}$ be a locally free sheaf and $\mathcal{F}$ be a coherent sheaf on $\mathcal{C}$. We define the **$\mathcal{E}$-degree** of $\mathcal{F}$ by

$$d_{\mathcal{E}}(\mathcal{F}) = \mathbf{deg}\, \pi_* \mathcal{H}om(\mathcal{E}, \mathcal{F}) - \mathtt{rank}\, \mathcal{F}\, \mathbf{deg}\, \pi_* \mathcal{H}om(\mathcal{E}, \mathcal{O}).$$

Note that the $\mathcal{E}$-degree is additive in short exact sequences in both entries. Moreover $d_{\mathcal{E}}(-) = d_{\mathcal{E}\otimes\pi^*L}$ for any line bundle L on the coarse space. It follows from Lemma 1.2.20 that the $\mathcal{E}$-degree only depends on the multiplicities of $\mathcal{E}$. We now give a notion of "weights", which is simply a repackaging of the multiplicities, that is useful for computations with $\mathcal{E}$-degrees.

Definition 1.3.14 Let $\mathcal{E}$ be a locally free sheaf with multiplicities $m_{p,j}$. We define the **weights** of $\mathcal{E}$ to be $w_{p,j} = w_{p,j}(\mathcal{E}) := \frac{\sum_{l=1}^{j} m_{p,l}(\mathcal{E})}{\mathtt{rank}\,\mathcal{E}}$, where j runs from 0 to $e_p - 1$.

Note that by construction $0 = w_{p,0} \leq w_{p,1} \leq \cdots \leq w_{p,e_p-1} \leq 1$. The inequalities are strict if and only if $\mathcal{E}$ is a generating sheaf.

Example 1.3.15 Let $\mathcal{E}_{\mathsf{fav}}$ be the standard generating sheaf, then $w_{p,i} = \frac{i}{e_p}$.

In fact we can find a locally free sheaf with arbitrary rational weights.

Example 1.3.16 Let $\mathcal{C}$ be a stacky curve and for each stacky point p, let $w_{p,i} = \frac{a_{p,i}}{d_p}$ be rational numbers with a common denominator d_p, such that the numerators satisfy

$$0 = a_{p,0} \leq a_{p,1} \leq \cdots \leq a_{p,e_p-1} \leq d_p.$$

Set $b_{p,i} = a_{p,i} - a_{p,i-1}$ for $0 < i \leq e_p - 1$ and $b_{p,0} = d_p - a_{p,e_p-1}$. The locally free sheaf $\mathcal{E} := \bigotimes_p \bigoplus_{i=0}^{e_p-1} \mathcal{O}_{\mathcal{C}}(\frac{i}{e_p}p)^{\oplus b_{p,i}}$ has weights $w_{p,i}$.

The weights allow us to give a formula for the $\mathcal{E}$-degree in terms of invariants defined on the coarse space and multiplicities.

Proposition 1.3.17 Let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves on $\mathcal{C}$. We have

$$d_{\mathcal{E}}(\mathcal{F}) = \mathtt{rank}\, \mathcal{E}\, \mathbf{deg}(\pi_* \mathcal{F}) + \mathtt{rank}\, \mathcal{E} \sum_p \sum_{i=0}^{e-1} m_{p,i}(\mathcal{F}) w_{p,i}(\mathcal{E}).$$

In particular, when $\mathcal{E} = \mathcal{E}_{\mathsf{fav}}$ is the standard generating sheaf, $\frac{d_{\mathcal{E}}(\mathcal{F})}{\mathtt{rank}\, \mathcal{E}} = \mathbf{deg}\, \mathcal{F}$, for any coherent sheaf $\mathcal{F}$.

Proof. Note that all the terms of the formula are additive in short exact sequences of vector bundles, for both $\mathcal{E}$ and $\mathcal{F}$, so we may assume $\mathcal{E}$ and $\mathcal{F}$ are line bundles. The case of line bundles is immediate from the description in Corollary 1.2.11. For the case of the standard generating sheaf the result follows from Proposition 1.2.30 and the fact that the formula $d_{\mathcal{E}}(\mathcal{F}) = \mathtt{rank}\, \mathcal{E}\, \mathbf{deg}\, \mathcal{F}$ is additive in all short exact sequences for $\mathcal{F}$. ⬡

Now we state a more refined version of the Riemann-Roch theorem.

Theorem 1.3.18 (Stacky Riemann-Roch) Let $\mathcal{C}$ be a projective stacky curve, $\mathcal{E}$ be a locally free sheaf and $\mathcal{F}$ be a coherent sheaf on $\mathcal{C}$. We have

$$\mathtt{ext}^0(\mathcal{E},\mathcal{F})-\mathtt{ext}^1(\mathcal{E},\mathcal{F}) = d_{\mathcal{E}}(\mathcal{F})+\mathtt{rank}(\mathcal{F})\big(\mathtt{ext}^0(\mathcal{E},\mathcal{O}_{\mathcal{C}})-\mathtt{ext}^1(\mathcal{E},\mathcal{O}_{\mathcal{C}})\big).$$

In particular when $\mathcal{E} = \mathcal{E}_{\mathsf{fav}}$, we have

$$\frac{\mathtt{ext}^0(\mathcal{E}_{\mathsf{fav}},\mathcal{F}) - \mathtt{ext}^1(\mathcal{E}_{\mathsf{fav}},\mathcal{F})}{\mathtt{rank}\,\mathcal{E}_{\mathsf{fav}}} = \mathbf{deg}\,\mathcal{F} + \mathtt{rank}(\mathcal{F})(1-g_{\mathcal{C}}).$$

We will give a proof that is analogous to the classical case, to explain the appearance of the different terms. A shorter way to prove the theorem would be to apply the classical Riemann-Roch theorem to $\pi_*\mathcal{H}om(\mathcal{E},\mathcal{F})$ and $\pi_*(\mathcal{E}^\vee)$ and combine the results.

Proof. Since everything is additive in short exact sequences, we may assume $\mathcal{F}$ is a line bundle. Assume $\mathcal{F} = \mathcal{O}_{\mathcal{C}}$, then $d_{\mathcal{E}}(\mathcal{O}_{\mathcal{C}}) = 0$, so the formula holds. Assume the formula holds for a line bundle $\mathcal{L}$ and we have a non-zero map $\mathcal{L} \to \mathcal{L}'$. Denote the cokernel, which is a torsion sheaf, by $\mathcal{T}$. From the additivity of $\mathcal{E}$-degrees we get $d_{\mathcal{E}}(\mathcal{L}') - d_{\mathcal{E}}(\mathcal{L}) = d_{\mathcal{E}}(\mathcal{T})$. We also get the long exact sequence

$$\begin{aligned}0 \to \mathtt{Ext}^0(\mathcal{E},\mathcal{L}) \to \mathtt{Ext}^0\big(\mathcal{E},\mathcal{L}'\big) \to \mathtt{Ext}^0(\mathcal{E},\mathcal{T}) \to \\ \to \mathtt{Ext}^1(\mathcal{E},\mathcal{L}) \to \mathtt{Ext}^1\big(\mathcal{E},\mathcal{L}'\big) \to \mathtt{Ext}^1(\mathcal{E},\mathcal{T}) = 0.\end{aligned}$$

The last Ext group is 0 because $\mathtt{Ext}^1(\mathcal{E},\mathcal{T}) = H^1(\pi_*(\mathcal{T}\otimes\mathcal{E}^\vee)) = 0$. Also $\mathtt{ext}^0(\mathcal{E},\mathcal{T}) = h^0(\pi_*(\mathcal{T}\otimes\mathcal{E}^\vee)) = d_{\mathcal{E}}(\mathcal{T})$, since $\mathcal{T}$ is torsion. Now taking the Euler characteristic of the long exact sequence, we see that the formula also holds for $\mathcal{L}'$. A completely analogous argument works when we have a non-zero map $\mathcal{L}' \to \mathcal{L}$.

Now any line bundle has the form $\mathcal{L} \simeq \mathcal{O}_{\mathcal{C}}(D)$ for some Weil-divisor D. Let D_+ be the positive part of D. We have a non-zero map $\mathcal{O}_{\mathcal{C}} \to \mathcal{O}(D_+)$ and a non-zero map $\mathcal{O}_{\mathcal{C}}(D) \to \mathcal{O}_{\mathcal{C}}(D_+)$ showing that the formula holds for $\mathcal{L}$.

Finally by Proposition 1.3.17, we have $d_{\mathcal{E}_{\mathsf{fav}}}(\mathcal{F}) = \mathtt{rank}(\mathcal{E}_{\mathsf{fav}})\,\mathbf{deg}\,\mathcal{F}$ and by the naive Riemann-Roch theorem

$$\mathtt{ext}^0(\mathcal{E}_{\mathsf{fav}}, \mathcal{O}_{\mathcal{C}}) - \mathtt{ext}^1(\mathcal{E}_{\mathsf{fav}}, \mathcal{O}_{\mathcal{C}}) = \mathbf{deg}(\pi_* \mathcal{E}_{\mathsf{fav}}^\vee) + \mathtt{rank}(\mathcal{E}_{\mathsf{fav}})(1 - g_C) =$$
$$- \mathtt{rank}(\mathcal{E}_{\mathsf{fav}}) \frac{1}{2} \left(\sum_{p_i} \frac{e_i - 1}{e_i} \right) + \mathtt{rank}(\mathcal{E}_{\mathsf{fav}})(1 - g_C) = \mathtt{rank}(\mathcal{E}_{\mathsf{fav}})(1 - g_{\mathcal{C}}).$$

Plugging these two computations into the general equation gives the result. ⬠

The numerical Grothendieck group

Many computations with coherent sheaves can be reduced to computations on the Grothendieck group. In fact some computations can be reduced to an even smaller group, called the numerical Grothendieck group, which is the group to which the Riemann-Roch theorem most naturally applies.

> **Definition 1.3.19** We define the **Euler pairing** of two coherent sheaves $\mathcal{E}, \mathcal{F}$ to be
> $$\langle \mathcal{E}, \mathcal{F} \rangle := \mathtt{ext}^0(\mathcal{E}, \mathcal{F}) - \mathtt{ext}^1(\mathcal{E}, \mathcal{F}).$$

The Euler pairing is additive in short exact sequences in both coordinates, so descends to give a bilinear form $\mathsf{K}_0(\mathcal{C}) \times \mathsf{K}_0(\mathcal{C}) \to \mathbb{Z}$.

> **Definition 1.3.20** Let R be the right radical of $\langle \_\,, \_ \rangle$, i.e. the kernel of the map $[\mathcal{F}] \mapsto \langle \_\,, \mathcal{F} \rangle$. We define the **numerical Grothendieck group** to be $\mathsf{K}_0^{\mathsf{num}}(\mathcal{C}) := \mathsf{K}_0(\mathcal{C})/R$. The elements of $\mathsf{K}_0^{\mathsf{num}}(\mathcal{C})$ are called **numerical invariants**.

By Serre duality
$$\langle \mathcal{E}, \mathcal{F} \rangle = -\langle \mathcal{F}, \mathcal{E} \otimes \omega_{\mathcal{C}} \rangle,$$
so the right radical is equal to the left radical. It follows that the Euler pairing descends to a pairing $\mathsf{K}_0^{\mathsf{num}}(\mathcal{C}) \times \mathsf{K}_0^{\mathsf{num}}(\mathcal{C}) \to \mathbb{Z}$. In fact $\mathsf{K}_0^{\mathsf{num}}(\mathcal{C})$ is the maximal quotient of $\mathsf{K}_0(\mathcal{C})$ such that the Euler pairing descends to a **non-degenerate** bilinear form.

Proposition 1.3.21 The right radical of $\langle \_ , \_ \rangle$ is given by $\mathtt{Pic}^0_C \hookrightarrow \mathsf{K}_0(\mathcal{C})$. As a consequence we have an injection of Abelian groups induced by $\mathbf{deg} \circ \pi_*$, $\mathbf{rank}$, and $m_{p,i}$

$$\mathsf{K}_0^{\mathsf{num}}(\mathcal{C}) \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \bigoplus_{p \in \underline{p}} \mathsf{K}_0(\mathcal{G}_p).$$

This induces a non-canonical isomorphism

$$\mathsf{K}_0^{\mathsf{num}}(\mathcal{C}) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \bigoplus_{p \in \underline{p}} \mathbb{Z}^{e_p - 1}.$$

Proof. To compute the right radical, we use Example 1.3.16, Proposition 1.3.17, and Theorem 1.3.18. The degree defines a short exact sequence $0 \to \mathtt{Pic}^0_C \to \mathtt{Pic}_C \to \mathbb{Z} \to 0$, so the result follows from Proposition 1.2.27. ⬠

Definition 1.3.22 We say that a numerical invariant $\boldsymbol{\alpha}$ is **positive** if $\mathbf{rank}\, \boldsymbol{\alpha} \geq 0$ and $m_{p,i}\boldsymbol{\alpha} \geq 0$ for each p and i. A numerical invariant is **generating** if the inequalities are strict.

Note that an invariant $\boldsymbol{\alpha}$ is positive if and only if there exists a vector bundle $\mathcal{F}$ such that $\boldsymbol{\alpha} = [\mathcal{F}]$ and an invariant is generating if and only if there exists a generating sheaf $\mathcal{E}$ such that $\boldsymbol{\alpha} = [\mathcal{E}]$.

Generalized Hilbert polynomials and stability conditions

We will now explain a way to define Hilbert polynomials for sheaves on stacky curves.

Definition 1.3.23 Let $\mathcal{C}$ be a projective stacky curve. We define a **polarization** of $\mathcal{C}$ to be a pair $(\mathcal{E}, \mathcal{O}_C(1))$, where $\mathcal{E}$ is a generating sheaf for $\mathcal{C}$ and $\mathcal{O}_C(1)$ is a polarizing line bundle on the coarse space C. For a coherent sheaf $\mathcal{F}$, we write $\mathcal{F}(m) := \mathcal{F} \otimes \pi^*\mathcal{O}_C(m)$.

In [FL21] the authors explain how a generating sheaf together with a polarization of the coarse space induces an embedding of the stacky curve into a twisted

Grassmanian stack. The twisted Grassmanians are simultaneous generalizations of weighted projective spaces and Grassmanians. This justifies calling the pair $(\mathcal{E}, \mathcal{O}_C(1))$ a polarization.

Definition 1.3.24 Let $\mathcal{C}$ be a projective stacky curve together with a polarization $(\mathcal{E}, \mathcal{O}_C(1))$. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{C}$. We define the $\mathcal{E}$**-Hilbert polynomial** of $\mathcal{F}$ to be

$$P_{\mathcal{E}}(\mathcal{F})(m) := \chi(\mathcal{H}om(\mathcal{E}, \mathcal{F} \otimes \pi^* \mathcal{O}_C(m))) = \langle \mathcal{E}, \mathcal{F}(m) \rangle.$$

We define the reduced $\mathcal{E}$-Hilbert polynomial $p_{\mathcal{E}}(\mathcal{F})$ to be $P_{\mathcal{E}}(\mathcal{F})$ divided by its leading coefficient.

Note that $P_{\mathcal{E}}(\mathcal{F})$ is the Hilbert polynomial of $F_{\mathcal{E}}(\mathcal{F})$, so it is a polynomial. More explicitly, using Theorem 1.3.18 it follows that

$$P_{\mathcal{E}}(\mathcal{F})(m) = \operatorname{rank}(\mathcal{F}) \operatorname{rank}(\mathcal{E}) \deg(\mathcal{O}_C(1)) \cdot m + d_{\mathcal{E}}(\mathcal{F}) + \operatorname{rank}(\mathcal{F}) \cdot C_{\mathcal{E}},$$

where $C_{\mathcal{E}}$ is a constant that does not depend on $\mathcal{F}$. It follows that we can completely reconstruct the Hilbert polynomial if we know the rank, degree and multiplicities of $\mathcal{F}$.

Definition 1.3.25 ([Nir09, Definition 3.14]) Let $\mathcal{C}$ be a stacky curve with generating sheaf $\mathcal{E}$. We say that a coherent sheaf $\mathcal{F}$ is **Gieseker-(semi)stable** with respect to $\mathcal{E}$ if for every proper subsheaf $\mathcal{F}' \subset \mathcal{F}$ we have $p_{\mathcal{E}}(\mathcal{F}') \underset{(-)}{<} p_{\mathcal{E}}(\mathcal{F})$. Define the **slope** of $\mathcal{F}$ to be $\mu_{\mathcal{E}}(\mathcal{F}) := \frac{d_{\mathcal{E}}\mathcal{F}}{\operatorname{rank}\mathcal{F}}$. We say that $\mathcal{F}$ is $\mathcal{E}$**-slope-(semi)stable** if for every proper subsheaf we have $\mu_{\mathcal{E}}(\mathcal{F}') \underset{(-)}{<} \mu_{\mathcal{E}}(\mathcal{F})$.

Since the slope only depends on the numerical class $\boldsymbol{\alpha}$ of $\mathcal{E}$ we may also write $\mu_{\boldsymbol{\alpha}} := \mu_{\mathcal{E}}$.

Remark 1.3.26 We have that $\mu_{\mathcal{E}}(\mathcal{F}) = \frac{\langle \mathcal{E}, \mathcal{F} \rangle}{\operatorname{rank}\mathcal{F}} - C_{\mathcal{E}} = p_{\mathcal{E}}(\mathcal{F})(0) - C_{\mathcal{E}}$, so slope (semi)stability is equivalent to Gieseker-(semi)stability. In practice it is sometimes more convenient to compare inequalities using Euler pairings than to compare slopes i.e. $\mathcal{F}$ is $\mathcal{E}$-semistable if and only if $\frac{\langle \mathcal{E}, \mathcal{F}' \rangle}{\operatorname{rank}\mathcal{F}'} \leq \frac{\langle \mathcal{E}, \mathcal{F} \rangle}{\operatorname{rank}\mathcal{F}}$ for every proper subbundle $\mathcal{F}' \subset \mathcal{F}$. In Chapter 4 formulas are massively simplified by working with the Euler pairings.

1.4 Parabolic vector bundles

One important reason to study vector bundles on stacky curves is their close relation to parabolic bundles. Parabolic bundles where originally considered in [MS80, Definition 1.5] to give a generalization of the Narasimhan-Seshadri correspondence for punctured curves. In this section we start by recalling the basic concepts surrounding parabolic bundles. The goal of this section is to give a dictionary between the parabolic language and the stacky curve language.

Definition 1.4.1 ([MS80, Definition 1.5]) Let C be a classical curve and $\underline{p}$ be a finite set of points on C. A **quasi-parabolic vector bundle** $\mathbb{F}$ on $(C, \underline{p})$ is a vector bundle F on C together with filtrations $F = F_0^p \supset F_1^p \supset \ldots \supset F_{e_p}^p = F \otimes \mathcal{O}_C(-p)$ for each $p \in \underline{p}$. The integer e_p is called the length of the parabolic structure at p. The collection of quasi-parabolic vector bundles of fixed length forms a category $\mathfrak{qpar}(C, \underline{p}, \underline{e})$, where the morphisms are given by morphism of the underlying vector bundles respecting the filtration. Explicitly the morphisms are morphisms $\phi\colon F \to G$ such that $\phi(F_j^p) \subset \phi(G_j^p)$ for all p, j.

Remark 1.4.2 Instead of a filtration of sheaves, one can equivalently give a flag of quotients of the fiber $F|_p = V_0^p \twoheadrightarrow V_1^p \twoheadrightarrow \cdots \twoheadrightarrow V_{e_p-1}^p \twoheadrightarrow V_{e_p}^p = 0$ at each point p. To see this, send a filtration $F_\bullet$ to $V_i^p = \texttt{coker}(F_{e_p-i}^p \to F_0^p)|_p$. To obtain a flag of injections $F|_p = W_0^p \supset W_1^p \supset \cdots W_{e_p}^p = 0$, instead simply consider $W_i^p = \texttt{ker}(V_0^p \twoheadrightarrow V_{e_p-i}^p)$.

Contrary to the classical definition, we do not require the inclusions of the filtrations to be strict. One reason is that this gives much better categorical properties. For example, a parabolic subbundle is simply a subobject in the category $\mathfrak{qpar}(C, \underline{p}, \underline{e})$, whereas classically subbundles might have shorter length filtrations, as the length would be bounded by the rank.

We now describe how to obtain a quasi-parabolic vector bundle from a vector bundle on a stacky curve.

Definition 1.4.3 Let $\mathcal{C}$ be a stacky curve with stacky points $\underline{p}$ of degree $\underline{e}$. We define a functor $\mathtt{par}\colon \mathfrak{Vect}(\mathcal{C}) \to \mathfrak{qpar}(C, \underline{p}, \underline{e})$ as follows. Let $\mathcal{F}$ be a vector bundle on $\mathcal{C}$. Then $\mathtt{par}(\mathcal{F})$ is the vector bundle $\pi_*\mathcal{F}$ together with the filtrations

$$\pi_*\mathcal{F} \supset \pi_*(\mathcal{F} \otimes \mathcal{O}_{\mathcal{C}}\left(-\frac{1}{e_p}p\right)) \supset \cdots \supset \pi_*(\mathcal{F} \otimes \mathcal{O}_{\mathcal{C}}\left(-\frac{e_p}{e_p}p\right)),$$

for each $p \in \underline{p}$. A morphism $f\colon \mathcal{F} \to \mathcal{G}$ gets sent to $\mathtt{par}(f) := \pi_* f\colon \pi_*\mathcal{F} \to \pi_*\mathcal{G}$.

There is also an inverse functor, but it is much harder to define, so we will omit it here.

Theorem 1.4.4 ([Bor07, Théorème 4]) The functor $\mathtt{par}$ defines an equivalence of categories.

We will now look at how the functor $\mathtt{par}$ interacts with multiplicities.

Definition 1.4.5 Let $\mathbb{F}$ be a quasi-parabolic bundle. We define the multiplicities

$$m_{p,i}(\mathbb{F}) := \mathtt{dim}\,\mathtt{coker}(F^p_{i+1} \to F^p_i)|_p,$$

where $0 \leq i < e_p$.

In the surjective flag picture, we have $m_{p,i} = \mathtt{dim}\, V^p_{e_p-i-1} - \mathtt{dim}\, V^p_{e_p-i}$ or in the injective flag picture $m_{p,i} = \mathtt{dim}\, W^p_i - \mathtt{dim}\, W^p_{i+1}$.

Proposition 1.4.6 Let $\mathcal{F}$ be a vector bundle on a stacky curve $\mathcal{C} := \sqrt[\underline{e}]{\underline{p}/C}$; then $m_{p,i}(\mathcal{F}) = m_{p,i}(\mathtt{par}(\mathcal{F}))$.

Proof. We see that $m_{p,i}(\mathtt{par}\,\mathcal{F})$ is additive in short exact sequences, so it suffices to show this for line bundles. Then for a line bundle $\mathcal{L} = \pi^* L \otimes \bigotimes_{p\in\underline{p}} \mathcal{O}_{\mathcal{C}}(\frac{n_p}{e_p}p)$ we see that the filtrations of $\mathtt{par}(\mathcal{L})$ are given by $L^p_i = L$ for $0 \leq i \leq n_p$ and

$L_i^p = L(-p)$ for $n_p < i \leq e_p$. This shows that $m_{p,n_p}(\mathtt{par}(\mathcal{L})) = 1$ and the other multiplicities are 0 as required. ⬡

Now we will discuss the notion of weights and (semi)stability for quasi-parabolic bundles, following [MS80, Definition 1.5].

Definition 1.4.7 Let C be a classical curve and $\underline{p}$ be a finite set of points of C. A **parabolic bundle** on C is a quasi-parabolic bundle together with a set $\underline{\alpha}$ of **parabolic weights** consisting of $\alpha_{p,j} \in \mathbb{R}$ for $p \in \underline{p}$ and $0 \leq j < e_p$, satisfying

$$0 \leq \alpha_{p,0} < \cdots < \alpha_{p,e_p-1} < 1.$$

The **parabolic degree** of a parabolic bundle $(\mathbb{F}, \underline{\alpha})$ is $\mathtt{pardeg}(\mathbb{F}, \underline{\alpha}) := \mathbf{deg}\, F + \sum_p \sum_{i=1}^{e_p-1} \alpha_{p,i} m_{p,i}(\mathbb{F})$ and the **parabolic slope** is defined by $\mu(\mathbb{F}, \underline{\alpha}) := \frac{\mathtt{pardeg}(\mathbb{F},\underline{\alpha})}{\mathtt{rank}(F)}$. We say that a parabolic bundle $(\mathbb{F}, \underline{\alpha})$ is **(semi)stable** if for every proper quasi-parabolic subbundle $\mathbb{F}' \subset \mathbb{F}$ we have $\mu(\mathbb{F}', \underline{\alpha}) \underset{(-)}{\leq} \mu(\mathbb{F}, \underline{\alpha})$.

The functor $\mathtt{par}$ respects stability.

Lemma 1.4.8 Let $\mathcal{F}$ be a vector bundle on a stacky curve $\mathcal{C} := \sqrt[\underline{e}]{\underline{p}/C}$. Let $\mathcal{E}$ be a generating sheaf; then $\mathcal{F}$ is $\mathcal{E}$-(semi)stable if and only if $(\mathtt{par}(\mathcal{F}), w_{p,i}(\mathcal{E}))$ is a (semi)stable parabolic bundle.

Proof. This is immediate from the fact that $\mathbf{deg}_{\mathcal{E}}(\mathcal{F}) = \mathtt{pardeg}(\mathtt{par}(\mathcal{F}), \underline{w}(\mathcal{E}))$, which is obtained by combining Propositions 1.3.17 and 1.4.6. ⬡

Theorem 1.4.9 Let $\mathfrak{qpar}(C, \underline{p}, \underline{e})^{\underline{\alpha}-(s)s} \subset \mathfrak{qpar}(C, \underline{p}, \underline{e})$ be the full subcategory of quasi-parabolic bundles that are (semi)stable when endowed with the parabolic weights $\underline{\alpha}$. Then there exists a generating sheaf $\mathcal{E}$ on $\mathcal{C} = \sqrt[\underline{e}]{\underline{p}/C}$, such that the category of (semi)stable vector bundles $\mathfrak{Vect}(\mathcal{C})^{\mathcal{E}-(s)s}$ is equivalent to $\mathfrak{qpar}(C, \underline{p}, \underline{e})^{\underline{\alpha}-(s)s}$.

Proof. By [MS80, Corollary 2.9] we can always perturb the weights $\underline{\alpha}$ to be rational without changing the notion of stability. Secondly we can shift the parabolic weights by a constant without changing the notion of (semi)stability by [MS80, Remark 2.10], so we might as well assume that $\alpha_{p,0} = 0$. This means we can pick $\mathcal{E}$ as in Example 1.3.16. ⬠

We end this section with some comments on "strongly" parabolic homomorphisms and Higgs fields.

Definition 1.4.10 Let $\mathbb{F}, \mathbb{G} \in \mathfrak{qpar}(C, \underline{p}, \underline{e})$ be quasi-parabolic bundles. We define a **strongly parabolic morphism** to be a morphism $f\colon F \to G$, such that $f(F_i^p) \subset G_{i+1}^p$ for every p, i. The set of strongly parabolic morphisms is denoted by $\mathtt{sHom}(\mathbb{F}, \mathbb{G})$.

Let $D_{\underline{p}} := \sum_{p \in \underline{p}} p$ be the **parabolic divisor**. A **Higgs field** on $\mathbb{F}$ is a strongly parabolic parabolic morphism $\phi\colon \mathbb{F} \to \mathbb{F} \otimes \omega_C(D_{\underline{p}})$. (Here the tensor product should be done term-wise on every term of the filtrations of $\mathbb{F}$.)

The notion of a strongly parabolic morphisms might seem quite ad-hoc. In fact the only reason that it shows up is that the "logarithmic" canonical sheaf $\omega_C(D_{\underline{p}})$ has the wrong parabolic structure. On the level of stacky curves this will be apparent.

Proposition 1.4.11 Let $\mathcal{C} = \sqrt[\underline{e}]{\underline{p}/C}$ and let $\mathcal{F}, \mathcal{G} \in \mathfrak{Vect}(\mathcal{C})$ be two vector bundles. We have a natural isomorphism

$$\phi : \mathtt{Hom}\left(\mathcal{F}, \mathcal{G} \otimes \bigotimes_p \mathcal{O}_{\mathcal{C}}\left(-\frac{1}{e_p}p\right)\right) \to \mathtt{sHom}(\mathtt{par}(\mathcal{F}), \mathtt{par}(\mathcal{G})).$$

In particular we have a correspondence of Higgs fields

$$\mathtt{sHom}(\mathtt{par}(\mathcal{F}), \mathtt{par}(\mathcal{F}) \otimes \omega_C(D)) = \mathtt{Hom}(\mathcal{F}, \mathcal{F} \otimes \omega_{\mathcal{C}}).$$

Proof. Denote by ι the inclusion $\iota\colon \mathcal{G} \otimes \bigotimes_p \mathcal{O}_{\mathcal{C}}(-\frac{1}{e_p}p) \hookrightarrow \mathcal{G}$. We define ϕ by sending a morphism $f\colon \mathcal{F} \to \mathcal{G} \otimes \bigotimes_p \mathcal{O}_{\mathcal{C}}(-\frac{1}{e_p}p)$ to $\phi(f) := \mathtt{par}(\iota \circ f)$.

By definition this defines a strongly parabolic morphism and clearly ϕ is injective. To see that ϕ is surjective, take any strongly parabolic morphism $h\colon \mathtt{par}(\mathcal{F}) \to \mathtt{par}(\mathcal{G})$; by Theorem 1.4.4 h lifts to a unique morphism $\tilde{h}\colon \mathcal{F} \to \mathcal{G}$. We need to show that $\tilde{h}$ factors through ι. To see this consider the generating sheaf $\mathcal{E} = \bigoplus_{p \in \underline{p}} \bigoplus_{i=0}^{e_p-1} \mathcal{O}_{\mathcal{C}}(\frac{i}{e_p}p)$. The fact that h is strongly parabolic ensures that $F_{\mathcal{E}}(\tilde{h})$ factors through $F_{\mathcal{E}}(\mathcal{G} \otimes \bigotimes_p \mathcal{O}_{\mathcal{C}}(-\frac{1}{e_p}p))$, where $F_{\mathcal{E}}$ is as in Definition 1.3.9. Now consider the following commutative diagram.

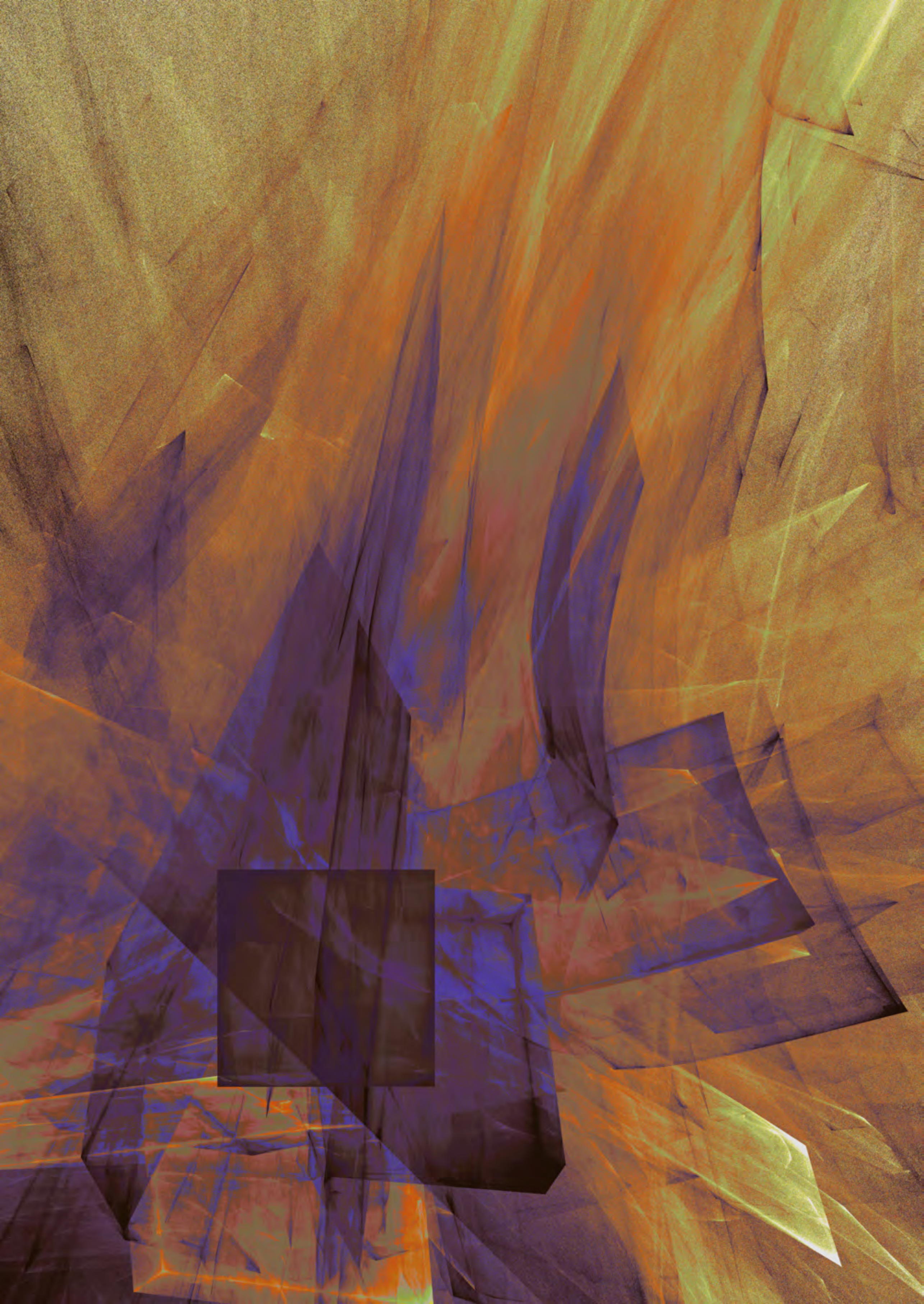
$$\begin{tikzcd}
G_{\mathcal{E}}F_{\mathcal{E}}(\mathcal{F}) \arrow[r] \arrow[d, two heads] \arrow[rr, bend left, "G_{\mathcal{E}}F_{\mathcal{E}}(\tilde{h})"] & G_{\mathcal{E}}F_{\mathcal{E}}(\mathcal{G} \otimes \bigotimes_p \mathcal{O}_{\mathcal{C}}(-\frac{1}{e_p}p)) \arrow[r, hook] \arrow[d, two heads] & G_{\mathcal{E}}F_{\mathcal{E}}(\mathcal{G}) \arrow[d, two heads] \\
\mathcal{F} \arrow[r, dotted] \arrow[rr, bend right, "\tilde{h}"'] & \mathcal{G} \otimes \bigotimes_p \mathcal{O}_{\mathcal{C}}(-\frac{1}{e_p}p) \arrow[r, hook] & \mathcal{G}
\end{tikzcd}$$

This shows that the image of $\tilde{h}$ lies inside $\mathcal{G} \otimes \bigotimes_p \mathcal{O}_{\mathcal{C}}(-\frac{1}{e_p}p)$. ⬡

The theorem above also explains why Serre duality [Yok95, Proposition 3.7] for parabolic bundles is perhaps not what we would expect naively. Namely we have

$$\begin{aligned}
\mathtt{Ext}^1_{\mathtt{par}}(\mathtt{par}(\mathcal{F}), \mathtt{par}(\mathcal{G})) &= \mathtt{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\mathcal{F}, \mathcal{G}) \\
&= \mathtt{Hom}_{\mathcal{O}_{\mathcal{C}}}(\mathcal{G}, \mathcal{F} \otimes \omega_{\mathcal{C}})^{\vee} \\
&= \mathtt{sHom}(\mathtt{par}(\mathcal{G}), \mathtt{par}(\mathcal{F}) \otimes \omega_C(D))^{\vee}.
\end{aligned}$$

All the equivalences in this section are on the level of categories, but we will see in Chapter 3 that they also hold on the level of moduli stacks.



CHAPTER 2

Spherical curves

Those who build walls
are their own prisoners.
I'm going to go fulfil my proper
function in the social organism.
I'm going to go unbuild walls.

Shevek, in The Disposessed
Ursula K. Le Guin

Classically curves fall into a trichotomy defined by their genus: namely the spherical curves of genus 0, the flat curves of genus 1 and the hyperbolic curves of genus ≥ 2. This trichotomy also exists for stacky curves and in this chapter we will study the most well behaved case, that of spherical stacky curves. Since the genus of a stacky curve is a rational number the first question is how to classify the curves of genus $0 < g_{\mathcal{C}} < 1$ and $1 < g_{\mathcal{C}} < 2$. We will take the following notion.

> **Definition 2.0.1** A **spherical** stacky curve is a projective stacky curve $\mathcal{C}$ with genus $g_{\mathcal{C}} < 1$ and a k-point.

The condition that $\mathcal{C}$ has a k-point is there for convenience, ensuring that the coarse space of $\mathcal{C}$ is $\mathbb{P}^1_k$. The reason we make this restriction is that want to apply the theory to stacky curves that are obtained as quotients of $\mathbb{P}^1_k$, which always have a

k-point. It would be interesting to consider also projective stacky curves without k-points, however this also forces all the stacky points to have residue field of degree 2, giving a much smaller class of curves.

These curves have been studied over $k = \mathbb{C}$ in [BN06], allowing for stacky curves that have generic stacky structure as well. They show that spherical curves are all quotients of weighted projective lines $\mathcal{P}(m, n)$. We generalize this to a separably closed field, but since we do not allow generic stacky structure, we will see that all spherical curves are quotients of the football spaces $\mathcal{F}(m, n)$, with m and n coprime.

The categories of sheaves on spherical curves have also been thoroughly studied by representation theorists, using completely different techniques. We highlight one such result [GL87, p. 5.4.1.].

2.1 Classification of spherical curves

We start by giving a complete list of all the spherical stacky curves. We then construct a natural root system attached to every spherical curve, thus giving a natural correspondence between spherical curves and Dynkin diagrams.

Let $\mathcal{C}$ be a projective stacky curve with stacky points $\underline{p}$. From the inequality

$$\frac{1}{4}\sum_{p\in\underline{p}}[\kappa(p):k] \leq g_{\mathbb{P}^1_k} + \frac{1}{2}\sum_{p\in\underline{P}}[\kappa(p):k]\cdot\frac{e_p - 1}{e_p} = g_{\mathcal{C}} < 1,$$

it follows that spherical curves can have at most 3 stacky points, weighted by the degree of their residue field. We recall the following lemma, which generalizes the 3-transitivity of PGL_2.

> **Lemma 2.1.1** (Strong 3-transitivity) Let D, D' be two smooth closed codimension 1 subschemes of $\mathbb{P}^1_k$, such that $\deg(D), \deg(D') \leq 3$ and $\kappa(D) \simeq \kappa(D')$; then there exists a k-automorphism of $\mathbb{P}^1_k$ sending D to D'.

We take the convention that when D is disconnected $\kappa(D)$ is the product of the residue fields of the connected components.

Proof. Let $F, G \in k[X,Y]$ be the separable homogeneous polynomials defining D and D'. By assumption, they are polynomials of degree at most 3 over k, with the same splitting field l. Choose a $\mathtt{Gal}_{l/k}$-equivariant bijection between the linear factors of F and G. By 3-transitivity of $\mathsf{PGL}_2(l)$, there exists a unique transformation $\phi \in \mathsf{PGL}_2(l)$ which sends D_l to D'_l respecting the chosen bijection. Now the action of $\mathtt{Gal}_{l/k}$ fixes F, G and the bijection of the roots, so by the uniqueness of ϕ the action must also fix ϕ. It follows that $\phi \in \mathsf{PGL}_2(k)$. ⬡

It follows that spherical stacky curves are completely determined by the order and the residue field of their stacky points.

Corollary 2.1.2 Let $\mathcal{C}$ be a spherical stacky curve with stacky points $\underline{p}$. Then $\mathcal{C}$ falls into one of three groups.

(I) All points $\underline{p}$ are k-points, in which case we may assume $\underline{p} \subset \{0, 1, \infty\}$ and the isomorphism class of $\mathcal{C}$ is determined by the orders $\underline{e}$.

(II) The set $\underline{p}$ contains a single point p with $[\kappa(p) : k] = 2$ and at most one k-point, in which case the isomorphism class of $\mathcal{C}$ is determined by the quadratic field extension $\kappa(p)$ and $\underline{e}$.

(III) There is a single stacky point q with $[\kappa(q) : k] = 3$, in which case the isomorphism class of $\mathcal{C}$ is determined by the field extension $\kappa(q)$ and e_q.

Using the above corollary we give a list of all isomorphism classes of spherical curves in Table 2.1.

Remark 2.1.3 Recall that by Remark 1.1.34 the residue fields of stacky points of a smooth curve are separable. It follows that over that over the separable closure k_{sep} all curves are of type (I), i.e. $(\mathcal{B}_{n,\kappa(p)})_{k_{\mathsf{sep}}} \simeq (\mathcal{F}(n,n))_{k_{\mathsf{sep}}}$, $(\mathcal{C}_{n,\kappa(p)})_{k_{\mathsf{sep}}} \simeq (\mathcal{D}_n)_{k_{\mathsf{sep}}}$, $(\mathcal{F}_{4,\kappa(p)})_{k_{\mathsf{sep}}} \simeq (\mathcal{E}_6)_{k_{\mathsf{sep}}}$ and $(\mathcal{G}_{2,\kappa(q)})_{k_{\mathsf{sep}}} \simeq (\mathcal{D}_4)_{k_{\mathsf{sep}}}$.

Root systems and Dynkin diagrams

As the suggestive naming of our curves indicates, there is a natural way to obtain a Dynkin diagram from a spherical stacky curve. To see this we will construct a natural root system on (a quotient of) the Grothendieck group $\mathsf{K}_0(\mathcal{C})$. The Grothendieck

Type	Orders			$g_{\mathcal{C}}$	$\sqrt[\underline{e}]{\underline{p}/\mathcal{P}^1_k}$
	e_0	e_∞	e_1		
(I)	m	$n \leq m$		$1-\frac{m+n}{2mn}$	$\mathcal{F}(m,n)$
	2	2	n	$1-\frac{1}{2n}$	$\mathcal{D}_n$
	2	3	3	$1-\frac{1}{12}$	$\mathcal{E}_6$
	2	3	4	$1-\frac{1}{24}$	$\mathcal{E}_7$
	2	3	5	$1-\frac{1}{60}$	$\mathcal{E}_8$
	e_p		e_0		
(II)	n			$1-\frac{1}{n}$	$\mathcal{B}_{n,\kappa(p)}$
	2		n	$1-\frac{1}{2n}$	$\mathcal{C}_{n,\kappa(p)}$
	3		2	$1-\frac{1}{12}$	$\mathcal{F}_{4,\kappa(p)}$
		e_q			
(III)		2		$1-\frac{1}{4}$	$\mathcal{G}_{2,\kappa(q)}$

Table 2.1: The isomorphism classes of spherical stacky curves, split up into the types of Corollary 2.1.2. Each row describes a stacky curve, giving the orders $\underline{e}$, the genus $g_{\mathcal{C}}$ and the final column indicates how we will denote each curve.

group comes with a the natural bilinear form $\mathsf{K}_0(\mathcal{C}) \times \mathsf{K}_0(\mathcal{C}) \to \mathbb{Z}$.

$$\langle \alpha, \beta \rangle \coloneqq \mathtt{ext}^0(\alpha,\beta) - \mathtt{ext}^1(\alpha,\beta).$$

From this we obtain a symmetric bilinear form

$$\langle \alpha, \beta \rangle_{\mathsf{sym}} = \frac{1}{2}\left(\langle \alpha,\beta\rangle + \langle \beta,\alpha\rangle\right)$$

This form is degenerate precisely on the subspace $\mathtt{Pic}_C \subset \mathsf{K}_0(\mathcal{C})$ and we let $E(\mathcal{C})$ be the $\mathbb{R}$-vector space $(\mathsf{K}_0(\mathcal{C})/\mathtt{Pic}_C) \otimes \mathbb{R}$, together with the inner product $\langle -,- \rangle_{\mathsf{euc}}$ induced by $\langle -,- \rangle_{\mathsf{sym}}$. There is a natural root system $\Phi_{\mathcal{C}}$ inside $E(\mathcal{C})$, generated by the classes of indecomposable torsion sheaves and $\mathcal{O}_{\mathcal{C}}$. Note that there are only finitely many classes of indecomposable torsion sheaves in $E(\mathcal{C})$. Indeed any indecomposable torsion sheaf is equivalent to a class of the form $[\mathcal{O}_{\mathcal{C}}(\frac{n}{e_p})p] - [\mathcal{O}_{\mathcal{C}}(\frac{m}{e_p})p]$ for some point p and $m < n$. As $[\mathcal{O}_{\mathcal{C}}(p)] - [\mathcal{O}_{\mathcal{C}}] = 0$

by construction, m and n are only defined modulo e_p. We choose a set of simple roots by letting

$$\alpha_{p,i} \coloneqq [\mathcal{O}_{\mathcal{C}}(\frac{i}{e_p})p] - [\mathcal{O}_{\mathcal{C}}(\frac{i-1}{e_p}p)] = \mathcal{O}_{\frac{i}{e_p}p} - \mathcal{O}_{\frac{i-1}{e_p}p}$$

for $1 \leq i \leq e_p - 1$. We then set $\Delta_{\mathcal{C}} = \{\alpha_{p,i}\} \cup \{[\mathcal{O}_{\mathcal{C}}]\}$. Notice that the set of positive roots $\Phi^+_{\mathcal{C}}$ contains in particular all the classes corresponding to vector bundles and torsion sheaves of the form $\mathcal{O}_D$ for a divisor D. We suspect that this property uniquely specifies our choice of simple roots. We compute

$$\begin{aligned}
\langle \mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}} \rangle_{\mathsf{euc}} &= 1, \\
\langle \alpha_{p,i}, \alpha_{p,i} \rangle_{\mathsf{euc}} &= [\kappa(p) : k], \\
\langle \alpha_{p,i}, \alpha_{p,i+1} \rangle_{\mathsf{euc}} = \langle \mathcal{O}_{\mathcal{C}}, \alpha_{p,1} \rangle_{\mathsf{euc}} &= -\frac{[\kappa(p) : k]}{2},
\end{aligned}$$

and all other inner products are 0. By comparing Cartan matrices we can see that our correspondence is as follows.

$$\begin{array}{rcl}
\mathcal{F}(m,n) & \mapsto & A_{m+n-1} \\
\mathcal{B}_{n,\kappa(p)} & \mapsto & B_n \\
\mathcal{C}_{n,\kappa(p)} & \mapsto & C_{n+1} \\
\mathcal{D}_n & \mapsto & D_{n+2} \\
\mathcal{E}_i & \mapsto & E_i \\
\mathcal{F}_{4,\kappa(p)} & \mapsto & F_4 \\
\mathcal{G}_{2,\kappa(q)} & \mapsto & G_2
\end{array}$$

In Figures 2.1 and 2.2 we have visualized this correspondence for some of the low dimensional root systems. We also obtain a geometric interpretation of the folding phenomenon. Namely, the following isomorphisms induce foldings of root systems.

$$\begin{array}{l}
(\mathcal{B}_{n,\kappa(p)})_{\kappa(p)} \simeq (\mathcal{F}(n,n))_{\kappa(p)} \\
(\mathcal{C}_{n,\kappa(p)})_{\kappa(p)} \simeq (\mathcal{D}_n)_{\kappa(p)} \\
(\mathcal{F}_{4,\kappa(p)})_{\kappa(p)} \simeq (\mathcal{E}_6)_{\kappa(p)} \\
(\mathcal{G}_{2,\kappa(q)})_{\kappa(q)} \simeq (\mathcal{D}_4)_{\kappa(p)}
\end{array}
\quad \to \quad
\begin{array}{rcl}
A_{2n-1} & \dashrightarrow & B_n \\
D_{n+1} & \dashrightarrow & C_n \\
E_6 & \dashrightarrow & F_4 \\
D_4 & \dashrightarrow & G_2
\end{array}$$

To see this, let $\mathcal{C}$ be a type (II) or (III) curve and let p be the non-rational stacky point. Let l be the Galois closure of $\kappa(p)$, so the map $\mathcal{C}_l \to \mathcal{C}_k$ induces an embedding

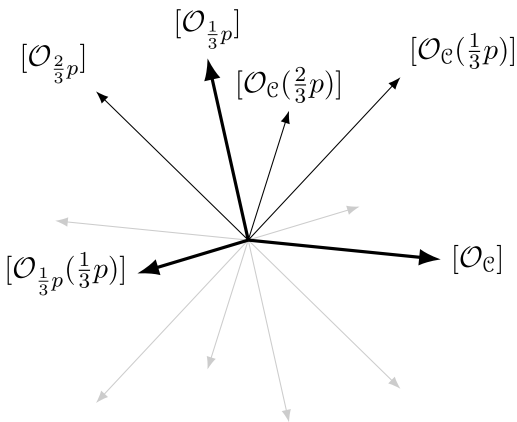


Figure 2.1: The root system A_3 corresponding to $\mathcal{C} = \mathcal{F}(3,1)$. The thick black arrows are the simple roots, the thin black arrows are the positive roots, and the gray arrows are the negative roots.

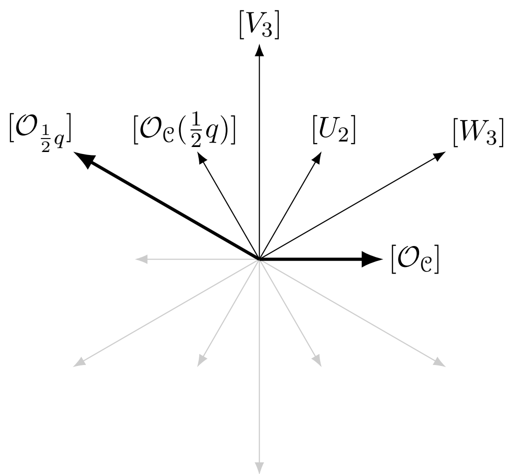


Figure 2.2: The root system G_2 corresponding to $\mathcal{C} = \mathcal{G}_{2,\kappa(q)}$. The thick black arrows are the simple roots, the thin black arrows are the positive roots, and the gray arrows are the negative roots. Do the classes $[U_2]$, $[V_3]$, $[W_3]$ have interesting representatives?

$E(\mathcal{C}_k) \to E(\mathcal{C}_l)$. Note that $\Phi_{\mathcal{C}_l}$ corresponds to a **simply laced** Dynkin diagram. The Galois group $G = \mathtt{Gal}_{l/k}$ acts on $\mathcal{C}_l$ by permuting the points lying over p, hence acts on the simple roots $\Delta_{\mathcal{C}_l}$ by switching the $\alpha_{p,i}$, inducing a diagram automorphism of $\Phi_{\mathcal{C}_l}$. The invariants under this action naturally correspond to the image of $E(\mathcal{C}_k)$. Moreover, if we set

$$\Phi_{\mathcal{C}_l}^G := \left\{ \frac{1}{\mathtt{ord}(G_x)} \sum_{\sigma \in G} \sigma(x) \right\}_{x \in \Phi_{\mathcal{C}_l}},$$

the induced map is in fact an isomorphism of root systems $(E(\mathcal{C}_k), \Phi_{\mathcal{C}_k}) \simeq (E(\mathcal{C}_l)^G, \Phi_{\mathcal{C}_l}^G)$. See Figure 2.3 for a visualization of this folding isomorphism in the lowest dimensional case.

2.2 Finite subgroups of the projective linear group

We will give a quick application of this classification of spherical stacky curves to the classification of conjugacy classes of subgroups of $\mathsf{PGL}_2(k)$ over a separably closed field. This classification was first obtained for $k = \mathbb{C}$ by Klein [Kle88]. In [Beau10] Beauville gives a classification for an arbitrary field, for groups whose order is not divisible by the characteristic of k. Finally in [Fab23] the classification was completed by characterizing the groups whose order is divisible by the characteristic. Interestingly the proofs over an arbitrary field depend on first proving the case of separably closed fields.

Definition 2.2.1 We define Q to be the map

$$Q : \left\{ \begin{array}{l} \text{Conjugacy classes of subgroups} \\ \text{of } \mathsf{PGL}_2 \text{ with order not divisible} \\ \text{by the characteristic of } k. \end{array} \right\} \to \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{spherical stacky curves} \\ \text{with a } k\text{-point.} \end{array} \right\},$$

given by sending $[H] \mapsto \mathcal{C}_H := [\mathbb{P}^1_k / H]$.

To see that the map is well defined, let $H' = gHg^{-1}$ for some $g \in \mathsf{PGL}_2(k)$. Now the composite map $\mathbb{P}^1_k \xrightarrow{g} \mathbb{P}^1_k \to [\mathbb{P}^1_k / H']$ is H-invariant and defines an isomorphism $\mathcal{C}_H \to \mathcal{C}_{H'}$.

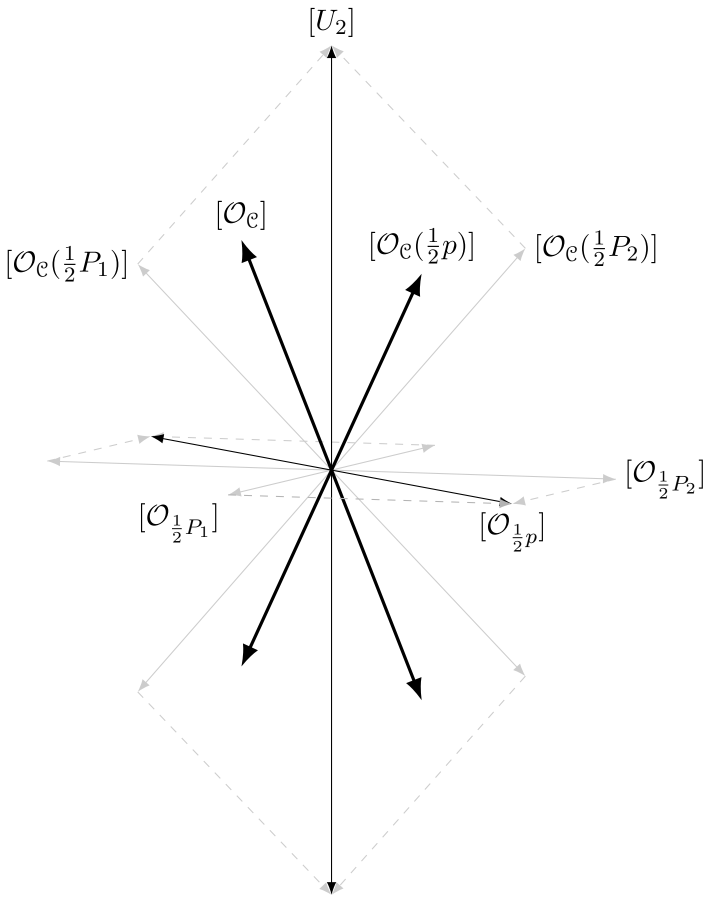


Figure 2.3: The folding of A_3 into B_2 induced by the isomorphism $(\mathcal{B}_{2,\kappa(p)})_{\kappa(p)} \simeq (\mathcal{F}(2,2))_{\kappa(p)}$. The point p splits into the two points P_1, P_2. The thick black arrows are the Galois-invariant roots of A_3 and the gray arrows are the roots of A_3 that are folded into roots of B_3. The labeled roots are exactly the positive roots of A_3 and B_3.

Lemma 2.2.2 Assume that k is separably closed; then the map Q is injective.

Proof. Let H and H' be two subgroups of $\mathsf{PGL}_2(k)$ such that there exists an isomorphism $g\colon [\mathbb{P}^1_k/H] \simeq [\mathbb{P}^1_k/H']$. Since $\mathbb{P}^1_k$ is simply connected, the isomorphism lifts to an isomorphism of covering spaces $\overline{g}\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$. We have $H' = \mathsf{Aut}_{[\mathbb{P}^1_k/H']}(\mathbb{P}^1) = g^{-1}\,\mathsf{Aut}_{[\mathbb{P}^1_k/H]}(\mathbb{P}^1_k)g = \overline{g}^{-1}H\overline{g}$. It follows that H and H' are conjugate. ⬠

To understand the image of Q we will compute some explicit quotients.

Lemma 2.2.3 Let ζ_n denote a primitive n-th root of unity. Let $\mathbb{F}$ denote the Fermat quadric $V(X^2+Y^2+Z^2) \subset \mathbb{P}^2$. We have the following quotients.

Let $\alpha \in k$. For

$$C_2 \simeq \left\langle \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} \right\rangle,$$

we have $[\mathbb{P}^1_k/C_2] \simeq \mathcal{B}_{2,k(\sqrt{\alpha})}$.

Assume $n > 2$ and $\zeta_n + \zeta_n^{-1} \in k$. For

$$C_n \simeq \left\langle \begin{bmatrix} 2+\zeta_n+\zeta_n^{-1} & (\zeta_n+\zeta_n^{-1})^2-4 \\ 1 & 2+\zeta_n+\zeta_n^{-1} \end{bmatrix} \right\rangle,$$

we have $[\mathbb{P}^1_k/C_n] \simeq \mathcal{B}_{n,k(\zeta_n)}$.

Assume $\zeta_n \in k$. For

$$D_{2n} \simeq \left\langle \begin{bmatrix} 1 & 0 \\ 0 & \zeta_n \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} \right\rangle,$$

we have

$$[\mathbb{P}^1_k/D_{2n}] \simeq \begin{cases} \mathcal{D}_n & \text{if } n \text{ is even} \\ \mathcal{C}_{n,k(\sqrt{\alpha})} & \text{if } n \text{ is odd} \end{cases}.$$

For
$$A_4 \simeq \left\langle \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle \subset \mathsf{PGL}_3(k),$$
we have $[\mathbb{F}/A_4] \simeq \mathcal{F}_{4,k(\zeta_3)}$.

For
$$S_4 \simeq \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle \subset \mathsf{PGL}_3(k),$$
we have $[\mathbb{F}/S_4] \simeq \mathcal{E}_7$.

Assume $\sqrt{5} \in k$ and let ϕ be a root of the polynomial $x^2 - x - 1$ (the golden ratio) and $\omega = \phi - 1 = \frac{1}{\phi}$. For
$$A_5 \simeq \left\langle \begin{bmatrix} 1 & \omega & \phi \\ \omega & \phi & -1 \\ -\phi & 1 & \omega \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle \subset \mathsf{PGL}_3(k),$$
we have $[\mathbb{F}/A_5] \simeq \mathcal{E}_8$.

Proof. The quotients are automatically spherical curves, so it suffices to compute the stabilizer and inertia of each action. This is an elementary exercise in linear algebra, which we leave to the reader. ⬠

Note that the tameness of $\mathcal{F}(n,n), \mathcal{D}_n, \mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8$ restricts the characteristic of k, so we may assume that $\zeta_n, \sqrt{5}$ etc. are inside k_{sep} when appropriate. It follows that we obtain the following quotient presentations over $k = k_{\mathsf{sep}}$.

$$\begin{aligned} \mathcal{F}(n,n) &\simeq [\mathbb{P}^1_k/C_n], \\ \mathcal{D}_n &\simeq [\mathbb{P}^1_k/D_{2n}], \\ \mathcal{E}_6 &\simeq [\mathbb{P}^1_k/A_4], \\ \mathcal{E}_7 &\simeq [\mathbb{P}^1_k/S_4], \\ \mathcal{E}_8 &\simeq [\mathbb{P}^1_k/A_5]. \end{aligned}$$

Corollary 2.2.4 Assume that $k = k_{\mathsf{sep}}$ is separably closed; then the finite subgroups of $\mathsf{PGL}_2(k)$, whose order is not divisible by the characteristic of k, are isomorphic to one of C_n, D_{2n}, A_4, S_4, A_5 and each isomorphism class has exactly one conjugacy class.

Proof. By Proposition 1.3.5 the image of Q does not contain $\mathcal{F}(m, n)$ for $m \neq n$. On the other hand Lemma 2.2.3 and Table 2.1 show that there is a unique curve corresponding to the groups C_n, D_{2n}, A_4, S_4, A_5, hence there is only one conjugacy class for each of the groups. ⬠

2.3 Equivariant structures on the projective line

Let $G \subset \mathsf{PGL}_2(k)$ be a finite subgroup acting on $\mathbb{P}^1_k$, then [BM24] asks which vector bundles on $\mathbb{P}^1_k$ admit a G-equivariant structure, i.e. what is the image of $\mathfrak{Vect}\big([\mathbb{P}^1/G]\big) \to \mathfrak{Vect}\big(\mathbb{P}^1\big)$. In [BM24] they answer this question for Abelian groups G and $k = \mathbb{C}$. Here we will generalize this to an arbitrary group G over a separably closed field $k = k_{\mathsf{sep}}$, such that the order of G is invertible in k.

We start by showing some basic results about Harder-Narasimhan filtrations on spherical curves.

Lemma 2.3.1 Let $G \subset \mathsf{PGL}_2(k)$ be a finite group whose order is invertible in k and write $\pi \colon \mathbb{P}^1 \to [\mathbb{P}^1/G] =: \mathcal{C}$. Let $\mathcal{F}$ be a vector bundle on $\mathcal{C}$ and let

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F},$$

be the Harder-Narasimhan filtration with respect to the slope function $\mu_{\mathcal{C}} = \frac{\mathsf{deg}}{\mathtt{rank}}$. Then

$$\pi^*\mathcal{F}_0 \subset \pi^*\mathcal{F}_1 \subset \cdots \subset \pi^*\mathcal{F}$$

is the Harder-Narasimhan filtration of $\pi^*\mathcal{F}$.

Proof. Let $F' \subset \pi^*\mathcal{F}$ be the maximally destabilizing sheaf; by maximality it must be preserved by the G-equivariant structure on $\pi^*\mathcal{F}$. It follows that $F' = \pi^*\mathcal{F}'$ for some $\mathcal{F}' \subset \mathcal{F}$. Since $\mu_{\mathcal{C}} \circ \pi^* = \mathtt{ord}\, G \cdot \mu$ it follows that $\mathcal{F}'$ is the maximally destabilizing sheaf for $\mathcal{F}$. ⬠

By the classification of vector bundles on $\mathbb{P}^1_k$, the Harder-Narasimhan filtration of any vector bundle on $\mathbb{P}^1_k$ splits. This splitting descends to the stacky curve.

Lemma 2.3.2 With the above notation, the Harder-Narasimhan filtration of $\mathcal{F}$ splits.

Proof. Let $\mathcal{F}' \hookrightarrow \mathcal{F}$ be the maximally destabilizing sheaf and let $\mathcal{Q}$ be the quotient. By the previous lemma we have a splitting $f\colon \pi^*\mathcal{Q} \to \pi^*\mathcal{F}$. For $\gamma \in G$, let ${}^\gamma f$ denote the natural composition $\pi^*\mathcal{Q} \to \gamma^*\pi^*\mathcal{Q} \to \gamma^*\pi^*\mathcal{F} \to \pi^*\mathcal{F}$ induced by the equivariant structures. The map

$$\tilde{f} = \frac{1}{\operatorname{\texttt{ord}} G} \sum_{\gamma \in G} {}^\gamma f$$

is then a G-equivariant splitting, so it descends to a splitting $\mathcal{Q} \to \mathcal{F}$. ◯

Corollary 2.3.3 Every vector bundle $\mathcal{F}$ on $\mathcal{C}$ can be written as a direct sum $\mathcal{F} = \bigoplus_i \mathcal{F}_i$, such that $\pi^*\mathcal{F}_i \simeq \mathcal{O}_{\mathbb{P}^1}(i)^{n_i}$.

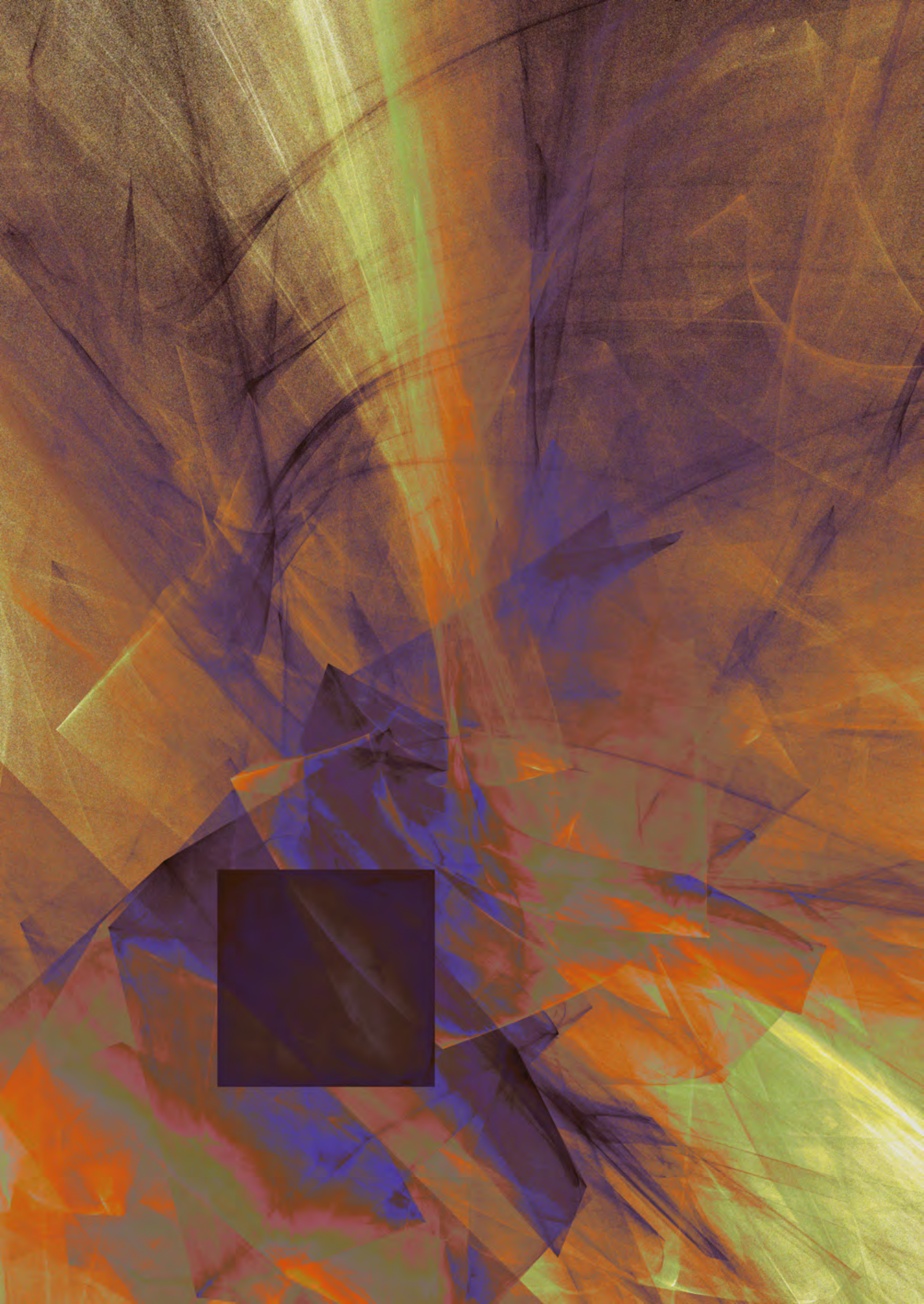
We are now ready to classify the vector bundles which admit a G-equivariant structure.

Theorem 2.3.4 Let G be either D_{2n} with n odd or C_n; then every vector bundle on $\mathbb{P}^1$ admits an equivariant structure. For every other group G, a vector bundle admits a G-equivariant structure if and only if it can be written as

$$\bigoplus_i \mathcal{O}_{\mathbb{P}^1}(2i)^{\oplus n_i} \oplus \bigoplus_j \mathcal{O}_{\mathbb{P}^1}(2j-1)^{\oplus 2n_j}.$$

Proof. We will denote the quotient map by $\pi\colon \mathbb{P}^1 \to [\mathbb{P}^1/G] =: \mathcal{C}$. If $G = C_n$ we note that $[\mathbb{P}^1/G] \simeq \mathcal{F}(n,n)$ and let p be one of the stacky points of $\mathcal{F}(n,n)$. Then $\pi^*\mathcal{O}_{\mathcal{F}(n,n)}(\frac{i}{n}p) = \mathcal{O}_{\mathbb{P}^1}(i)$. As every bundle on $\mathbb{P}^1$ is a direct sum of bundles of the form $\mathcal{O}_{\mathbb{P}^1}(i)$ we conclude that the map $\mathfrak{Vect}([\mathbb{P}^1/C_n]) \to \mathfrak{Vect}(\mathbb{P}^1)$ is surjective. Similarly for D_{2n} with n odd let p be the point of order n and q be a point of order 2 on $\mathcal{D}_n$, then $\pi^*\mathcal{O}_{\mathcal{D}_n}(\frac{(1-n)/2)}{n}p + \frac{1}{2}q) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$.

For every other group we notice that $\mathsf{deg}\colon \mathfrak{Vect}(\mathcal{C}) \to \mathbb{Q}$ lands inside $\mathbb{Z}[\frac{2}{\mathtt{ord}\, G}]$ by Definition 1.2.29. It follows that $\mathsf{deg} \circ \pi^*$ must land in $2\mathbb{Z} \subset \mathbb{Z}$, so a bundle can only admit an equivariant structure if it has even degree. By Corollary 2.3.3 any equivariant bundle is of the sum of equivariant bundles of the form $\mathcal{O}_{\mathbb{P}^1}(i)^{n_i}$, so if i is odd, we must have n_i even. Now notice that $\pi^*\omega_{\mathcal{C}} = \omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$. Next we take a non-split extension $\omega_{\mathcal{C}} \to E \to \mathcal{O}_{\mathcal{C}}$, so π^* defines a non-split extension $\omega_{\mathbb{P}^1} \to \pi^*E \to \mathcal{O}_{\mathbb{P}^1}$. It follows that π^*E is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Since π^* preserves direct sums and tensor products the result follows. ⬠



CHAPTER 3

Moduli Stacks

J'en suis convaincu,
mais je ne peux pas le prouver
parce que rien d'important
ne peut être prouvé;
on peut simplement
le ressentir, le deviner.

Will we continue scientific research?
Alexander Grothendieck

In this chapter we will introduce several moduli stacks that are related to the study of sheaves on stacky curves. We will give basic properties of these moduli stacks and morphisms between them. We end by upgrading the categorical result of Chapter 1 and show that moduli stacks of (semistable) parabolic bundles are isomorphic to moduli stacks of (semistable) vector bundles on stacky curves.

3.1 Moduli of sheaves

We start with a big definition containing the main moduli problems that we will study.

Definition 3.1.1 Let $\mathcal{C}$ be a stacky curve. We denote by $\mathsf{Coh}(\mathcal{C})$ the stack of coherent sheaves on $\mathcal{C}$. Explicitly the objects over $T \to \mathsf{Spec}(k)$ are flat families of sheaves on $\mathcal{C}$ over T and a morphism from an object $\mathcal{F}/S$ to an object $\mathcal{G}/T$ is a pair (f, ϕ), where $f\colon S \to T$ is an fppf morphism of schemes and $\phi\colon f^*\mathcal{G} \to \mathcal{F}$ is an isomorphism of coherent sheaves.

We denote by $\mathsf{Bun}(\mathcal{C})$ and $\mathsf{Bun}^{\mathcal{E}\text{-ss}}(\mathcal{C})$ the substacks of vector bundles and $\mathcal{E}$-semistable vector bundles respectively i.e. the stacks of families that are fiber-wise (semistable) vector bundles. For fixed rank and twisted degrees $(n, \underline{d})$, we denote by

$$\mathsf{Coh}_{n,\underline{d}}(\mathcal{C}) \supset \mathsf{Bun}_{n,\underline{d}}(\mathcal{C}) \supset \mathsf{Bun}^{\mathcal{E}\text{-ss}}_{n,\underline{d}}(\mathcal{C})$$

the substacks with fixed invariants. We will drop $\mathcal{C}$ from the notation when it is clear from context. Since $n, \underline{d}$ defines a unique numerical class $\boldsymbol{\alpha} \in \mathsf{K}_0^{\mathsf{num}}(\mathcal{C})$ we can also write $\mathsf{Coh}_{\boldsymbol{\alpha}}$ instead. When it is more natural, we will sometimes refer to $\mathsf{Bun}_{n,\underline{d}}$ as $\mathsf{Bun}_{n,d,\underline{m}}$.

Being torsion-free is an open condition, so $\mathsf{Bun} \subset \mathsf{Coh}$ is an open substack. By [Nir09, Corollary 4.16], $\mathsf{Bun}^{\mathcal{E}\text{-ss}} \subset \mathsf{Coh}$ is an open substack. By [OS03, Lemma 4.3], $\mathsf{Coh}_{n,\underline{d}} \subset \mathsf{Coh}$ is an open and closed substack and Coh is the disjoint union of the $\mathsf{Coh}_{n,\underline{d}}$, running over all the possible invariants. By [Nir09, Corollary 2.27], Coh is an algebraic stack, locally of finite presentation over k. It follows that all the stacks in the definition are algebraic and locally of finite presentation.

Vector bundle stacks

We will now introduce a class of moduli stacks that admit the structure of a vector bundle stack, the stackified notion of a vector bundle. The definition of a vector bundle stack first appeared in [BF97].

Definition 3.1.2 A **vector bundle stack** over a stack $\mathcal{X}$ is a morphism $\mathcal{V} \to \mathcal{X}$, such that exists an smooth cover $U \to \mathcal{X}$ and a two term complex of vector bundles $V_0 \to V_1$ on U and an isomorphism $[V_1/V_0] \simeq \mathcal{V} \times_{\mathcal{X}} U$.

Note that when we write V_1/V_0 we interpret $\mathbb{V}(V_0^\vee)$ as an additive group acting on $\mathbb{V}(V_1^\vee)$ and we secretly mean $[\mathbb{V}(V_1^\vee)/\mathbb{V}(V_0^\vee)]$. We could also work with a more restrictive notion of vector bundle stacks, asking for étale or even open covers, but this looser definition is enough for our purposes. The most important example of a vector bundle stack is the following.

Definition 3.1.3 Denote by $\mathsf{SES}(\mathcal{C})$ the stack of short exact sequences of coherent sheaves i.e. the objects over T are given by a triple $\mathcal{E}, \mathcal{F}, \mathcal{G}$ of coherent sheaves on $\mathcal{C} \times T$, all flat over T together with a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0.$$

The morphisms are morphisms of short exact sequences.

When we consider $\mathsf{SES}(\mathcal{C})$ as a stack over $\mathsf{Coh}(\mathcal{C}) \times \mathsf{Coh}(\mathcal{C})$ via the forgetful map that forgets everything except for the outer two sheaves we get a different perspective of the objects. Namely for an object $T \to \mathsf{Coh}(\mathcal{C}) \times \mathsf{Coh}(\mathcal{C})$ corresponding to the pair of sheaves $(\mathcal{E}, \mathcal{G})$ on $\mathcal{C} \times T$ we see that $(\mathsf{SES}(\mathcal{C}) \times_{\mathsf{Coh}(\mathcal{C}) \times \mathsf{Coh}(\mathcal{C})} T)(T)$ consists of short exact sequences $\mathcal{E}' \to \mathcal{F}' \to \mathcal{G}'$ together with isomorphisms $\mathcal{E} \simeq \mathcal{E}'$ and $\mathcal{G} \simeq \mathcal{G}'$. The morphisms are morphisms of short exact sequences $(\mathcal{E}' \to \mathcal{F}' \to \mathcal{G}') \to (\mathcal{E}'' \to \mathcal{F}'' \to \mathcal{G}'')$ that respect the isomorphisms on the outer terms. In other words the objects are extensions and the morphisms are morphisms of extensions. This implies in particular that the fibers of $\mathsf{SES}(\mathcal{C}) \to \mathsf{Coh}(\mathcal{C}) \times \mathsf{Coh}(\mathcal{C})$ are given by $[\mathsf{Ext}^1(\mathcal{G}, \mathcal{E}) / \mathsf{Ext}^0(\mathcal{G}, \mathcal{E})]$. This is why this stack is sometimes said to be "the stack classifying extensions", see for example [GHS14, Section 3]. Strictly speaking this is incorrect, since morphisms of extensions are more restrictive than morphisms of short exact sequences.

Theorem 3.1.4 The forgetful map $p\colon \mathsf{SES}(\mathcal{C}) \to \mathsf{Coh}(\mathcal{C}) \times \mathsf{Coh}(\mathcal{C})$, sending a short exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ to the pair $(\mathcal{E}, \mathcal{G})$ is a vector bundle stack.

Proof. Pick an ample sheaf $\mathcal{O}_C(1)$ on C and consider the open substack $U_d \subset \mathsf{Coh}(\mathcal{C}) \times \mathsf{Coh}(\mathcal{C})$ consisting of pairs $(\mathcal{E}, \mathcal{G})$, such that $\mathcal{H}om(\mathcal{G}, \mathcal{E})(d)$ has no higher cohomology. Clearly the U_d cover $\mathsf{Coh}(\mathcal{C}) \times \mathsf{Coh}(\mathcal{C})$. Let $p\colon U_d \times \mathcal{C} \to U_d$ and $q\colon U_d \times \mathcal{C} \to \mathcal{C}$ denote the projection maps. Let $(\mathcal{E}_{\mathsf{univ}}, \mathcal{G}_{\mathsf{univ}})$ be the universal pair of sheaves on $U_d \times \mathcal{C}$ and set $Y := \mathcal{H}om(\mathcal{G}_{\mathsf{univ}}, \mathcal{E}_{\mathsf{univ}})$. We have a short exact sequence

$$0 \to Y \to Y(d) \to Q \to 0,$$

where Q is defined to be the quotient. We claim that $\mathsf{SES}(\mathcal{C})|_{U_d} \simeq [p_*Q/p_*Y(d)]$.

First of all notice that Q is the twist of Y by the relative effective divisor defined by $\mathcal{O}_{U_d} \to q^*\mathcal{O}_{\mathcal{C}}(d)$, thus Q is flat over U_d. Applying Rp_* to the short exact sequence we get the long exact sequence

$$\begin{aligned} 0 \to R^0p_*Y \to R^0p_*Y(d) \to R^0p_*Q \to R^1p_*Y \to \\ \to R^1p_*Y(d) \to R^1p_*Q \to 0. \end{aligned}$$

From the definition of U_d it follows that $R^1p_*Y(d) = 0$ and hence also $R^1p_*Q = 0$. It follows that $R^0p_*Y(d) = p_*Y(d)$ and $R^0p_*Q = p_*Q$. By the cohomology and base change theorem [Hal14, Theorem A], it follows that $p_*Y(d)$ and p_*Q are vector bundles.

Let T be an affine scheme and $t\colon T \to U_d$ correspond to an object $(\mathcal{E}, \mathcal{G})$, then by [GHS14, Proposition 3.1] the objects of $([p_*Q/p_*Y(d)] \times_{U_d} T)(T)$ are given by

$$H^1(T, t^*\left(R^0p_*Y(d) \to R^0p_*Q\right)) = t^*R^1p_*Y = R^1p'_*t'^*Y = \mathsf{Ext}^1(\mathcal{G}, \mathcal{E})$$

and the morphisms are given by

$$H^0(T, t^*\left(R^0p_*Y(d) \to R^0p_*Q\right)) = \mathsf{Ext}^0(\mathcal{G}, \mathcal{E}).$$

By the discussion above we have $\mathsf{SES}(\mathcal{C})|_{U_d} \simeq [p_*Q/p_*Y(d)]$. ◯

It follows that $\mathsf{SES}(\mathcal{C})$ is also an Artin stack, locally of finite presentation over $\mathsf{Spec}(k)$. The forgetful map $\mathsf{SES}(\mathcal{C}) \to \mathsf{Coh}(\mathcal{C}) \times \mathsf{Coh}(\mathcal{C})$ also lets us define many natural variants of $\mathsf{SES}(\mathcal{C})$ coming from the different substacks of $\mathsf{Coh}(\mathcal{C})$ defined before. We note that the proof given in [GHS14, Corollary 3.2] also works perfectly well for stacky curves. We give this alternative proof because we think the ideas can be useful in other situations as well, such as in the construction of Section 4.3.

Definition 3.1.5 We define $\mathsf{SES}_{(n_1,\underline{d}_1),(n_2,\underline{d}_2)}(\mathcal{C})$ to be the fiber product

$$\begin{tikzcd} \mathsf{SES}_{(n_1,\underline{d}_1),(n_2,\underline{d}_2)}(\mathcal{C}) \arrow[r] \arrow[d] & \mathsf{SES}(\mathcal{C}) \arrow[d] \\ \mathsf{Coh}_{n_1,\underline{d}_1}(\mathcal{C}) \times \mathsf{Coh}_{n_2,\underline{d}_2}(\mathcal{C}) \arrow[r] & \mathsf{Coh}(\mathcal{C}) \times \mathsf{Coh}(\mathcal{C}). \end{tikzcd}$$

Equivalently $\mathsf{SES}_{(n_1,\underline{d}_1),(n_2,\underline{d}_2)}(\mathcal{C})$ is the stack of short exact sequences, where we specify the invariants of the first and last term. By construction the projection to $\mathsf{Coh}_{n_1,\underline{d}_1}(\mathcal{C}) \times \mathsf{Coh}_{n_2,\underline{d}_2}(\mathcal{C})$ is again a vector bundle stack.

Smoothness

We will study the smoothness of the stacks defined above using the tangent bundle stack. We take the definition as in [LM00, Définition 17.13]

Definition 3.1.6 Let $D \coloneqq \mathsf{Spec}\left(k[\epsilon]/\epsilon^2\right)$ be the spectrum of the dual numbers. For a stack T, we set $T[\epsilon] \coloneqq T \times D$. We denote the natural maps by $\iota\colon T \to T[\epsilon]$ and $\rho\colon T[\epsilon] \to T$.

Let $\mathcal{X}$ be an algebraic stack. We define the **tangent bundle** $\mathrm{T}_{\mathcal{X}}$ of $\mathcal{X}$ by setting $\mathrm{T}_{\mathcal{X}}(T) \coloneqq \mathcal{X}(T[\epsilon])$. The tangent bundle comes with a natural projection $\mathrm{T}_{\mathcal{X}} \to \mathcal{X}$ and a zero section $\mathcal{X} \to \mathrm{T}_{\mathcal{X}}$ induced by the maps ι and ρ respectively.

Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of stacks, then there is a natural morphism $\mathrm{T}_{\mathcal{X}} \to \mathrm{T}_{\mathcal{Y}}$ and we define the **relative tangent bundle** to be $\mathrm{T}_{\mathcal{X}} \times_{\mathrm{T}_{\mathcal{Y}}} \mathcal{Y}$.

Classically smoothness is closely related to the tangent bundle being a vector bundle; this generalizes nicely to algebraic stacks when we consider vector bundle stacks instead.

Proposition 3.1.7 Let $\mathcal{X}$ be a reduced algebraic stack locally of finite presentation over an algebraically closed field k; then $\mathcal{X}$ is smooth if and only if $\mathrm{T}_{\mathcal{X}}$ is a vector bundle stack.

Proof. Take a smooth atlas $u\colon X \to \mathcal{X}$. By the proof of [LM00, Théorème 17.16], we have

$$u^* \mathrm{T}_{\mathcal{X}} \simeq [\mathbb{V}(\Omega_{X/k})/\mathbb{V}(\Omega_{X/\mathcal{X}})].$$

Assume $\mathcal{X}$ is smooth, then $X \to \mathcal{X}$ and $X \to \mathsf{Spec}(k)$ are smooth and we have that $\Omega_{X/\mathcal{X}}$ and $\Omega_{X/k}$ are locally free, so this presents $\mathrm{T}_{\mathcal{X}}$ as a quotient of vector bundles.

Assume $\mathrm{T}_{\mathcal{X}}$ is a vector bundle stack, then so is $u^* \mathrm{T}_{\mathcal{X}}$ and

$$\mathtt{rank}\, \mathbb{V}(\Omega_{X/k}) - \mathtt{rank}\, \mathbb{V}(\Omega_{X/\mathcal{X}})$$

is constant. Since $\Omega_{X/\mathcal{X}}$ is locally free it follows that $\mathtt{rank}\, \mathbb{V}(\Omega_{X/k})$ is constant, so X is smooth. ⬠

We will now compute the tangent bundle of **Coh** explicitly.

> **Theorem 3.1.8** The tangent bundle $\mathrm{T}_{\mathsf{Coh}(\mathcal{C})}$ is isomorphic to the stack of short exact sequences $\mathcal{E} \to \widetilde{\mathcal{E}} \to \mathcal{E}$, where the outer two terms are explicitly identified. The morphisms are morphisms of short exact sequences that respect the identification of the outer terms. In other words, we have the following 2-Cartesian square.
>
> $$\begin{tikzcd} \mathrm{T}_{\mathsf{Coh}(\mathcal{C})} \arrow[r] \arrow[d] & \mathsf{SES}(\mathcal{C}) \arrow[d] \\ \mathsf{Coh}(\mathcal{C}) \arrow[r, "\Delta"] & \mathsf{Coh}(\mathcal{C}) \times \mathsf{Coh}(\mathcal{C}) \end{tikzcd}$$
>
> It follows that $\mathrm{T}_{\mathsf{Coh}\,\mathcal{C}}$ is a vector bundle stack.

Proof. Let $\mathcal{E} \in \mathrm{T}_{\mathsf{Coh}(\mathcal{C})}(T)$; then $\mathcal{E}$ is a $T[\epsilon]$-flat family of sheaves on $\mathcal{C} \times T[\epsilon]$. We can tensor $\mathcal{E}$ with the short exact sequence

$$\epsilon \mathcal{O}_T \to \mathcal{O}_{T[\epsilon]} \to \mathcal{O}_T$$

of $\mathcal{O}_{T[\epsilon]}$-modules to get a short exact sequence

$$\mathcal{E} \otimes \mathcal{O}_T \to \mathcal{E} \to \mathcal{E} \otimes \mathcal{O}_T$$

on $\mathcal{C} \times T[\epsilon]$. Then we can push this forward along ρ to get a short exact sequence on $\mathcal{C} \times T$.

Starting with a short exact sequence $\mathcal{E} \to \widetilde{\mathcal{E}} \to \mathcal{E}$ on $\mathcal{C} \times T$, we can take the inverse image $\rho^{-1}(\mathcal{E} \to \widetilde{\mathcal{E}} \to \mathcal{E})$, which is an exact sequence of $\rho^{-1}\mathcal{O}_T$-modules. Now $\rho^{-1}\widetilde{\mathcal{E}}$ obtains a $\mathcal{O}_{T[\epsilon]}$-module structure by defining the action of ϵ as $\rho^{-1}\widetilde{\mathcal{E}} \to \rho^{-1}\mathcal{E} \to \rho^{-1}\widetilde{\mathcal{E}}$.

We leave it to the reader to show that these two constructions give well defined functors that are inverse to each other. ⬠

Corollary 3.1.9 The stack $\mathsf{Coh}(\mathcal{C})$ is smooth, hence so are

$$\mathsf{Bun}(\mathcal{C}), \mathsf{SES}, \mathsf{Coh}_{\boldsymbol{\alpha}}, \mathsf{Bun}_{\boldsymbol{\alpha}}(\mathcal{C}), \mathsf{Bun}_{\boldsymbol{\alpha}}^{\mathcal{E}\text{-ss}}(\mathcal{C}), \mathsf{SES}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\,.$$

Using the Euler pairing (Definition 1.3.19) we compute the dimensions to be

$$\begin{aligned} \mathtt{dim}(\mathsf{Coh}_{\boldsymbol{\alpha}}(\mathcal{C})) &= -\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle, \\ \mathtt{dim}(\mathsf{SES}_{\boldsymbol{\alpha},\boldsymbol{\beta}}) &= -\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle - \langle \boldsymbol{\beta}, \boldsymbol{\beta} \rangle - \langle \boldsymbol{\beta}, \boldsymbol{\alpha} \rangle. \end{aligned}$$

We can give an explicit formula for the expression $\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle$ in terms of the rank, degree and multiplicities, but we think it is more informative to give the following upper and lower bound.

Proposition 3.1.10 Let $\boldsymbol{\alpha}$ be the numerical class of a vector bundle, then

$$(g_C - 1)\,\mathtt{rank}(\boldsymbol{\alpha})^2 \leq -\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle \leq (g_{\mathcal{C}} - 1)\,\mathtt{rank}(\boldsymbol{\alpha})^2.$$

Moreover, the left hand bound is attained whenever $\boldsymbol{\alpha} = [\pi^* F \otimes L]$, for some vector bundle F on C and a line bundle L on $\mathcal{C}$. The right hand bound is attained whenever the multiplicities are balanced, i.e. $m_{p,i}(\boldsymbol{\alpha}) = m_{p,j}(\boldsymbol{\alpha})$ for all p and i, j.

Proof. As $\boldsymbol{\alpha}$ is positive, we can choose a representative $F = \bigoplus_i \mathcal{L}_i$ of the class $\boldsymbol{\alpha}$, which is a sum of line bundles. As in Corollary 1.2.11, we can write $\mathcal{L}_i \simeq \pi^* L_i \otimes$

$\mathcal{O}(\sum_p \frac{a_{p,i}}{e_p} e_p)$. Then we calculate

$$\langle \mathcal{L}_i, \mathcal{L}_j \rangle = \left\langle \mathcal{O}_{\mathfrak{C}}, \pi^*(L_j \otimes L_i^\vee) \otimes \left(\sum_p \frac{a_{p,j} - a_{p,i}}{e_p} e_p \right) \right\rangle$$
$$= \langle \mathcal{O}_C, L_j \otimes L_i^\vee \rangle - \sum_{p \,:\, a_{p,j} < a_{p,i}} 1.$$

We have $m_{p,\ell}(\mathcal{L}_i) = 1$ if and only if $a_{p,i} = \ell$ and otherwise $m_{p,\ell}(\mathcal{L}_i) = 0$. Thus $m_{p,\ell}(F)$ counts the number of $a_{p,i}$'s such that $a_{p,i} = \ell$. Using the fact that the Euler pairing is additive, we obtain

$$-\langle F, F \rangle = (g_C - 1)\,\mathtt{rank}(F)^2 + \sum_p \sum_{j=0}^{e_p - 2} \left(m_{p,j}(F) \cdot \sum_{i=j+1}^{e_p - 1} m_{p,i}(F) \right).$$

The result now follows from the following combinatorial statement:
Let $n = \sum_{i=0}^{e-1} m_i$ be a partition of n into e terms. Then

$$S \coloneqq \sum_{i=0}^{e-2} \left(m_i \cdot \sum_{j=i+1}^{e} m_j \right) \leq \frac{e-1}{2e} n^2,$$

moreover the bound is attained precisely when $m_i = n/e$ for every i. To see this note that

$$2S + \sum_{i=0}^{e-1} m_i^2 = \left(\sum m_i \right)^2 = n^2,$$

so S is maximal when $\sum m_i^2$ is minimal. This happens when $m_i = \frac{n}{e}$, in which case we find $2S + e\frac{n}{e}^2 = n^2$ or $S = \frac{e-1}{2e} n^2$. ⬠

Connected components

The following theorem will show that our discrete invariants really are the discrete invariants, i.e. they uniquely identify a connected component of **Coh**. Since **Coh** is smooth the connected components are the same as the irreducible components. First we show the result for $\mathbf{Coh}_{0,\underline{d}}$ using the interpretation of torsion sheaves as quiver representations 1.2.23.

Lemma 3.1.11 The stack of torsion sheaves $\mathsf{Coh}_{0,\underline{d}}$, with $\underline{d} \geq 0$, is irreducible.

Proof. We first show the case $\underline{d} = (1, \ldots, 1)$. Consider the open embedding

$$\iota\colon \ \mathsf{Coh}_{0,1}(C) \to \mathsf{Coh}_{0,\underline{1}}(\mathcal{C}),$$

given by $T \mapsto \pi^* T$. We claim that $\mathsf{Coh}_{0,1}(C)$ is dense inside $\mathsf{Coh}_{0,\underline{1}}(\mathcal{C})$. To see this we consult Fig. 3.1. The locus of torsion sheaves which are supported away from the stacky points is isomorphic to $\mathsf{Coh}_{0,1}(C - \{p_i\})$. On the right we see the zoomed in locus L_p of torsion sheaves supported at a stacky point p. This locus is isomorphic to $\mathsf{Spec}\big(k[x_1, \ldots, x_e]/(x_1 \cdot \dots \cdot x_{e_p} = 0)\big)/(\mathbb{G}_m)^{\times e_p}$ using the quiver interpretation. Now the family

$$\mathtt{coker}\left(\mathcal{O}_{\mathcal{C}}(-q + \frac{a}{e}p) \to \mathcal{O}_{\mathcal{C}}(\frac{a}{e}p)\right), q \to p$$

converges to the point corresponding to the orbit $x_a = 0$ and $\prod_{b \neq a} x_b \neq 0$. (These are the outer points in the diagram). Since the union of these orbits lie dense in L_p the claim follows. Since $\mathsf{Coh}_{0,1}(C)$ is irreducible it follows that $\mathsf{Coh}_{0,\underline{1}}(\mathcal{C})$ is irreducible.

Next we consider the case $\underline{d}$ contains a zero degree $d_{p,i} = 0$, we see that all the sheaves must be supported at p. The corresponding quiver representations are automatically nilpotent as the i-th vector space is 0. It follows that $\mathsf{Coh}_{0,\underline{d}}$ is isomorphic to the quotient of an affine space, thus it is irreducible.

For the general case we proceed by induction. Let $\underline{d} = \underline{d'} + (1, \ldots, 1)$ and assume that $\mathsf{Coh}_{0,\underline{d'}}$ is irreducible. There are maps,

$$\begin{tikzcd} \mathsf{SES}_{(0,\underline{d'}),(0,(1,\ldots,1))} \arrow[r] \arrow[d] & \mathsf{Coh}_{0,\underline{d}} \\ \mathsf{Coh}_{0,\underline{d'}} \times \mathsf{Coh}_{0,(1,\ldots,1)} & \end{tikzcd}$$

where the vertical arrow is a vector bundle stack, so $\mathsf{SES}_{(0,\underline{d'}),(0,(1,\ldots,1))}$ is irreducible. The horizontal arrow is surjective, so $\mathsf{Coh}_{0,\underline{d}}$ is irreducible for every $\underline{d}$ by induction. ⬠

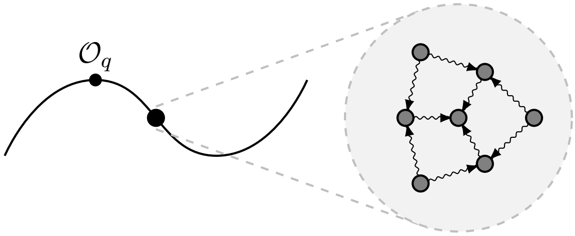


Figure 3.1: The topology of $\mathsf{Coh}_{0,(1,1,1)}(\mathcal{C})$ for a curve $\mathcal{C}$ with a stacky point of order 3. Generically $\mathsf{Coh}_{0,(1,1,1)}(\mathcal{C})$ looks like C, via the correspondence $q \mapsto \mathcal{O}_q$. A more complicated specialization structure appears at the stacky point p and is described further in Figure 3.2.

$$\begin{tikzcd}
& \mathcal{O}_p \arrow[dl] \arrow[d] & \mathcal{O}_p(\tfrac{1}{3}p) \arrow[dll] \arrow[dr] & \mathcal{O}_p(\tfrac{2}{3}p) \arrow[dl] \arrow[d] \\
\mathcal{O}_{\frac{2}{3}p} \oplus \mathcal{O}_{\frac{1}{3}p}(\tfrac{2}{3}p) \arrow[dr] & \mathcal{O}_{\frac{2}{3}p}(\tfrac{1}{3}p) \oplus \mathcal{O}_{\frac{1}{3}p} \arrow[d] & & \mathcal{O}_{\frac{2}{3}p}(\tfrac{2}{3}p) \oplus \mathcal{O}_{\frac{1}{3}p}(\tfrac{1}{3}p) \arrow[dll] \\
& \mathcal{O}_{\frac{1}{3}p} \oplus \mathcal{O}_{\frac{1}{3}p}(\tfrac{1}{3}p) \oplus \mathcal{O}_{\frac{1}{3}p}(\tfrac{2}{3}p) & &
\end{tikzcd}$$

Figure 3.2: The specialization structure of sheaves supported at p with twisted degrees $(1, 1, 1)$. We will give a general method to obtain these specialization structures in Section 3.3.

Theorem 3.1.12 The stack $\mathsf{Coh}_{n,\underline{d}}$ is irreducible; hence so are the stacks $\mathsf{Bun}_{n,\underline{d}}$ and $\mathsf{Bun}^{\mathcal{E}\text{-ss}}_{n,\underline{d}}$, whenever they are non-empty.

Proof. By Lemma 3.1.11 the result holds for $n = 0$. We proceed by induction on the rank n. Consider the maps,

$$\begin{tikzcd}
\mathsf{SES}' \arrow[r] \arrow[d] & \mathsf{Coh}_{n,\underline{d}}(\mathcal{C}) \\
\mathsf{Coh}_{n-1,\underline{d}'-\underline{i}}(\mathcal{C}) \times \mathsf{Bun}_{1,i}(C) &
\end{tikzcd}$$

where SES' is the stack of short exact sequences of the form $\pi^* L \to \mathcal{F} \to \mathcal{G}$, where $L \in \mathsf{Bun}_{1,i}(C)$ and $\mathcal{G} \in \mathsf{Coh}_{n-1,\underline{d}'-i}(\mathcal{C})$. The vertical arrow is again a vector bundle stack, so SES' is irreducible by induction. As $i \to -\infty$ the images of the horizontal maps define a filtration by open substacks of $\mathsf{Coh}_{n,\underline{d}}$, each of which is irreducible, hence $\mathsf{Coh}_{n,\underline{d}}$ is irreducible. ⬠

3.2 Parabolic Moduli and Flag bundles

The goal of this section is to generalize the categorical equivalence between parabolic bundles on classical curves and bundles on stacky curves of Theorem 1.4.9 to an equivalence of stacks. As a consequence we will see that stacks of vector bundles on a stacky curve are iterated flag bundles over stacks of vector bundles on the coarse curve. We start by introducing the stack of quasi-parabolic vector bundles.

Definition 3.2.1 Let $\mathcal{C}$ be a smooth projective stacky curve and $\underline{p}$ be a collection of non-stacky points, $\underline{e}$ be corresponding (parabolic) lengths and $\underline{m}$ be a set of (parabolic) multiplicities. We define the stack of quasi-parabolic bundles $\mathtt{QPar}^{\underline{p},\underline{e},\underline{m}}(\mathcal{C})$ whose objects over T are pairs $(\mathcal{F}, \mathcal{F}_\bullet)$, where $\mathcal{F}$ is an object of $\mathsf{Bun}(\mathcal{C})(T)$ and $\mathcal{F}_\bullet$ is a set of filtrations for each $p \in \underline{p}$

$$\mathcal{F} = \mathcal{F}_0^p \supseteq \mathcal{F}_1^p \supseteq \cdots \supseteq \mathcal{F}_e^p = \mathcal{F} \otimes \mathcal{O}_{\mathcal{C}}(-p \times T),$$

such that $\mathcal{F}_i^p / \mathcal{F}_{i+1}^p$ is flat over T and $\mathtt{rank}((\mathcal{F}_i^p/\mathcal{F}_{i+1}^p)|_p) = \underline{m}_{p,i}$. We note that the flatness condition guarantees that this rank is constant along T. The morphisms are the natural ones. We let $\mathtt{QPar}^{\underline{p},\underline{e},\underline{m}}_{n,d,\underline{m}'}$ be the substack where we fix the invariants of $\mathcal{F}$.

We can obtain a natural projection $\mathtt{QPar}^{\underline{p},\underline{e},\underline{m}}(\mathcal{C}) \to \mathsf{Bun}(\mathcal{C})$ by forgetting the quasi-parabolic structure. When we consider a single parabolic point it is a “well known fact” that the forgetful map is a fibration by flag varieties. We will make precise what this means and explain how to generalize the result to the case with more then one parabolic point.

Definition 3.2.2 Let $\mathcal{V}$ be a vector bundle on a stack $\mathcal{X}$ of rank n and $\underline{m} \in \mathbb{N}^e_{\geq 0}$, such that $\sum_{m_i \in \underline{m}} m_i = n$. A **flag of type** $\underline{m}$ is a filtration by subbundles $\mathcal{V} = \mathcal{V}_0 \supset \mathcal{V}_1 \supset \cdots \supset \mathcal{V}_e = 0$, such that the successive quotients $\mathcal{V}_i/\mathcal{V}_{i+1}$ are vector bundles of $\mathtt{rank}(\mathcal{V}_i/\mathcal{V}_{i+1}) = m_i$.

We denote by $\mathtt{Flag}_{\underline{m}}(\mathcal{V}) \to \mathcal{X}$ the **flag bundle stack** of type $\underline{m}$ associated to $\mathcal{V}$, which is defined as follows. The objects over T are given by (x, F), where x is an object of $\mathcal{X}(T)$ and F is a flag of $x^*\mathcal{V}$, such that the successive quotients are flat over T and for every $t \in T$ the flag F_t has type $\underline{m}$.

Applying the definition to the most simple situation we recover flag varieties.

Example 3.2.3 Taking $\mathcal{X} = \mathtt{Spec}(k)$ and $\mathcal{V} = k^n$, the stack $\mathtt{Flag}_{\underline{m}}(k^n)$ is a smooth projective variety called a (partial) flag variety. In general we have $\mathtt{Flag}_{\underline{m}}(\mathcal{O}^n_{\mathcal{X}}) \simeq \mathtt{Flag}_{\underline{m}}(k^n) \times \mathcal{X}$.

We can always take a Zariski local covering $U \to \mathcal{X}$ that trivializes the vector bundle $\mathcal{V}$. Then we have $\mathtt{Flag}_{\underline{m}}(\mathcal{V}) \times_{\mathcal{X}} U \simeq \mathtt{Flag}_{\underline{m}}(k^n) \times U$. In other words flag bundle stacks are always (Zariski-local) fibrations by flag varieties.

Lemma 3.2.4 Let $\mathcal{C}$ be a stacky curve and p be a non-stacky point on $\mathcal{C}$. Let $\mathcal{E}_{\text{univ}}$ be the universal vector bundle on $\mathsf{Bun}(\mathcal{C}) \times \mathcal{C}$. There is an isomorphism

$$\mathtt{QPar}^{p,e,\underline{m}_p}(\mathcal{C}) \simeq \mathtt{Flag}_{\underline{m}_p}(p^*\mathcal{E}_{\text{univ}})$$

as stacks over $\mathsf{Bun}(\mathcal{C})$.

Proof. Note that an object of $\mathtt{Flag}_{\underline{m}_p}(p^*\mathcal{E}_{\text{univ}})(T)$ consist of a vector bundle $\mathcal{F}$ on $\mathcal{C} \times T$, together with a flag of the vector bundle $p^*\mathcal{F}$ over T. Let

$$\phi\colon \mathtt{QPar}^{p,e,\underline{m}_p}(\mathcal{C}) \to \mathtt{Flag}_{\underline{m}}(p^*\mathcal{E}_{\text{univ}})$$

be defined by sending an object $(\mathcal{F}, \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \cdots \supseteq \mathcal{F}_e)$ to $(\mathcal{F}, (\mathcal{F}_0/\mathcal{F}_e)|_p \supseteq (\mathcal{F}_1/\mathcal{F}_e)|_p \supseteq \cdots \supseteq (\mathcal{F}_e/\mathcal{F}_e)|_p = 0)$. There is an inverse ψ defined by sending $(\mathcal{F}, F_0 \supseteq F_1 \supseteq \cdots \supseteq F_e)$ to the filtration $\mathcal{F}_0 \supseteq \cdots \supseteq \mathcal{F}_e$, where $\mathcal{F}_i$ is the kernel of $\mathcal{F} \to (F_0/F_i) \otimes \mathcal{O}_p$. We leave it to the reader to check that these two functors are actually inverse to each other. ◯

Theorem 3.2.5 Let $\mathcal{C}$ be a stacky curve, p be a non-stacky point of $\mathcal{C}$ and let $\mathcal{D} := \sqrt[e]{p/\mathcal{C}}$. Let $(n, d, \underline{m})$ be invariants for vector bundles on $\mathcal{D}$ and set $\underline{m}' = \underline{m} \setminus \underline{m}_p$. The functor $\mathtt{par}$ can be extended to an isomorphism of stacks.

$$\begin{tikzcd}
\mathsf{Bun}_{n,d,\underline{m}}(\mathcal{D}) \arrow[rr, dotted, "\sim"] \arrow[dr] & & \mathtt{QPar}^{p,e,\underline{m}_p}_{n,d,\underline{m}'}(\mathcal{C}) \arrow[dl] \\
& \mathsf{Bun}_{n,d,\underline{m}'}(\mathcal{C}) &
\end{tikzcd}$$

Proof. By [Nir09, Lemma 7.9] the functor $\mathtt{par}$ of Definition 1.4.3 and its inverse send flat families to flat families whenever $\mathcal{C}$ is a scheme, however the proofs still apply when $\mathcal{C}$ is a DM-stack. By Proposition 1.4.6 the multiplicities are preserved. ◯

Corollary 3.2.6 Let $\mathcal{C}$ be a stacky curve with stacky points $p_1, \dots p_l$ and let $\pi\colon \mathcal{C} \to C$ be the coarse space map. Let $(n, d, \underline{m})$ be discrete invariants on $\mathcal{C}$. The induced map of moduli stacks

$$\pi_* : \mathsf{Bun}_{n,d,\underline{m}}(\mathcal{C}) \to \mathsf{Bun}_{n,d}(C),$$

is an iterated flag bundle. Explicitly there is a factorization

$$\mathsf{Bun}_{n,d,\underline{m}}(\mathcal{C}) = B_l \to B_{l-1} \to \cdots \to B_0 = \mathsf{Bun}_{n,d} C,$$

such that the maps $B_i \to B_{i-1}$ are Zariski locally of the form

$$U \times \mathtt{Flag}_{\underline{m}_{p_i}}(k^n) \to U.$$

Proof. By viewing $\mathcal{C}$ as an iterated root stack over C as in Example 1.1.38, we can apply Lemma 3.2.4 to Theorem 3.2.5 iteratively. ◯

Corollary 3.2.7 Let C be a curve, $\underline{p}$ be a set of points, $\underline{e}$ be a set of lengths, $\underline{m}$ be a set of parabolic multiplicities and $\underline{\alpha}$ be a set of parabolic weights. Consider the open substack

$$^{\underline{\alpha}-\mathrm{ss}}\mathtt{QPar}_{n,d}^{\underline{p},\underline{e},\underline{m}}(C) \subset \mathtt{QPar}_{n,d}^{\underline{p},\underline{e},\underline{m}}(C)$$

of bundles that are semistable when endowed with the weights $\underline{\alpha}$. Then there exists a generating sheaf $\mathcal{E}$ on $\mathcal{C} = \sqrt[\underline{e}]{\underline{p}/C}$ such that

$$^{\underline{\alpha}-\mathrm{ss}}\mathtt{QPar}_{n,d}^{\underline{p},\underline{e},\underline{m}}(C) \simeq \mathsf{Bun}_{n,d,\underline{m}}^{\mathcal{E}\text{-ss}}(\mathcal{C}).$$

Proof. Applying Theorem 3.2.5 iteratively we see $\mathtt{QPar}_{n,d}^{\underline{p},\underline{e},\underline{m}}(C) \simeq \mathsf{Bun}_{n,d,\underline{m}}(\mathcal{C})$, and by Lemma 1.4.8 this isomorphism respects semistability. ◯

3.3 Stratifications

In this section we will describe several stratifications on the connected components of $\mathsf{Coh}_{n,\underline{d}}$. The goal is to stratify this stack into completely elementary parts, e.g. smooth curves and their symmetric products, lines and classifying spaces of algebraic groups. As a consequence we will be able to make qualitative statements about the Voevodsky motive of Coh. We start of by recalling some basic definitions and lemmas about stratifications. We will follow the naming conventions of [Stacks, Section 09XY].

Definition 3.3.1 Let $\mathcal{X}$ be a stack. A **partition** of $\mathcal{X}$ is a collection of locally closed substacks $\mathcal{X}_i \to \mathcal{X}$, indexed by $i \in \mathbb{I}$ such that $\coprod_{i\in\mathbb{I}} \mathcal{X}_i \to \mathcal{X}$ is a bijection. A **stratification** of $\mathcal{X}$ is a partition, together with a partial order $\leq$ on $\mathbb{I}$, such that $\overline{\mathcal{X}_i} \subset \bigcup_{j\leq i} \mathcal{X}_j$. The locally closed substacks $\mathcal{X}_i$ are called **parts** or **strata**.

Note that this is a relatively weak notion of a stratification, in particular we **do not** require equality, $\overline{\mathcal{X}_i} = \bigcup_{j\leq i} \mathcal{X}_i$. We present several ways to construct more stratifications from some given stratifications.

Definition 3.3.2 Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of stacks and $\coprod_{i\in\mathbb{I}} \mathcal{X}_i \to \mathcal{X}$ a stratification. Denote $\mathcal{Y}_i := \mathcal{X}_i \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$, then the **pullback** $\coprod_{i\in\mathbb{I}} \mathcal{Y}_i \to \mathcal{Y}$ is also a stratification.

Let $\coprod_{a\in A} \mathcal{X}_a \to \mathcal{X}$ and $\coprod_{b\in B} \mathcal{X}_b \to \mathcal{X}$ be two stratifications. We define the **intersection** to be the stratification

$$\coprod_{(a,b)\in A\times B} \mathcal{X}_a \times_{\mathcal{X}} \mathcal{X}_b \to \mathcal{X},$$

where the ordering on $A \times B$ is the product order.

Let $\coprod_{i\in\mathbb{I}} \mathcal{X}_i \to \mathcal{X}$ be a stratification, together with stratifications $\coprod_{j\in\mathbb{I}_i} \mathcal{X}_{i,j} \to \mathcal{X}_i$ of each stratum. We define the **refinement** to be the stratification

$$\coprod_{(i,j)\in\bigcup_{i\in\mathbb{I}}\{i\}\times\mathbb{I}_i} \mathcal{X}_{i,j} \to \mathcal{X},$$

where the ordering on $\bigcup_{i\in\mathbb{I}}\{i\} \times \mathbb{I}_i$ is the lexicographical ordering (reading i first and then j).

It is an elementary exercise in topology to check that these are all well defined stratifications. For any stratification we can consider the map $|\mathcal{X}| \to \mathbb{I}$, which sends a point to the index of the strata it is in. When we endow $\mathbb{I}$ with the upper topology (generated by the closed sets $\{j \mid j \leq i\}$), this map is continuous if and only if the sets $\coprod_{j\leq i} X_j$ are closed. This happens for example when the stratification is **good**, i.e. we have equality $\overline{\mathcal{X}_i} = \bigcup_{j\leq i} \mathcal{X}_j$. By [Stacks, Remark 09Y2] this is also the case whenever the stratification is locally finite in the following sense.

Definition 3.3.3 Let $\coprod_{i\in\mathbb{I}} \mathcal{X}_i \to \mathcal{X}$ be a stratification. We say that the stratification is **locally finite** if for every $x \in \mathcal{X}$ there exists an open subset $U \subset \mathcal{X}$ such that the pullback of this stratification to U has finitely many non-empty strata.

The pullback of a locally finite stratification is locally finite by construction and the intersection of locally finite stratifications is again locally finite. A refinement of a locally finite stratification $\mathbb{I}$ by finite $\mathbb{I}_i$ is also locally finite.

The following technical lemma lets us descend stratifications along finite group quotients.

Lemma 3.3.4 Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks that is a torsor for a finite group Γ. Let $\coprod_{i\in\mathbb{I}} \mathcal{X}_i \to \mathcal{X}$ be a (locally finite) stratification, such that Γ permutes the strata and the action descends to an order preserving action on $\mathbb{I}$. Then the orbit space $\mathbb{I}/\Gamma$ admits a partial order by setting

$$j\Gamma \leq i\Gamma \Leftrightarrow \exists \gamma \in \Gamma : \gamma(j) \leq i.$$

We set $\mathcal{Y}_{i\Gamma} := (\mathcal{X}_i \cdot \Gamma)/\Gamma = f(\mathcal{X}_i)$, then $\coprod_{i\Gamma\in\mathbb{I}/\Gamma} \mathcal{Y}_{i\Gamma} \to \mathcal{Y}$ is a (locally finite) stratification.

Proof. Since Γ is finite, $\mathcal{X}_i \cdot \Gamma$ is locally closed in $\mathcal{X}$ and since $\mathcal{X} \to \mathcal{Y}$ is open, $(\mathcal{X}_i \cdot \Gamma)/\Gamma$ is locally closed in $\mathcal{Y}$. Next we notice that

$$\overline{\mathcal{Y}_{\Gamma_i}} = \overline{f(\mathcal{X}_i \cdot \Gamma)} = f(\overline{\mathcal{X}_i} \cdot \Gamma) \subset f\left(\coprod_{j\leq i} \mathcal{X}_j \cdot \Gamma\right) = \coprod_{j\Gamma\leq i\Gamma} \mathcal{Y}_{j\Gamma}.$$

Finally assume $\coprod_{i\in\mathbb{I}} \mathcal{X}_i \to \mathcal{X}$ is locally finite. Let $U \subset \mathcal{X}$ be an open containing finitely many strata, then $U \cdot \Gamma$ is also an open containing finitely many strata, and $(U \cdot \Gamma)/\Gamma = f(U)$ is an open of $\mathcal{Y}$ containing finitely many strata. It follows that $\coprod_{i\Gamma\in\mathbb{I}/\Gamma} \mathcal{Y}_{i\Gamma} \to \mathcal{Y}$ is locally finite. ⬠

We now construct two basic stratifications of the symmetric power $C^{(d)}$ which will be used later.

Example 3.3.5 Let p be a point on a classical curve C. We can stratify C into two parts

$$C = C \setminus \{p\} \amalg \{p\}.$$

We can pull back this stratification along the n projection maps $C^n \to C$ and intersect these to get a stratification of C^n. The natural action of S_n on C^n permutes the strata, so we obtain a stratification of $[C^n/S_n]$. The coarse space morphism $[C^n/S_n] \to C^{(n)}$ is a homeomorphism, so we obtain a stratification of $C^{(n)}$. The strata are given by $(C \setminus \{p\})^{(l)} \to C^{(n)}$ for $l \leq n$, where the map is defined by $q_1 + \cdots + q_l \mapsto (n-l)p + q_1 + \cdots + q_l$.

Example 3.3.6 Let $\Delta \subset C^2$ be the diagonal, so that we have a stratification

$$C^2 = \Delta \amalg C^2 \setminus \Delta.$$

We will inductively construct a stratification on C^n. Consider the n maps $C^n \to C^{n-1}$ obtained by forgetting one coordinate. We pullback the stratification on C^{n-1} along the different maps and intersect them to obtain a stratification of C^n.

The open stratum consists of tuples of n **distinct** points and we will denote it by $(C^n)^\circ$. The natural S_n action again permutes the strata, so we also obtain stratifications of $[C^n/S_n]$ and $C^{(n)}$ for every n. The strata are indexed by (unordered) partitions $n = n_1 + \cdots + n_l$, with $n_1 \geq \ldots \geq n_l$ and are given by the images of

$$(C^l)^\circ \to C^{(n)} \colon (p_1, \ldots, p_l) \mapsto n_1 p_1 + \cdots + n_l p_l.$$

The strata in $C^{(n)}$ themselves are isomorphic to the free quotient $(C^l)^\circ/\Gamma$, where Γ is the group which permutes the i-th and j-th coordinate if $n_i = n_j$.

Stratification by torsion type

Every coherent sheaf on a stacky curve contains a unique maximal torsion subsheaf, and we will first stratify by the discrete invariants of this torsion subsheaf. This is a generalization of the stratification considered in [Hei12, Section 3].

Theorem 3.3.7 Let $\mathcal{C}$ be a stacky curve with stacky points $\underline{p}$ of order $\underline{e}$. Consider the partially ordered set $\mathbb{I} := \bigoplus_{p \in \underline{p}} \mathbb{N}^{e_p}$, where $\underline{c} \leq \underline{c}'$ if all the entries of $\underline{c}$ are less than or equal to the entries of $\underline{c}'$. Let $\mathsf{Coh}_{n,\underline{d}}^{\mathrm{tor}=\underline{c}}$ be the substack of coherent sheaves where the torsion part has twisted degrees $\underline{c}$. The decomposition

$$\coprod_{\underline{c} \in \mathbb{I}^{\mathrm{op}}} \mathsf{Coh}_{n,\underline{d}}^{\mathrm{tor}=\underline{c}}(\mathcal{C}) \to \mathsf{Coh}_{n,\underline{d}}(\mathcal{C}),$$

is a locally finite stratification.

Note that our partial order is the opposite of what you might expect, i.e. the maximal stratum, corresponding to $\underline{0}$, is the **open** stratum $\mathbf{Bun}_{n,\underline{d}} \subset \mathbf{Coh}_{n,\underline{d}}$.

Proof. Let $\mathcal{F}_{\mathsf{univ}}$ be the universal sheaf on $\mathbf{Coh}_{n,\underline{d}}(\mathcal{C}) \times \mathcal{C}$ and consider its torsion subsheaf $\mathcal{T} := (\mathcal{F}_{\mathsf{univ}})_{\mathsf{tor}} \subset \mathcal{F}_{\mathsf{univ}}$. Then our decomposition is simply the flattening stratification for $\mathcal{T}$ with respect to the projection $\mathbf{Coh}_{n,\underline{d}}(\mathcal{C}) \times \mathcal{C} \to \mathbf{Coh}_{n,\underline{d}}(\mathcal{C})$ as in Theorem A.2.8, so it is a well defined partition. A priori the flattening stratification is only a partition, so we are left to show that

$$\overline{\mathbf{Coh}_{n,\underline{d}}^{\mathsf{tor}=\underline{c}}} \subset \mathbf{Coh}_{n,\underline{d}}^{\mathsf{tor}\geq\underline{c}} := \bigcup_{\underline{c}' \leq_{\mathsf{op}} \underline{c}} \mathbf{Coh}_{n,\underline{d}}^{\mathsf{tor}=\underline{c}'} .$$

For $p \in \underline{p}$ and $0 \leq i < e_p$, consider the maps

$$\pi_{p,i} \colon \mathbf{Coh}_{n,\underline{d}}(\mathcal{C}) \to \mathbf{Coh}_{n,d_{p,i}}(C),$$

defined by $\mathcal{F} \mapsto \pi_* \left(\mathcal{F} \otimes \mathcal{O}_{\mathcal{C}} \left(\frac{i}{e_p} p \right) \right)$. By exactness of $\pi_{p,i}$ we have $(\pi_{p,i}\mathcal{F})_{\mathsf{tor}} = \pi_{p,i} \left(\mathcal{F}_{\mathsf{tor}} \right)$, so by continuity we get

$$\overline{\mathbf{Coh}_{n,\underline{d}}^{\mathsf{tor}=\underline{c}}(\mathcal{C})} \subset \pi_{p,i}^{-1} \overline{\pi_{p,i}(\mathbf{Coh}_{n,\underline{d}}^{\mathsf{tor}=\underline{c}})(\mathcal{C})}.$$

Notice that

$$\pi_{p,i}(\mathbf{Coh}_{n,\underline{d}}^{\mathsf{tor}=\underline{c}}(\mathcal{C})) \subset \mathbf{Coh}_{n,d_{p,i}}^{\mathsf{tor}=c_{p,i}}(C) \quad \text{and} \quad \overline{\mathbf{Coh}_{n,d_{p,i}}^{\mathsf{tor}=c_{p,i}}(C)} = \mathbf{Coh}_{n,d_{p,i}}^{\mathsf{tor}\geq c_{p,i}}(C),$$

so

$$\overline{\mathbf{Coh}_{n,\underline{d}}^{\mathsf{tor}=\underline{c}}(\mathcal{C})} \subset \bigcap_{p,i} \pi_{p,i}^{-1} \mathbf{Coh}_{n,d_{p,i}}^{\mathsf{tor}\geq c_{p,i}}(C) = \bigcup_{\underline{c}' \leq_{\mathsf{op}} \underline{c}} \mathbf{Coh}_{n,\underline{d}}^{\mathsf{tor}=\underline{c}'}(\mathcal{C}).$$

Finally $\pi_{p,i}^{-1} \mathbf{Coh}_{0,d}^{\mathsf{tor}\leq c}(C)$ is open for any d and c, so $\mathbf{Coh}_{0,\underline{d}}^{\mathsf{tor}\leq\underline{c}}(\mathcal{C})$ is open and contains finitely many strata. ⬡

Each stratum admits the structure of a vector bundle stack as follows.

Proposition 3.3.8 The map $\mathsf{Coh}(\mathcal{C})_{n,\underline{d}}^{\mathrm{tor}=\underline{c}} \to \mathsf{Coh}_{0,\underline{c}}(\mathcal{C}) \times \mathsf{Bun}_{n,\underline{d}-\underline{c}}(\mathcal{C})$ sending a family of sheaves $\mathcal{F}$ to the pair $(\mathcal{F}_{\mathrm{tor}}, \mathcal{F}_{\mathrm{free}} := \mathcal{F}/\mathcal{F}_{\mathrm{tor}})$ is a vector bundle stack. In particular each stratum $\mathsf{Coh}(\mathcal{C})_{n,\underline{d}}^{\mathrm{tor}=\underline{c}}$ is smooth.

Proof. We have a commutative square,

$$\begin{tikzcd}
\mathsf{Coh}_{n,\underline{d}}^{\mathrm{tor}=\underline{c}} \arrow[r] \arrow[d] & \mathsf{SES} \arrow[d] \\
\mathsf{Coh}_{0,\underline{c}} \times \mathsf{Bun}_{n,\underline{d}-\underline{c}} \arrow[r] & \mathsf{Coh} \times \mathsf{Coh}
\end{tikzcd}$$

where the top arrow sends a family $\mathcal{F}$ to the short exact sequence $\mathcal{F}_{\mathrm{tor}} \hookrightarrow \mathcal{F} \twoheadrightarrow \mathcal{F}_{\mathrm{free}}$. Notice that $\mathcal{F}_{\mathrm{tor}}$ is a flat family, by the universal property of the flattening stratification, so the maps are well defined. We claim that this is a 2-Cartesian square. This is true since any isomorphism $\mathcal{F} \to \mathcal{F}'$ of sheaves restricts to an isomorphism $\mathcal{F}_{\mathrm{tor}} \to \mathcal{F}'_{\mathrm{tor}}$ of torsion parts and thus lifts to an isomorphism of short exact sequences $(\mathcal{F}_{\mathrm{tor}} \hookrightarrow \mathcal{F} \twoheadrightarrow \mathcal{F}_{\mathrm{free}}) \to (\mathcal{F}'_{\mathrm{tor}} \hookrightarrow \mathcal{F}' \twoheadrightarrow \mathcal{F}'_{\mathrm{free}})$. The result follows as $\mathsf{SES} \to \mathsf{Coh} \times \mathsf{Coh}$ is a vector bundle stack by Theorem 3.1.4. ⬡

These stratifications and vector bundle results can be neatly summarized in the following motivic statement. (See Appendix B for our setup.)

Corollary 3.3.9 Let $\mathcal{C}$ be a projective stacky curve with coarse space C. The motive $M(\mathsf{Coh}_{n,\underline{d}}(\mathcal{C}))$ in $\mathtt{DM}(k, \mathbb{Q})$ lies in the thick tensor subcategory generated by $M(C)$ and $M(\mathsf{Coh}_{0,\underline{e}})$ for all $\underline{e} \geq 0$.

Proof. Applying Proposition B.2.8, to the stratification of Theorem 3.3.7, we see that $M(\mathsf{Coh}_{n,\underline{d}})$ lies in the category generated by $\mathsf{Coh}_{n,d}^{\mathrm{tor}=\underline{e}}$. Applying Example B.3.4 to Proposition 3.3.8 we see that

$$M(\mathsf{Coh}_{n,d}^{\mathrm{tor}=\underline{e}}) \simeq M(\mathsf{Coh}_{0,\underline{c}}(\mathcal{C})) \otimes M(\mathsf{Bun}_{n,\underline{d}-\underline{c}}(\mathcal{C})).$$

Applying B.2.5 to Lemma 3.2.4 we find

$$M(\mathsf{Bun}_{n,\underline{d}-\underline{c}}(\mathcal{C})) = M(\mathsf{Bun}_{n,d_0-c_0}(C)) \otimes \bigotimes_{p \in \underline{p}} M(\mathtt{Flag}_{\underline{m}_p}(k^n)),$$

where $\underline{m}_p$ are the multiplicities corresponding to the twisted degrees $\underline{d}_p - \underline{c}_p$. Now $M(\mathtt{Flag}_{\underline{m}_p}(k^n))$ is pure Tate [Hab12, Proposition 4.4.11] and $M(\mathsf{Bun}_{n,d_0-c_0}(C))$ lies in the thick tensor subcategory generated by $M(C)$ by [HP21b]. ◇

Stratification by support type

For a classic curve C the stack of torsion sheaves $\mathsf{Coh}_{0,d}(C)$ is well understood. For example Laumon constructs a proper small map $\widetilde{\mathsf{Coh}}_{0,d}(C) \to \mathsf{Coh}_{0,d}(C)$, where $\widetilde{\mathsf{Coh}}_{0,d}(C)$ is the stack containing filtrations of torsion sheaves $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \cdots \subset \mathcal{T}_d$, such that $\mathcal{T}_i$ has degree i [Lau87, Section 3]. This construction specializes to the Grothendieck-Springer resolution when $C = \mathbb{A}^1$. For a stacky curve $\mathcal{C}$, the geometry of $\mathsf{Coh}_{0,\underline{d}}(\mathcal{C})$ is more complicated, for example the natural analogue of the Grothendieck-Springer resolution is not small [Hei04]. In an attempt to describe the geometry of this stack we will completely stratify it into elementary parts: symmetric powers of (open) curves and classifying spaces of linear algebraic groups. We start by defining some relevant partially ordered sets.

Definition 3.3.10 Let d be a positive integer. A **marked** partition is a partition $d = \boldsymbol{n_0} + n_1 + n_2 + \cdots + n_l$, such that $n_1 \geq n_2 \geq \ldots \geq n_l$. The positive integer $\boldsymbol{n_0}$ is considered marked. We will say that a marked partition is smaller than a second marked partition if we can obtain the first by adding together parts of the second; when adding the marked part to another part the result is considered marked.

As an example we have given the marked partitions of 3 in Figure 3.3. The arrows point towards the smaller marked partition.

Definition 3.3.11 Let C be a classical curve with a marked point p. We define a stratification of the d-th symmetric power of C

$$\coprod_{\boldsymbol{n_0}+n_1+\cdots n_l=d} X_{\boldsymbol{n_0}+n_1+\cdots+n_l} \to C^{(d)},$$

indexed by marked partitions of d as follows. Let $(C^l)^\star \subset C^l$ be the open subset of l-tuples of distinct points, none of which are p. We define $X_{\boldsymbol{n_0}+n_1+\cdots+n_l}$

$$\begin{tikzcd}
 & & \mathbf{0}+1+1+1 \arrow[dl] \arrow[dr] & & \\
 & \mathbf{1}+1+1 \arrow[dl] \arrow[dr] & & \mathbf{0}+2+1 \arrow[lll] \arrow[dl] \arrow[dr] & \\
\mathbf{2}+1 \arrow[drr] & & \mathbf{1}+2 \arrow[d] & & \mathbf{0}+3 \arrow[dll] \\
 & & \mathbf{3} & &
\end{tikzcd}$$

Figure 3.3: The marked partitions of 3.

to be the image of $(C^l)^\star \to C^{(d)}$ given by

$$(q_1, \ldots, q_l) \mapsto n_0 p + n_1 q_1 + \cdots + n_l q_l.$$

Note that this is just the intersection of the stratifications of Example 3.3.5 and Example 3.3.6. We will use this stratification of $C^{(d)}$ to construct a stratification by support type for torsion sheaves.

Definition 3.3.12 Let $\mathcal{C}$ be a stacky curve with a single stacky point p of order e. Consider for $0 \leq i \leq e-1$ the "twisted support" maps $\mathsf{supp}_i\colon \mathsf{Coh}_{0,\underline{d}} \to C^{(d_i)}$, given by

$$\mathcal{T} \mapsto \mathsf{supp}\, \pi_* \left(\mathcal{T} \otimes \mathcal{O}_{\mathcal{C}} \left(\frac{i}{e} p \right) \right).$$

We define the stratification by **support type** to be the intersection over $0 \leq i < e$, of the pullbacks along supp_i of the stratifications of $C^{(d_i)}$ given in Definition 3.3.11. We will denote the strata by $\mathsf{Coh}^{\tau}_{0,\underline{d}}$ and a sheaf inside $\mathsf{Coh}^{\tau}_{0,\underline{d}}$ is said to have support type τ, where τ consists of marked partitions $d_i = \boldsymbol{n_{0,i}} + n_{1,i} + \cdots + n_{l_i,i}$ for $0 \leq i < e$.

Note that technically a support type τ consists of a set of e marked partitions, however the unmarked part of the partitions is not affected by twisting. In fact τ is determined by a single partition $n = n_1 + \cdots + n_l$, such that $n \leq \mathtt{min}_i(d_i)$. The marked parts can be recovered via $\boldsymbol{n_{0,i}} = d_i - n$ and $n_{j,i} = n_j$ for $0 \le i < e$ and $1 \le j \le l$.

In general the strata $\mathbf{Coh}^{\mathcal{T}}_{0,\underline{d}}$ are not smooth, in fact they contain many intersecting irreducible components. The next stratification will resolve this issue by further stratifying these irreducible components.

Stratification by graded Young diagrams

As a primer let us first discuss the relation between Young diagrams and torsion sheaves on classical curves.

> **Definition 3.3.13** A **Young diagram** Y of size d is a decreasing sequence of natural numbers $n_0, n_1, \ldots$ such that $\sum n_i = d$. Young diagrams of size d are partially ordered as follows
>
> $$Y \le Y' \Leftrightarrow \sum_{i=0}^{j} n_i \le \sum_{i=0}^{j} n'_i,$$
>
> for all j. We denote the set of Young diagrams of size d by $\mathbb{Y}_d$.

Let $\mathbf{Coh}_{0,d,q}(C)$ be the stack of degree d torsion sheaves on a classical curve C, supported at a fixed k-point q. By Proposition 1.2.23 we see that $\mathbf{Coh}_{0,d,q}(C)$ is isomorphic to the stack $\mathbf{NRep}_d(J)$, of nilpotent quiver representations of the Jordan quiver of dimension d. Representations of the Jordan quiver correspond to matrices up to conjugation and are thus in bijection with Jordan normal forms. The Jordan normal form of a nilpotent matrix is completely determined by its block sizes, so the points of this stack are in bijection with the Young diagrams of size d. In fact the bijection is very explicit. A torsion sheaf $\mathcal{T} = \bigoplus_i \mathcal{O}_{n_i q}$, simply gets sent to $n_0, n_1, \ldots$ up to reordering. Moreover the bijection $\mathbf{Coh}_{0,d,q}(C) \to \mathbb{Y}_d$ is a **homeomorphism**, when $\mathbb{Y}_d$ is endowed with the upper topology. The fact that this is a homeomorphism comes down to the fact that all specializations of torsion sheaves are iterations of the specialization

$$\mathcal{T} \oplus \mathcal{O}_{mq} \oplus \mathcal{O}_{nq} \rightsquigarrow \mathcal{T} \oplus \mathcal{O}_{(m-1)q} \oplus \mathcal{O}_{(n+1)q},$$

for $m \ge n+2$.

The above story has been generalized to the case of stacky points in [Kem82], using the fact that for a stacky point p of order e the stack $\mathsf{Coh}_{0,\underline{d},p}(\mathcal{C})$ is isomorphic to the stack of nilpotent quiver representations $\mathsf{NRep}_{\underline{d}}(Q_e)$ of the cyclic quiver Q_e with dimension vector $\underline{d}$. (See also [Joh10], for a more modern presentation.) The correct generalization of Young diagrams in this setting is the following.

> **Definition 3.3.14** An e-**graded Young diagram** is a decreasing sequence of positive numbers $n_i \in \mathbb{Z}$, together with a sequence of twists $\epsilon_i \in \mathbb{Z}/e\mathbb{Z}$. We take representatives $0 \leq \epsilon_i \leq e-1$ and we require that $n_i = n_{i+1} \Rightarrow \epsilon_i \leq \epsilon_{i+1}$ and if $n_i = 0 \Rightarrow \epsilon_i = 0$. We denote the set of e-graded Young diagrams with signature $\underline{d}$ by $\mathbb{Y}^e_{\underline{d}}$.

We think of e-graded Young diagrams as diagrams of marked boxes. In row i we draw n_i boxes and mark the last one with the twist ϵ_i. Working backwards in each row we mark the next box with $\epsilon_i+1, \epsilon_i+2, \ldots$ until we hit the start of the row. The **signature** $\underline{d}$ of an e-graded Young diagram is the vector $(d_0, d_1, \ldots, d_{e-1})$, where d_j is the number of times the marking j appears. See Figure 3.4 for an example. Kempken defines a partial order on $\mathbb{Y}^e_{\underline{d}}$ as follows. For $Y, Y' \in \mathbb{Y}^e_{\underline{d}}$ we say $Y \leq Y'$ if the first i columns of Y contain more markings j then the first i columns of Y', for every i and j.

2	1	0
0	2	1
1	0	
2		

Figure 3.4: The 3-graded Young tableau $\underline{n} = (3,3,2,1), \underline{\epsilon} = (0,1,0,2)$, with signature $\underline{d} = (3,3,3)$.

Let $\mathcal{C}$ be a stacky curve with a stacky point p of order e. There is a bijection $|\,\mathsf{Coh}_{0,\underline{d},p}\,| \to \mathbb{Y}^e_{\underline{d}}$. Namely we send

$$\mathsf{Coh}_{0,\underline{d},p} \ni \mathcal{T} \simeq \bigoplus_i \mathcal{O}_{\frac{n_i}{e}p} \otimes \mathcal{O}_{\mathcal{C}}(\frac{\epsilon_i}{e}p) \mapsto (n_i, \epsilon_i)_i.$$

Kempken shows that this bijection is in fact a homeomorphism when $\mathbb{Y}^e_{\underline{d}}$ is endowed with the upper topology [Kem82, 2.10 Korrolar 2]. In particular the decomposition of $\left| \mathsf{Coh}_{0,\underline{d},p} \right|$ into its points is a stratification. See Figure 3.5 for an example of this homeomorphism.

We are now ready to further stratify $\mathsf{Coh}^{\tau}_{0,\underline{d}}$.

Lemma 3.3.15 Let τ be a support type as in Definition 3.3.12 given by the partition $n = n_1 + \cdots + n_l$. Let Γ be the finite subgroup of S_l permuting i and j whenever $n_i = n_j$. Let $\widetilde{\mathsf{Coh}}^{\tau}_{0,\underline{d}} := \mathsf{Coh}^{\tau}_{0,\underline{d}} \times_{C^{(d_i)}} (C^l)^\star$, where $(C^l)^\star \to C^{(d_i)}$ is as in Definition 3.3.11, which is independent of the choice of $0 \leq i < e$. Then

$$\widetilde{\mathsf{Coh}}^{\tau}_{0,\underline{d}} \simeq (C^l)^\star \times \mathsf{NRep}_{\underline{d}-n\cdot\underline{1}}(Q_e) \times \mathsf{NRep}_{n_1}(J) \times \cdots \times \mathsf{NRep}_{n_l}(J)$$

and $\widetilde{\mathsf{Coh}}^{\tau}_{0,\underline{d}}$ can be stratified by taking the intersection of the pullback of the stratifications of $\mathsf{NRep}_{\underline{d}-n\cdot\underline{1}}(Q_e)$ and $\mathsf{NRep}_{n_i}(J)$ by (graded) Young diagrams. Moreover the natural Γ action on $\widetilde{\mathsf{Coh}}^{\tau}_{0,\underline{d}}$ permutes the strata in an order preserving way.

Proof. By construction $\widetilde{\mathsf{Coh}}^{\tau}_{0,\underline{d}}$ is the stack of torsion sheaves with support type τ, together with an ordering of the points in the support. The isomorphism then follows as $\mathsf{Coh}_{0,\underline{d},q}(\mathcal{C}) \simeq \mathsf{NRep}_{d_0}(J)$ is independent of $q \in \mathcal{C} \setminus p$. The action of Γ on $\mathbb{Y}^e_{\underline{d}-n\cdot\underline{1}} \times \mathbb{Y}_{n_1} \times \cdots \times \mathbb{Y}_{n_l}$ simply permutes Young diagrams of the same size, which is order preserving. ◯

By Lemma 3.3.4 we can stratify $\mathsf{Coh}^{\tau}_{0,\underline{d}}$ by **unordered** sets of (graded) Young diagrams

$$(Y_0, \{Y_1, \ldots, Y_n\}) \in \mathbb{Y}^e_{\underline{d}-n\cdot\underline{1}} \times (\mathbb{Y}_{n_1} \times \cdots \times \mathbb{Y}_{n_l}) / \Gamma.$$

See Figure 3.6 for a visualization of this stratification in the case of $\mathsf{Coh}_{0,(2,2)}(\mathcal{C})$.

Proposition 3.3.16 Let

$$\coprod \mathsf{Coh}_{Y_0,\{Y_1,\ldots,Y_l\}} \to \mathsf{Coh}_{0,\underline{d}},$$

be the refinement of the stratification by support type by the stratifications by unordered sets of (graded) Young diagrams. For each stratum there is a linear algebraic group G and finite group Γ such that

$$\mathsf{Coh}_{Y_0,\{Y_1,\dots,Y_l\}} \simeq \Big((C^l)^\star \times BG\Big)/\Gamma.$$

Proof. All the strata of $\mathsf{NRep}_n(J)$ and $\mathsf{NRep}_{\underline{d}}(Q_e)$ have a single point and since $\mathsf{NRep}_n(J)$ and $\mathsf{NRep}_{\underline{d}}(Q_e)$ are quotient stacks, the strata are isomorphic to BG for a linear algebraic group G. It follows that the strata of $\widetilde{\mathsf{Coh}}^\tau_{0,\underline{d}}$ are of the form $(C^l)^\star \times BG$ for some linear algebraic group G. By construction the strata of $\mathsf{Coh}^\tau_{0,\underline{d}}$ are of the form $\big((C^l)^\star \times BG\big)/\Gamma$ for a finite group Γ. ◯

Theorem 3.3.17 Let $\mathcal{C}$ be a tame stacky curve over an algebraically closed field $k = \bar{k}$. The motive $M(\mathsf{Coh}_{0,\underline{d}}(\mathcal{C}))$ in $\mathsf{DM}(k, \mathbb{Q})$ lies in the thick tensor subcategory generated by $M(C)$.

Proof. We only prove this in the case that $\mathcal{C}$ has a single stacky point. It should be clear that these arguments generalize to any amount of stacky points. The stratification by unordered (graded) Young diagrams shows that $M(\mathsf{Coh}_{0,\underline{d}}(\mathcal{C}))$ lies in the category generated by $M((C^l)^\star \times BG)/\Gamma)$. Since we are working with $\mathbb{Q}$-coefficients, the motive lies in the category generated by $M((C^l)^\star \times BG)$. As k is algebraically closed the motive $M(BG)$ is pure Tate by Example B.3.6 and it is clear that $M((C^l)^\star)$ lies in the category generated by $M(C)$. It follows that $M(\mathsf{Coh}_{0,\underline{d}}(\mathcal{C}))$ itself lies in the category generated by $M(C)$. ◯

Combining this result with Corollary 3.3.9 we obtain the following corollary.

Corollary 3.3.18 Let $\mathcal{C}$ be a tame stacky curve over an algebraically closed field $k = \bar{k}$. The motive $M(\mathsf{Coh}_{n,\underline{d}}(\mathcal{C}))$ in $\mathsf{DM}(k, \mathbb{Q})$ lies in the thick tensor subcategory generated by $M(C)$.

$$
\begin{tikzcd}[column sep=small]
& \begin{array}{|c|c|c|c|}\hline 0&1&0&1\\\hline\end{array} \arrow[ddl, bend right] \arrow[d] \arrow[drr] & & \begin{array}{|c|c|c|c|}\hline 1&0&1&0\\\hline\end{array} \arrow[dll] \arrow[d] \arrow[ddr, bend left] & \\
& \begin{array}{|c|c|c|}\hline 0&1&0\\\hline \multicolumn{1}{|c|}{1}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} \arrow[dr] & & \begin{array}{|c|c|c|}\hline 1&0&1\\\hline \multicolumn{1}{|c|}{0}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} \arrow[dl] & \\
\begin{array}{|c|c|}\hline 0&1\\\hline 0&1\\\hline\end{array} \arrow[dr] & & \begin{array}{|c|c|}\hline 0&1\\\hline 1&0\\\hline\end{array} \arrow[dl] \arrow[dr] & & \begin{array}{|c|c|}\hline 1&0\\\hline 1&0\\\hline\end{array} \arrow[dl] \\
& \begin{array}{|c|c|}\hline 0&1\\\hline \multicolumn{1}{|c|}{0}&\multicolumn{1}{c}{}\\\cline{1-1} \multicolumn{1}{|c|}{1}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} \arrow[dr] & & \begin{array}{|c|c|}\hline 1&0\\\hline \multicolumn{1}{|c|}{0}&\multicolumn{1}{c}{}\\\cline{1-1} \multicolumn{1}{|c|}{1}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} \arrow[dl] & \\
& & \begin{array}{|c|}\hline 0\\\hline 0\\\hline 1\\\hline 1\\\hline\end{array} & &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=tiny]
& \mathcal{O}_{2p}(\tfrac{1}{2}p) \arrow[ddl, bend right] \arrow[d] \arrow[drr] & & \mathcal{O}_{2p} \arrow[dll] \arrow[d] \arrow[ddr, bend left] & \\
& \mathcal{O}_{\frac{3}{2}p} \oplus \mathcal{O}_{\frac{1}{2}p}(\tfrac{1}{2}p) \arrow[dr] & & \mathcal{O}_{\frac{3}{2}p}(\tfrac{1}{2}p) \oplus \mathcal{O}_{\frac{1}{2}p} \arrow[dl] & \\
\mathcal{O}_p(\tfrac{1}{2}p)^{\oplus 2} \arrow[dr] & & \mathcal{O}_p \oplus \mathcal{O}_p(\tfrac{1}{2}p) \arrow[dl] \arrow[dr] & & \mathcal{O}_p^{\oplus 2} \arrow[dl] \\
& \mathcal{O}_p(\tfrac{1}{2}p) \oplus \mathcal{O}_{\frac{1}{2}p} \oplus \mathcal{O}_{\frac{1}{2}p}(\tfrac{1}{2}p) \arrow[dr] & & \mathcal{O}_p \oplus \mathcal{O}_{\frac{1}{2}p} \oplus \mathcal{O}_{\frac{1}{2}p}(\tfrac{1}{2}p) \arrow[dl] & \\
& & \mathcal{O}_{\frac{1}{2}p}^{\oplus 2} \oplus \mathcal{O}_{\frac{1}{2}p}(\tfrac{1}{2}p)^{\oplus 2} & &
\end{tikzcd}
$$

Figure 3.5: The homeomorphism between $\mathbb{Y}^2_{(2,2)}$ and $\left|\,\mathsf{Coh}_{0,(2,2),p}\,\right|$, where p is a stacky point of order 2.

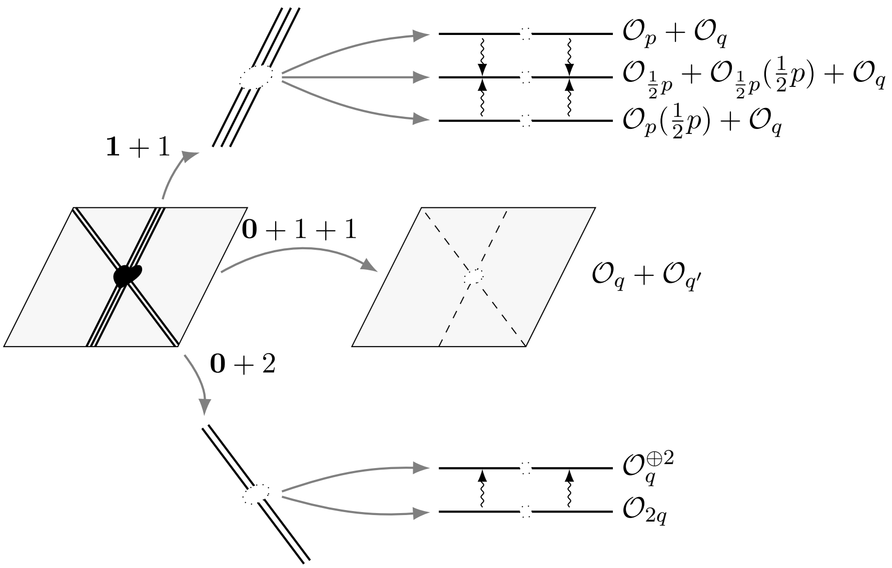
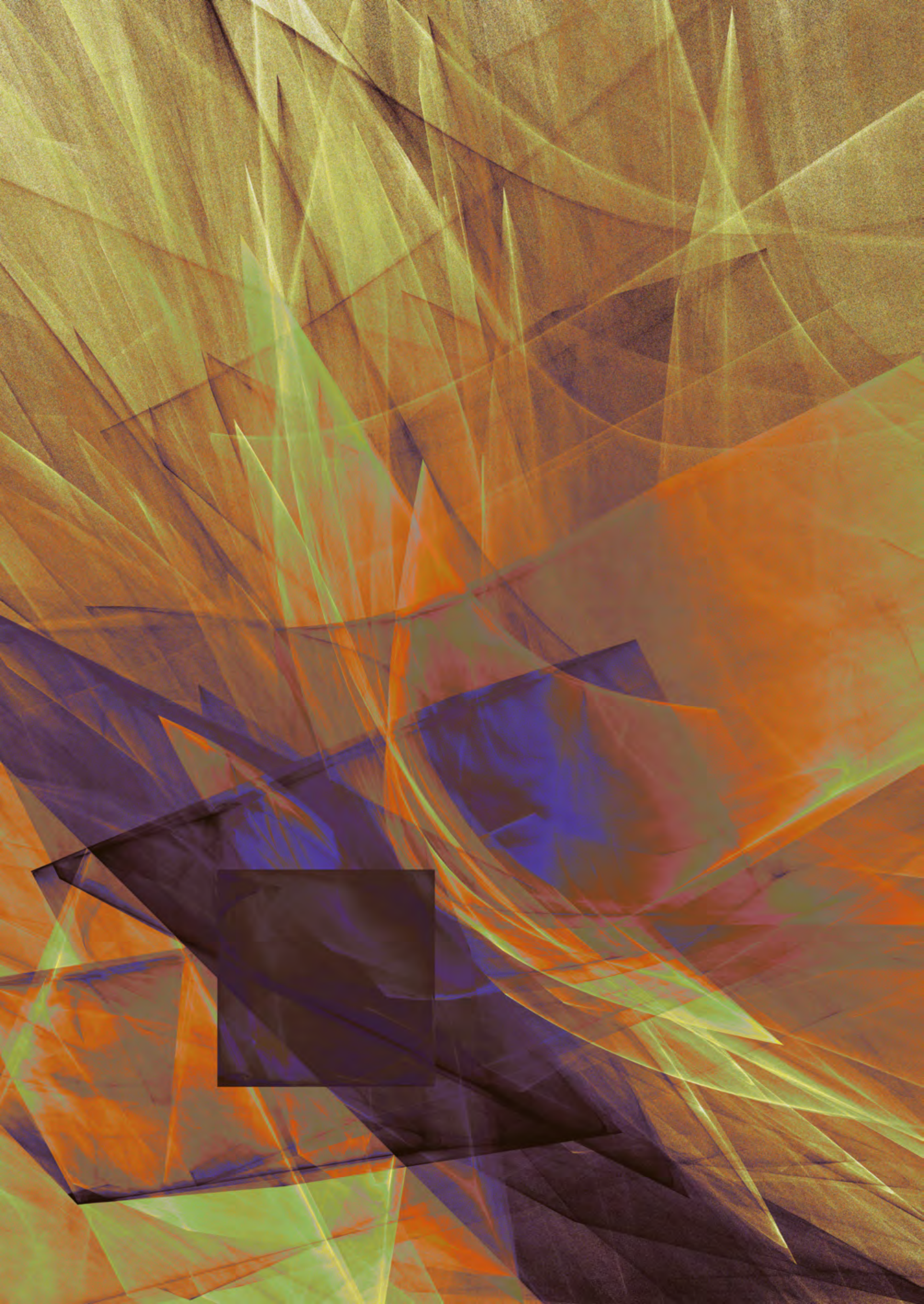


Figure 3.6: The stratification of $\mathsf{Coh}_{0,(2,2)}(\mathcal{C})$, by support type and refined by Young diagrams, where $\mathcal{C}$ has a single stacky point p of order 2. The strata with support type $\mathbf{2}$ are given in Figure 3.5.



CHAPTER 4

Good Moduli Spaces

In short, mathematics only exists in a living community of mathematicians that spreads understanding and breaths life into ideas both old and new. The real satisfaction from mathematics is in learning from others and sharing with others.

What's a mathematician to do?
Bill Thurston

4.1 Stacks of bundles on stacky curves

This chapter is based on the joint paper [DHMT24] with C. Damiolini, V. Hoskins, and S. Makarova.

The goal of this chapter is to show that the moduli stack of semistable vector bundles on a smooth projective stacky curve admits a projective good moduli space. This was first shown by Mehta and Seshadri [MS80], using geometric invariant theory and the language of parabolic vector bundles. We will instead give a "modern" proof in two steps. First we will apply the existence theorem [AHH23] to show that a good moduli space (a priori an algebraic space) exists. Next we will define an explicit line bundle on the stack and show that it induces a finite map from the good moduli space to a projective space. This implies that the good moduli space is in fact a projective variety.

This modern approach has been applied before to the moduli space of curves [Kol90], the moduli space of vector bundles on a curve [ABBLT22], and the moduli space of quiver representations [BDFHMT22]. The major advantage of the modern approach compared to GIT is the fact that we get **effective** bounds for the existence of sections of our line bundle. These effective bounds are a first step towards obtaining explicit embeddings of the moduli space in projective space.

We would also like to highlight the convenience of working with stacky curves over parabolic bundles. Once the correct generalization of the statements in [ABBLT22] are found it is quite straightforward to generalize proofs from classical curves to stacky curves. Note that the results in this chapter do not depend on the results of [ABBLT22], so the arguments specialize to the case of classical curves, and in fact provide a streamlined proof.

Finiteness

We start by proving that the stack of semistable bundles with fixed invariant is of finite type. Before we can do that, we cite a preliminary result about Quot schemes for Deligne-Mumford stacks and prove that semistable vector bundles on stacky curves can be expressed as quotients.

> **Theorem 4.1.1** [OS03, Theorem 1.5] Let $\mathcal{X}$ be a tame Deligne-Mumford stack, separated and of finite type over k. Assume that $\mathcal{X}$ is a global quotient and that its coarse moduli space X is a projective variety. Let E be a quasi-coherent sheaf on $\mathcal{X}$. Define the Quot stack Q to be the stack whose fiber over a base B are groupoids of locally finitely presented quotients of E that are flat and with proper support over B. Then the connected components of Q are projective.

For classical curves, the fact that semistable vector bundles can be described as quotients of a fixed sheaf is key to constructing moduli spaces via GIT. The next result generalizes this quotient description to stacky curves. Recall the definition of the slope (semi)stability given in Definition 1.3.25.

Lemma 4.1.2 Let $\boldsymbol{\alpha} \in \mathsf{K}_0^{\mathsf{num}}(\mathcal{C})$ be a generating numerical invariant (Definition 1.3.22), represented by a generating sheaf E. Let $\boldsymbol{\beta} \in \mathsf{K}_0^{\mathsf{num}}(\mathcal{C})$ be another invariant.

(i) For any sheaf F on $\mathcal{C}$, there exists $\mu_{\boldsymbol{\alpha},\mathsf{max}}(F)$ such that for all subbundles $F' \subset F$, we have $\mu_{\boldsymbol{\alpha}}(F') \leq \mu_{\boldsymbol{\alpha},\mathsf{max}}(F)$.

(ii) If F is an $\boldsymbol{\alpha}$-semistable sheaf and $\mu_{\boldsymbol{\alpha}}(F) > \mu_{\boldsymbol{\alpha},\mathsf{max}}(E \otimes \omega_{\mathcal{C}})$, then $\mathtt{Ext}^1(E, F) = 0$.

(iii) If F is an $\boldsymbol{\alpha}$-semistable sheaf with $\mu_{\boldsymbol{\alpha}}(F) > \mu_{\boldsymbol{\alpha},\mathsf{max}}(E \otimes \omega_{\mathcal{C}}) + \mathtt{rank}(E)$, then the map $\mathtt{ev}\colon \mathtt{Hom}(E, F) \otimes E \to F$ is surjective.

Proof. Part (i) follows as F' is a subobject of F, so the degree and multiplicities of F' are bounded above, and the rank is non-negative.

We now prove part (ii). By semistability of F and the assumption on the slopes, it follows that $\mathtt{Hom}(F, E \otimes \omega_{\mathcal{C}}) = 0$. Then Serre duality implies that $\mathtt{Ext}^1(E, F) = 0$.

In order to prove part (iii), we will adapt a classical argument (for example, see [New78, Chapter 5]). For any point $x \in \mathcal{C}$, let e_x be the order of x (which will be equal to 1 if x is chosen to be non-stacky). Note that tensoring by $\mathcal{O}(-x)$ doesn't change the multiplicities, so $F(-x)$ is still semistable and $\mu_E(F(-x)) = \mu_{\boldsymbol{\alpha}}(F) - \mathtt{rank}(E)$, hence by part (ii), we have $\mathtt{Ext}^1(E, F(-x)) = 0$. Consider the long exact sequence obtained from applying $\mathtt{Hom}(E, \_)$ to the short exact sequence

$$0 \longrightarrow F(-x) \longrightarrow F\left(-\frac{1}{e_x}x\right) \longrightarrow T \longrightarrow 0,$$

where T is the quotient torsion sheaf. We can see that $\mathtt{Ext}^1(E, F(-x))$ surjects onto $\mathtt{Ext}^1\left(E, F(-\frac{1}{e_x}x)\right)$, hence it also vanishes.

Let $\iota_x\colon \mathcal{G}_x \to \mathcal{C}$ denote the inclusion of the residual gerbe (where $\mathcal{G}_x = \mathtt{Spec}(k)$ if x is non-stacky), and set $F_{\mathcal{G}_x} := \iota_{x*}\iota_x^* F$. Applying $\mathtt{Hom}(E, \_)$ to the short exact sequence

$$0 \longrightarrow F\left(-\frac{1}{e_x}x\right) \longrightarrow F \longrightarrow F_{\mathcal{G}_x} \longrightarrow 0$$

yields an exact sequence

$$\mathtt{Hom}(E,F) \longrightarrow \mathtt{Hom}(E,F_{\mathcal{G}_x}) \longrightarrow \mathtt{Ext}^1\left(E,F\left(-\frac{1}{e_x}x\right)\right).$$

We have already proved that $\mathtt{Ext}^1\left(E,F(-\frac{1}{e_x}x)\right) = 0$, hence we have a surjection

$$f\colon \mathtt{Hom}(E,F) \longrightarrow \mathtt{Hom}(E,F_{\mathcal{G}_x}).$$

We claim that the morphism obtained by adjunction is surjective as well:

$$\mathtt{ev}_x\colon \mathtt{Hom}(E,F)\otimes E \longrightarrow F_{\mathcal{G}_x}.$$

Indeed, pick any vector $v \in F_{\mathcal{G}_x}$. By adjunction, we have

$$\mathtt{Hom}_{\mathcal{C}}(E,F_x) = \mathtt{Hom}_{\mu_{e_x}}(\iota_x^* E, \iota_x^* F),$$

and since E is generating, there is a morphism of $\mathbb{Z}/e\mathbb{Z}$-graded vector spaces $g\colon \iota_x^* E \to \iota_x^* F$ such that $v = g(w)$ for some section w in a neighborhood of x. Since f is surjective, there is a morphism $h\colon E \to F$ such that $f(h) = g$. But now we observe that $v = \mathtt{ev}_x(h\otimes w)$, and we conclude that $\mathtt{ev}_x$ is surjective. ⬠

Proposition 4.1.3 If $\boldsymbol{\alpha}$ is a generating numerical invariant, then $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is of finite type.

Proof. Fix an ample line bundle $\mathcal{O}_C(1)$ on the good moduli space $\pi\colon \mathcal{C} \to C$, and for an arbitrary sheaf F on $\mathcal{C}$, denote by $F(n)$ the twist $F\otimes \pi^*\mathcal{O}_C(n)$. Pick a generating bundle E of class $\boldsymbol{\alpha}$. For a large enough $m\in\mathbb{Z}$, we have that

$$\mu_E(F(m)) > \mu_{E,\mathsf{max}}(E\otimes\omega_{\mathcal{C}}) + \mathtt{rank}(E)$$

for every $F \in \mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}(k)$. Therefore, by Lemma 4.1.2 (iii), we have that for every $F \in \mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$, the following map is surjective:

$$\mathtt{Hom}(E,F(m))\otimes E \to F(m).$$

By Lemma 4.1.2 (ii), we deduce that the dimension of $\mathtt{Hom}(E,F(m))$ is independent of $F \in \mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$; call this dimension N. Therefore, every $F \in \mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ can be written as a quotient

$$E(-m)^{\oplus N} \to F,$$

or in other words, realized as an element of the Quot scheme Q of quotients of $E(-m)^{\oplus N}$ that have a fixed numerical invariant $\boldsymbol{\beta}$. Since semistability is open, we find an open subscheme $Q^\circ \subset Q$ that surjects onto $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$. Since $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is connected by Theorem 3.1.12, we find a connected component Q' of Q such that $Q' \cap Q^\circ$ still surjects on $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$. But by Theorem 4.1.1, Q' is projective, so $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is bounded. ⬠

Remark 4.1.4 In fact there is an open subscheme $\mathcal{Q} \subset Q$ and an integer N, such that $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}} \simeq [\mathcal{Q}/\,\mathsf{GL}_N]$ [Nir09, Theorem 5.1]. It follows that $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ has affine diagonal.

4.2 Existence and properties of good moduli space

In this section we apply the existence criterion of Alper, Halpern-Leistner and Heinloth [AHH23, Theorem A] to prove that the stack $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ admits a good moduli space in the sense of Alper [Alp13]. In this section, we will assume that $\mathtt{char}(k) = 0$, as we only apply the existence criterion in characteristic zero due to the difference in positive characteristic between linearly reductive and reductive stabilizers (which requires a weaker notion of an adequate moduli space). In this section, $\boldsymbol{\alpha}$ will denote a generating numerical invariant.

Applying the existence theorem

Since $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is an algebraic stack of finite type over k with affine diagonal we are in the position to apply the following existence criterion for good moduli spaces. We will only state this criterion in characteristic zero, as we cannot verify the additional local reductivity assumption required in positive characteristic to obtain the étale local quotient description as in [AHR23] when the stabilizers of closed points are linearly reductive (in characteristic zero, this is always the case, as S-completeness implies these stabilizers are reductive).

Theorem 4.2.1 (Existence criteria for stacks, [AHH23, Theorem A]) Let $\mathcal{X}$ be an algebraic stack of finite type over a characteristic zero field k with affine diagonal. Then $\mathcal{X}$ admits a separated good moduli space if and only if $\mathcal{X}$ is Θ-complete and S-complete.

Let us give the definitions of the completeness conditions appearing here, which are valuative criteria involving verifying codimension 2 filling conditions.

Definition 4.2.2 A stack $\mathcal{X}$ is Θ-complete or S-complete if for every DVR R with uniformizer π, every morphisms from $\mathcal{T}_R \setminus \{\mathbf{0}\} \to \mathcal{X}$ extends to $\mathcal{T}_R$ where

$$\mathcal{T}_R = \Theta_R \coloneqq [\mathsf{Spec}(R[s])/\mathbb{G}_m] \quad \text{or}$$
$$\mathcal{T}_R = \overline{\mathsf{ST}}_R \coloneqq [\mathsf{Spec}(R[s,t]/(\pi - st))/\mathbb{G}_m] \quad \text{respectively,}$$

where $\mathbb{G}_m$ acts on s with weight $+1$ and t with weight -1 and $\mathbf{0}$ is the unique closed point of $\mathcal{T}_R$.

By definition, Θ_R is the base change of $\Theta \coloneqq [\mathsf{Spec}(\mathbb{Z}[s])/\mathbb{G}_m]$ to R. For a detailed discussion of these conditions, we refer to [Alp24, §6.8.2]. If $\mathcal{X}$ is a moduli stack of objects in an Abelian category, morphisms $\Theta_R \setminus \{\mathbf{0}\} \to \mathcal{X}$ can be viewed as a family over R with a filtration over the generic fiber $K = \mathtt{Frac}(R)$ whose associated graded object lies in $\mathcal{X}$, and such a morphism extends to Θ_R if the filtration and associated graded object extend to the special fiber $\kappa = R/\pi$. Similarly in this Abelian setting, a morphism $\overline{\mathsf{ST}}_R \setminus \{\mathbf{0}\} \to \mathcal{X}$ can be viewed as two families over R whose generic fibers are isomorphic and this extends to $\overline{\mathsf{ST}}_R$ if the special fibers admit opposite filtrations whose associated graded objects are isomorphic.

Proposition 4.2.3 The stack $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ admits a separated good moduli space $B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$.

Proof. By the above existence criterion [AHH23, Theorem A], it suffices to prove that $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is Θ-complete and S-complete. Throughout we let R be a discrete valuation ring with residue field κ, and denote by π its uniformizer and by K its fraction field. Note that when $\mathcal{A}$ is the category of quasi-coherent sheaves on $\mathcal{C}$, the stack of coherent sheaves Coh coincides with the stack $\mathcal{M}_{\mathcal{A}}$ introduced in [AHH23, §7, Example 7.1 and Definition 7.8], thus Coh is S-complete and Θ-complete by [AHH23, Lemma 7.16 and 7.17]. Alternatively, one can use the properness of the Quot scheme of sheaves on $\mathcal{C}$ ([OS03, Theorem 1.1]) to prove that Coh is Θ-complete.

We start by showing that $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is Θ-complete. We can identify $\Theta_R \setminus \{\mathbf{0}\}$ with

$\mathsf{Spec}(R) \bigsqcup_{\mathsf{Spec}(K)} \Theta_K$, so a morphism $\Theta_R \setminus \{\mathbf{0}\} \to \mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ corresponds to a semistable vector bundle F over $\mathcal{C}_R$ with a filtration

$$0 = F_K^{-m} \subset \cdots \subset F_K^{\ell-1} \subset F_K^{\ell} \subset F_K^{\ell+1} \subset \cdots \subset F_K^{n} = F_K$$

of the generic fiber whose associated graded object $\mathtt{gr}(F_K^{\bullet}) = \bigoplus_{\ell} F_K^{\ell}/F_K^{\ell-1}$ lies in $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$. In particular, we must have $\mu_{\boldsymbol{\alpha}}(F_K^{\ell}) = \mu_{\boldsymbol{\alpha}}(F_K)$ and all these sheaves are $\boldsymbol{\alpha}$-semistable. This morphism extends to Θ_R if the above filtration and associated graded object extends over the special fibre κ of R in $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$. By the discussion above, Coh is Θ-complete and so we can extend the above morphism to $\phi : \Theta_R \to \mathsf{Coh}$ which gives a filtration $0 = F^{-M} \subset \cdots \subset F^{\ell-1} \subset F^{\ell} \subset F^{\ell+1} \subset \cdots \subset F^{N} = F$ of coherent sheaves on $\mathcal{C}_R$ that restricts to the above filtration of $\boldsymbol{\alpha}$-semistable vector bundles over $\mathcal{C}_K$.

Since the subsheaves F^{ℓ} are flat over R, they have the same $\boldsymbol{\alpha}$-slope as the generic fibre. Hence we also have $\mu_{\boldsymbol{\alpha}}(F_{\kappa}^{\ell}) = \mu_{\boldsymbol{\alpha}}(F_{\kappa})$ over the special fibre and deduce each F_{κ}^{ℓ} is $\boldsymbol{\alpha}$-semistable from the semistability of F_{κ} using that $F_{\kappa}^{\ell} \subset F_{\kappa}$ have the same $\boldsymbol{\alpha}$-slope. Since the kernel and cokernel of a map between $\boldsymbol{\alpha}$-semistable vector bundles is again $\boldsymbol{\alpha}$-semistable (this is a formal consequence of the seesaw inequality [Joy07, Definition 4.1]), we deduce that $\mathtt{gr}(F_{\kappa}^{\bullet})$ is also $\boldsymbol{\alpha}$-semistable. This proves the image of ϕ is contained in $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$.

Next we show that $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is S-complete. Note that $\overline{\mathsf{ST}}_R \setminus \{\mathbf{0}\}$ can be identified with $\mathsf{Spec}(R) \bigsqcup_{\mathsf{Spec}(K)} \mathsf{Spec}(R)$, so a morphism $\overline{\mathsf{ST}}_R \setminus \{\mathbf{0}\} \to \mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ corresponds to two semistable vector bundles $F_{-\infty}$ and F_{∞} over $\mathcal{C}_R$ with a fixed isomorphism over $\mathcal{C}_K$. This extends to $\overline{\mathsf{ST}}_R$ if we can find a system of vector bundles $(F_{\ell})_{\ell \in \mathbb{Z}}$ which fit in a diagram

$$\begin{tikzcd} \cdots \arrow[r, bend right, "s_{\ell-2}"'] & F_{\ell-2} \arrow[l, bend right, "t_{\ell-3}"'] \arrow[r, bend right, "s_{\ell-1}"'] & F_{\ell-1} \arrow[l, bend right, "t_{\ell-2}"'] \arrow[r, bend right, "s_{\ell}"'] & F_{\ell} \arrow[l, bend right, "t_{\ell-1}"'] \arrow[r, bend right, "s_{\ell+1}"'] & F_{\ell+1} \arrow[l, bend right, "t_{\ell}"'] \arrow[r, bend right, "s_{\ell+2}"'] & F_{\ell+2} \arrow[l, bend right, "t_{\ell+1}"'] \arrow[r, bend right, "s_{\ell+3}"'] & \cdots \arrow[l, bend right, "t_{\ell+2}"'] \end{tikzcd}$$

where

(S1) the maps s_i and t_i are injections such that $s_i \circ t_{i-1}$ and $t_i \circ s_{i+1}$ are given by multiplication by π (occasionally we will omit the subscripts and denote these maps by s and t);

(S2) there exists an $N \in \mathbb{Z}$ such that for every $n \geq N$ one has isomorphisms $F_n \cong F_\infty$ and $F_{-n} \cong F_{-\infty}$ commuting with the morphisms $s_{n+1}\colon F_n \to F_{n+1}$ and $t_{-n-1}\colon F_{-n} \to F_{-n-1}$, respectively; in particular, s_n and t_{-n} are isomorphisms for $n > N$;

(S3) the map s induces an injection $F_{\ell-1}/t(F_\ell) \to F_\ell/t(F_{\ell+1})$, and analogously the map t induces an injection $F_{\ell+1}/s(F_\ell) \to F_\ell/s(F_{\ell-1})$;

(S4) the sheaf over $\mathcal{C}_\kappa$

$$\mathtt{gr}(F) \coloneqq \bigoplus_{\ell\in\mathbb{Z}} \frac{F_\ell/t(F_{\ell+1})}{s(F_{\ell-1}/t(F_\ell))} \cong \bigoplus_{\ell\in\mathbb{Z}} \frac{F_\ell/s(F_{\ell-1})}{t(F_{\ell+1}/s(F_\ell))} \cong \bigoplus_{\ell\in\mathbb{Z}} \frac{F_\ell}{s(F_{\ell-1})+t(F_{\ell+1})}$$

is an $\boldsymbol{\alpha}$-semistable vector bundle.

Since **Coh** is S-complete, we can find a unique system of coherent sheaves $(F_\ell)_{\ell\in\mathbb{Z}}$ as in (4.2) satisfying conditions (S1), (S2) and (S3). Since the maps s and t are injective, this implies that F_ℓ is a vector bundle for every ℓ and thus we are left to show that (S4) holds. Note that conditions (S1)–(S3) tell us that

$$0 = \frac{F_{-\infty}}{F_{-\infty}} = \frac{F_{-N-1}}{t(F_{-N})} \subset \frac{F_{-N}}{t(F_{-N+1})} \subset \cdots \\ \cdots \subset \frac{F_{N-1}}{t(F_N)} \subset \frac{F_N}{t(F_{N+1})} = \frac{F_\infty}{t(F_\infty)} = F_\infty|_\kappa \tag{4.1}$$

is a finite filtration of $F_\infty|_\kappa$ (and similarly for $F_{-\infty}|_\kappa$). Recall that F_∞ has numerical invariant $\boldsymbol{\beta}$ and it is $\boldsymbol{\alpha}$-semistable, with $\langle\boldsymbol{\alpha},\boldsymbol{\beta}\rangle = 0$. Combining semistability (as in Remark 1.3.26) together with (4.1), we obtain

$$0 = \langle\boldsymbol{\alpha}, F_\infty\rangle \geq \left\langle\boldsymbol{\alpha}, \frac{F_{\ell-1}}{t(F_\ell)}\right\rangle = \langle\boldsymbol{\alpha}, F_{\ell-1}\rangle - \langle\boldsymbol{\alpha}, t(F_\ell)\rangle \\ = \langle\boldsymbol{\alpha}, F_{\ell-1}\rangle - \langle\boldsymbol{\alpha}, F_\ell\rangle,$$

where the last equality follows from the fact that t is injective. Thus we have that $\langle\boldsymbol{\alpha}, F_{\ell-1}\rangle \leq \langle\boldsymbol{\alpha}, F_\ell\rangle$. Repeating the argument with $F_{-\infty}$ we obtain the reverse inequality $\langle\boldsymbol{\alpha}, F_{\ell-1}\rangle \geq \langle\boldsymbol{\alpha}, F_\ell\rangle$ which forces $\langle\boldsymbol{\alpha}, F_{\ell-1}\rangle = \langle\boldsymbol{\alpha}, F_\ell\rangle$ for every ℓ, and thus $\langle\boldsymbol{\alpha}, F_\ell\rangle = 0$. Since $F_\ell \subset F_\infty$ is a subbundle of the same $\boldsymbol{\alpha}$-slope and

F_∞ is $\boldsymbol{\alpha}$-semistable, we conclude F_ℓ is also $\boldsymbol{\alpha}$-semistable. Again, as the category of $\boldsymbol{\alpha}$-semistable vector bundles of fixed slope is Abelian, we deduce that $F_{\ell-1}/t(F_\ell)$ and

$$\mathtt{gr}(F)_\ell := \frac{F_\ell/t(F_{\ell+1})}{s(F_{\ell-1}/t(F_\ell))}$$

are $\boldsymbol{\alpha}$-semistable. By semistability of $\mathtt{gr}(F)_\ell$, this sheaf is torsion free, and thus a vector bundle, which completes the proof. ◯

Remark 4.2.4 Note that the above proof cannot be applied to show that Bun is S-complete (or Θ-complete), as the cokernel of an inclusion of locally free sheaves may not be locally free (see [Alp13, Proposition 6.8.31 and Remark 6.8.33]).

Corollary 4.2.5 The good moduli space $B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is a normal and proper algebraic space of finite type over $\mathtt{Spec}(k)$, which is irreducible if it is non-empty.

Proof. The stack $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is irreducible and smooth by Theorem 3.1.12, and Corollary 3.1.9. By [Alp13, Theorem 4.16], the irreducibility and normality of $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ descend to its good moduli space $B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$. We are left to prove properness which, in view of [AHH23, Theorem A], amounts to showing that the stack $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ satisfies the existence part of the valuative criterion of properness. For this, we can assume that k is algebraically closed. For a non-stacky curve, this is a classical result of Langton [Lan75, Theorem at page 99] which was extended to the case of stacky curves in [Hua23, Theorem 1.1]. ◯

The remainder of the chapter is devoted to proving that the good moduli space is projective, and thus in particular is a scheme rather than just an algebraic space. Our first step is to construct the line bundle from which we will obtain a projective embedding.

4.3 Determinantal line bundles

In this section we construct a determinantal line bundle $\mathcal{L}_V$ over $\mathsf{Bun}_{\boldsymbol{\beta}}$ naturally associated to a vector bundle V on $\mathcal{C}$. We will see that when $\langle [V], \boldsymbol{\beta} \rangle = 0$, this line bundle has a global section. The properties of this line bundle will be crucial to proving the projectivity of $B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ in Section 4.5.

Definition and main properties of determinantal line bundles

Consider the diagram

$$
\begin{tikzcd}
 & \mathcal{U}_{\boldsymbol{\beta}} \arrow[d, dash, dotted] & \\
V \arrow[d, dash, dotted] & \mathcal{C} \times \mathsf{Bun}_{\boldsymbol{\beta}} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\mathcal{C} & & \mathsf{Bun}_{\boldsymbol{\beta}}
\end{tikzcd}
$$

where $\mathcal{U}_{\boldsymbol{\beta}}$ is the universal vector bundle on $\mathcal{C} \times \mathsf{Bun}_{\boldsymbol{\beta}}$ and V is a vector bundle on $\mathcal{C}$. Then we define

$$\mathcal{L}_V := \mathtt{det}\left(\mathsf{R}p_*\mathcal{H}om(q^*V, \mathcal{U}_{\boldsymbol{\beta}})\right)^\vee \tag{4.2}$$

and we call this bundle the **determinantal line bundle** on $\mathsf{Bun}_{\boldsymbol{\beta}}$ associated to V. Concretely, by base change [HR17, Corollary 4.13], at a point $E \in \mathsf{Bun}_{\boldsymbol{\beta}}(k)$ the fiber is given by

$$\mathcal{L}_V|_E = \mathtt{det}\,\mathtt{Ext}^0(V, E)^\vee \otimes \mathtt{det}\,\mathtt{Ext}^1(V, E).$$

The complex $\mathsf{R}p_*\mathcal{H}om(q^*V, \mathcal{U}_{\boldsymbol{\beta}})$ is locally represented by a complex of vector bundles $K^0 \to K^1$ on $\mathsf{Bun}_{\boldsymbol{\beta}}$. To see this, let $d \geq 0$ be an integer and consider the open substack $\mathcal{X}_d \subset \mathsf{Bun}_{\boldsymbol{\beta}}$ consisting of F such that $\mathtt{Ext}^1(V(-d), F) = 0$. By base change, this means that the fibers of $\mathsf{R}^1p_*\mathcal{H}om(q^*V(-d), \mathcal{U}_\beta)_{|\mathcal{X}_d}$ are zero, hence this sheaf vanishes. It is clear that these substacks $\mathcal{X}_d$ cover $\mathsf{Bun}_{\boldsymbol{\beta}}$. Now consider the short exact sequence of coherent sheaves on $\mathcal{C} \times \mathcal{X}_d$

$$0 \to \mathcal{H}om(q^*V, \mathcal{U}_{\boldsymbol{\beta}})|_{\mathcal{X}_d} \to \mathcal{H}om(q^*V(-d), \mathcal{U}_{\boldsymbol{\beta}}))_{\mathcal{X}_d} \to Q_d \to 0.$$

Applying $\mathsf{R}p_*$ to the short exact sequence we get a long exact sequence

$$\begin{aligned} 0 \to \mathsf{R}^0p_*\mathcal{H}om(q^*V, \mathcal{U}_{\boldsymbol{\beta}})|_{\mathcal{X}_d} &\to \mathsf{R}^0p_*\mathcal{H}om(q^*V(-d), \mathcal{U}_{\boldsymbol{\beta}})|_{\mathcal{X}_d} \to \\ &\to \mathsf{R}^0p_*Q_d \to \mathsf{R}^1p_*\mathcal{H}om(q^*V, \mathcal{U}_{\boldsymbol{\beta}})|_{\mathcal{X}_d} \to 0 \to \mathsf{R}^1p_*Q_d \to 0. \end{aligned}$$

Notice that Q_d is the tensor product of a vector bundle $\mathcal{H}om(q^*V, \mathcal{U}_{\boldsymbol{\beta}})|_{\mathcal{X}_d}$ with $q^*\mathcal{O}_D(d)$, where D is a divisor on $\mathcal{C}$ corresponding to the embedding $\mathcal{O}_\mathcal{C} \to$

$\mathcal{O}_{\mathcal{C}}(d)$. Since $q^*\mathcal{O}_D(d)$ is flat over $\mathfrak{X}_d$, it follows that Q_d is too. By the cohomology and base change theorem [Hal14, Theorem A] it follows that

$$\mathsf{R}^0 p_* \mathcal{H}om(q^*V(-d), \mathcal{U}_{\boldsymbol{\beta}})|_{\mathfrak{X}_d} = p_*\mathcal{H}om(q^*V(-d), \mathcal{U}_{\boldsymbol{\beta}})|_{\mathfrak{X}_d}$$

and $\mathsf{R}^0 p_* Q_d = p_* Q_d$ and they are both vector bundles. In particular, we have a quasi-isomorphism of complexes

$$\mathsf{R}p_*\mathcal{H}om(q^*V, \mathcal{U}_{\boldsymbol{\beta}})|_{\mathfrak{X}_d} \simeq \left[p_*\mathcal{H}om(q^*V(-d), \mathcal{U}_{\boldsymbol{\beta}})|_{\mathfrak{X}_d} \xrightarrow{\delta_d} p_*Q_d\right]$$

where the latter is a two term complex of vector bundles.

Let V be a vector bundle such that $\langle V, \boldsymbol{\beta}\rangle = 0$, then from the local picture we can see that $\mathcal{L}_V$ comes with a natural section. Namely, we take $\mathtt{det}(\delta_d) \in H^0(\mathfrak{X}_d, \mathcal{L}_V|_{\mathfrak{X}_d})$ (the determinant exists since the source and target have the same rank), and these sections glue together to a global section σ_V. Note that on the locus $\mathfrak{X}_0$ the complex is given by the unique map $\delta_0 \colon 0_{\mathfrak{X}_0} \to 0_{\mathfrak{X}_0}$ between the zero vector bundles, so $\mathcal{L}_V$ trivializes on $\mathfrak{X}_0$ via the canonical section $\mathtt{det}(\delta_0) = 1$. Hence for $E \in \mathsf{Bun}_{\boldsymbol{\beta}}(k)$, we have

$$\sigma_V|_E \neq 0 \text{ if and only if } \mathtt{Hom}(V, E) = \mathtt{Ext}^1(V, E) = 0. \tag{4.3}$$

Given an exact sequence of vector bundles $0 \to V' \to V \to V'' \to 0$, we have by construction $\mathcal{L}_V \cong \mathcal{L}_{V'} \otimes \mathcal{L}_{V''}$. It follows that $\mathcal{L}_V$ only depends on the algebraic class $[V]$ of V in the Grothendieck ring $\mathsf{K}_0(\mathcal{C})$. Because of this we will write $\mathcal{L}_{[V]}$ rather than $\mathcal{L}_V$ from now on. Note that if V is a vector bundle with algebraic invariant $m\widetilde{\boldsymbol{\alpha}}$, then $\mathcal{L}_{[V]} \cong \mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}^{\otimes m}$. A key point is that the section σ_V **does** depend on V and we will leverage this fact to construct many different sections of $\mathcal{L}_{[V]}$ using different vector bundles W with the same class as V.

4.4 Vanishing results

We consider $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathsf{K}_0^{\mathsf{num}}(\mathcal{C})$ such that $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}\rangle = 0$. We fix an algebraic invariant $\widetilde{\boldsymbol{\alpha}} = (L, \boldsymbol{\alpha}) \in \mathsf{K}_0(\mathcal{C})$, defined by the numerical invariant $\boldsymbol{\alpha}$, together with a fixed determinant $L \in \mathtt{Pic}(C)$. Any algebraic invariant $\widetilde{\boldsymbol{\gamma}} \in \mathsf{K}_0(\mathcal{C})$ gives rise to a numerical invariant $\boldsymbol{\gamma}$ and determinant $L \in \mathtt{Pic}(C)$. Let $B\mathbb{G}_m \hookrightarrow \mathtt{Pic}(C)$ be the inclusion corresponding to L, then we define $\mathsf{Bun}_{\widetilde{\boldsymbol{\gamma}}} := B\mathbb{G}_m \times_{\mathtt{Pic}(C)} \mathsf{Bun}_{\boldsymbol{\gamma}}$.

We will consider the determinantal line bundle $\mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}$ on $\mathsf{Bun}_{\boldsymbol{\beta}}$ and produce sections σ_V of $\mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}^{\otimes m}$ using vector bundles $V \in \mathsf{Bun}_{m\widetilde{\boldsymbol{\alpha}}}$ with numerical invariant $m\boldsymbol{\alpha}$ and fixed determinant $L^{\otimes m}$. First, we show for $m \gg 0$ (and in fact we give an effective bound) and for any $\boldsymbol{\alpha}$-semistable vector bundle E with invariant $\boldsymbol{\beta}$, we can find a vector bundle $V \in \mathsf{Bun}_{m\widetilde{\boldsymbol{\alpha}}}(k)$ such that $\mathsf{Hom}(V, E) = 0$ or equivalently $\sigma_V(E) \neq 0$, which will allow us to prove that the restriction of $\mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}$ to $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is semiample in Theorem 4.5.2 below. Throughout this section, we assume $k = \bar{k}$ to have the existence of k-points of $\mathsf{Bun}_{m\widetilde{\boldsymbol{\alpha}}}$.

Hom-vanishing

We will describe the codimension of loci where Hom-vanishing fails by using **stacks of short exact sequences**. For any numerical invariants $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$, we let $\mathsf{SES}_{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2}$ be the stack of short exact sequences with fixed invariants as in Definition 3.1.5. This stack admits natural forgetful maps

$$\begin{tikzcd} & \mathsf{SES}_{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2} \arrow[dl, "\pi_{13}"'] \arrow[dr, "\pi_2"] & \\ \mathsf{Bun}_{\boldsymbol{\beta}_1} \times \mathsf{Bun}_{\boldsymbol{\beta}_2} & & \mathsf{Bun}_{\boldsymbol{\beta}_1+\boldsymbol{\beta}_2} \end{tikzcd} \qquad \begin{tikzcd} & E_1 \hookrightarrow F \twoheadrightarrow E_2 \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ (E_1, E_2) & & F \end{tikzcd}$$

By Corollary 3.1.9 $\mathsf{SES}_{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2}$ is smooth of dimension $-\langle \boldsymbol{\beta}_1, \boldsymbol{\beta}_1 \rangle - \langle \boldsymbol{\beta}_2, \boldsymbol{\beta}_2 \rangle - \langle \boldsymbol{\beta}_2, \boldsymbol{\beta}_1 \rangle$. In addition the projection π_2 is representable, which can be seen in two different ways: the fibers are Quot schemes, or the corresponding functor is faithful, as a morphism of short exact sequences which is the identity on F must also be the identity on E_1 and E_2.

The following proposition can be seen as an extension of [ABBLT22, Lemma 5.8]; however we simplify the proof by doing dimension counts directly on the stack of vector bundles (with fixed invariants and determinant). Because of this we do not need a result of the form of [ABBLT22, Lemma 5.7] and we do not need to consider projectivized Ext groups. Moreover we write our formula's in terms of Euler pairing to simplify computations.

> **Proposition 4.4.1** Let $\widetilde{\boldsymbol{\alpha}} \in \mathsf{K}_0(\mathcal{C})$ be a positive algebraic invariant, and let $\boldsymbol{\beta}$ be a positive numerical invariant such that $\langle \widetilde{\boldsymbol{\alpha}}, \boldsymbol{\beta} \rangle = 0$. Let $\widetilde{\boldsymbol{\eta}} \in \mathsf{K}_0(\mathcal{C})$ be an effective algebraic invariant. Then there exists a constant $\kappa = \kappa_{\widetilde{\boldsymbol{\alpha}},\boldsymbol{\beta},\widetilde{\boldsymbol{\eta}}}$ such

that for any $m > \kappa$ and any $E \in \mathsf{Bun}_{\boldsymbol{\beta}}(k)$, a general vector bundle $V \in \mathsf{Bun}_{m\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\eta}}}(k)$ satisfies the following conditions.

(i) Any non-zero morphism $f\colon V \to E$ satisfies $\langle \widetilde{\boldsymbol{\alpha}}, \mathtt{image}(f)\rangle \geq 0$.

(ii) If we assume E is $\widetilde{\boldsymbol{\alpha}}$-stable, then every non-zero map $f\colon V \to E$ is surjective.

(iii) If E is $\widetilde{\boldsymbol{\alpha}}$-stable and $\langle \widetilde{\boldsymbol{\eta}}, \boldsymbol{\beta}\rangle \leq 0$, then $\mathtt{Hom}(V, E) = 0$.

Proof. We first show that for the general vector bundle $V \in \mathsf{Bun}_{m\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\eta}}}(k)$ one has that $\pi_* V$ is ℓ-regular for some ℓ independent of m. (Recall that F is ℓ-regular if $H^1(F(\ell-1)) = 0$.) Clearly this condition is open, so we just have to show there exists such a bundle. Let $E_1 \in \mathsf{Bun}_{\widetilde{\boldsymbol{\alpha}}}(k)$ be such that $\pi_* E_1$ is ℓ_1-regular and $E_2 \in \mathsf{Bun}_{\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\eta}}}(k)$ such that $\pi_* E_2$ is ℓ_2-regular, for some ℓ_1 and ℓ_2. Then $V = E_1^{\oplus m-1} \oplus E_2 \in \mathsf{Bun}_{m\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\eta}}}(k)$ is such that $\pi_* V$ is ℓ-regular for $\ell := \mathtt{max}(\ell_1, \ell_2)$, which is independent of m.

For (i), we will show that the locus inside $\mathsf{Bun}_{m\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\eta}}}$ where the condition fails has positive codimension. We will stratify this locus by the possible algebraic invariants of the image of the non-zero maps $f\colon V \to E$ such that $\langle \widetilde{\boldsymbol{\alpha}}, \mathtt{image}(f)\rangle < 0$ and $\pi_* V$ is ℓ-regular. Let $\boldsymbol{\gamma}$ be the numerical invariant of $G := \mathtt{image}(f)$, and recall that $\boldsymbol{\alpha}, \boldsymbol{\eta}$ are the numerical invariants of $\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\eta}}$, respectively. We claim that there are only finitely many values of $\boldsymbol{\gamma}$ that can appear. Since $G \subset E$, we have $1 \leq \mathtt{rank}(\boldsymbol{\gamma}) \leq \mathtt{rank}(\boldsymbol{\beta})$ and the multiplicities of G are bounded by those of E, so it remains to bound the degree of G. In fact, we will bound the $\boldsymbol{\alpha}$-degree of G and see that our bounds are independent of m. Since $\pi_* V$ is ℓ-regular, $\pi_* G$ is ℓ-regular as well, and we have $\mathsf{deg}\, \pi_* G(\ell) \geq 0$, so $\mathsf{deg}(\pi_* G) \geq -\ell\, \mathtt{rank}\, G$. On the other hand, by our assumption on f, we have $\mathsf{deg}_{\boldsymbol{\alpha}}(\boldsymbol{\gamma}) + \mathtt{rank}(\boldsymbol{\gamma}) \cdot \langle \boldsymbol{\alpha}, \mathcal{O}\rangle = \langle \boldsymbol{\alpha}, \boldsymbol{\gamma}\rangle < 0$, so combining this with Proposition 1.3.17 as well as the inequality we already obtained from ℓ-regularity, we get

$$-\ell\, \mathtt{rank}(\boldsymbol{\gamma}) \leq \mathsf{deg}\, \pi_* \boldsymbol{\gamma} \leq \mathsf{deg}_{\boldsymbol{\alpha}}(\boldsymbol{\gamma}) < -\mathtt{rank}(\boldsymbol{\gamma}) \cdot \langle \boldsymbol{\alpha}, \mathcal{O}\rangle. \tag{4.4}$$

Hence there are finitely many possibilities for $\boldsymbol{\gamma}$. For each of these finitely many $\boldsymbol{\gamma}$ with $\langle \boldsymbol{\alpha}, \boldsymbol{\gamma}\rangle < 0$, we let $B_{\boldsymbol{\gamma}}$ be the locus of $V \in \mathsf{Bun}_{m\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\eta}}}$ where there is a non-zero morphism $f\colon V \to E$ whose image G has invariant $\boldsymbol{\gamma}$.

Now consider the diagram of vector bundles, with specified invariants.

$$\begin{tikzcd} 0 \arrow[r] & \overset{m\boldsymbol{\alpha}+\boldsymbol{\eta}-\boldsymbol{\gamma}}{K} \arrow[r] & \overset{m\boldsymbol{\alpha}+\boldsymbol{\eta}}{V} \arrow[r] \arrow[dr, "f"'] & \overset{\boldsymbol{\gamma}}{G} \arrow[r] \arrow[d, hook] & 0 \\ & & & \underset{\boldsymbol{\beta}}{E} & \end{tikzcd}$$

Let $\mathcal{E}_{\boldsymbol{\gamma}}$ be the substack of $\mathsf{SES}_{m\boldsymbol{\alpha}+\boldsymbol{\eta}-\boldsymbol{\gamma},\boldsymbol{\gamma}}$, given by short exact sequences where the determinant of the pushforward of the middle term is equal to $\mathbf{det}\,\pi_*(m\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\eta}})$. By definition $\mathcal{E}_{\boldsymbol{\gamma}}$ fits in a Cartesian square:

$$\begin{tikzcd} \mathcal{E}_{\boldsymbol{\gamma}} \arrow[r] \arrow[d] & \mathsf{SES}_{m\boldsymbol{\alpha}+\boldsymbol{\eta}-\boldsymbol{\gamma},\boldsymbol{\gamma}} \arrow[d] \\ B\mathbb{G}_m \arrow[r] & \mathtt{Pic}(C) \end{tikzcd}$$

The right vertical map sends $(E_1 \to F \to E_2) \mapsto \mathbf{det}\,\pi_* F$, and the bottom arrow is the inclusion of $\mathbf{det}\,\pi_*(m\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\eta}})$ in $\mathtt{Pic}(C)$. Therefore we can compute the dimension

$$\mathtt{dim}\,\mathcal{E}_{\boldsymbol{\gamma}} = \mathtt{dim}\,\mathsf{SES}_{m\boldsymbol{\alpha}+\boldsymbol{\eta}-\boldsymbol{\gamma},\boldsymbol{\gamma}} - g_C.$$

Since the middle projection $\mathcal{E}_{\boldsymbol{\gamma}} \to \mathsf{Bun}_{m\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\eta}}}$ is representable and its image contains the locus $B_{\boldsymbol{\gamma}}$, it follows that

$$\begin{aligned} \mathtt{codim}\,B_{\boldsymbol{\gamma}} &\geq \mathtt{dim}\,\mathsf{Bun}_{m\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\eta}}} - \mathtt{dim}\,\mathcal{E}_{\boldsymbol{\gamma}} \\ &= -\langle m\boldsymbol{\alpha}+\boldsymbol{\eta}, m\boldsymbol{\alpha}+\boldsymbol{\eta}\rangle - g_C + \langle m\boldsymbol{\alpha}+\boldsymbol{\eta}-\boldsymbol{\gamma}, m\boldsymbol{\alpha}+\boldsymbol{\eta}-\boldsymbol{\gamma}\rangle \\ &\quad + \langle \boldsymbol{\gamma},\boldsymbol{\gamma}\rangle + \langle \boldsymbol{\gamma}, m\boldsymbol{\alpha}+\boldsymbol{\eta}-\boldsymbol{\gamma}\rangle + g_C \\ &= \langle \boldsymbol{\gamma},\boldsymbol{\gamma}\rangle - m\langle \boldsymbol{\alpha},\boldsymbol{\gamma}\rangle - \langle \boldsymbol{\eta},\boldsymbol{\gamma}\rangle. \end{aligned} \tag{4.5}$$

Since $\langle \boldsymbol{\alpha},\boldsymbol{\gamma}\rangle < 0$ by assumption, this codimension is positive for sufficiently large m, namely for $m > \frac{\langle \boldsymbol{\gamma}-\boldsymbol{\eta},\boldsymbol{\gamma}\rangle}{\langle \boldsymbol{\alpha},\boldsymbol{\gamma}\rangle}$.

For statement (ii), if $\mathtt{image}(f)$ is a proper subbundle, then by $\boldsymbol{\alpha}$-stability of E we conclude $\langle \boldsymbol{\alpha}, \mathtt{image}(f)\rangle < 0$, which contradicts (i). Hence $\mathtt{image}(f)$ is either 0 or E.

Finally to prove statement (iii), let $f\colon V \to E$ be a non-zero map. By (ii) we may assume that f is surjective, so we get an exact sequence

$$0 \longrightarrow \overset{m\boldsymbol{\alpha}+\boldsymbol{\eta}-\boldsymbol{\beta}}{K} \longrightarrow \overset{m\boldsymbol{\alpha}+\boldsymbol{\eta}}{V} \xrightarrow{f} \overset{\boldsymbol{\beta}}{E} \longrightarrow 0.$$

Let $\mathcal{E}$ be the substack of $\mathsf{SES}_{m\boldsymbol{\alpha}+\boldsymbol{\eta}-\boldsymbol{\beta},\boldsymbol{\beta}}$ where the final term is abstractly isomorphic to E and the determinant of the pushforward of the middle term is equal to $\mathtt{det}\,\pi_*(m\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\eta}})$. Then $\mathcal{E}$ has dimension

$$\mathtt{dim}\,\mathcal{E} = \mathtt{dim}\,\mathsf{Bun}_{m\boldsymbol{\alpha}+\boldsymbol{\eta}-\boldsymbol{\beta}} - \langle\boldsymbol{\beta}, m\boldsymbol{\alpha}+\boldsymbol{\eta}-\boldsymbol{\beta}\rangle - \mathtt{dim}\,\mathsf{Aut}(E) - g,$$

and the morphism $\mathcal{E} \to \mathsf{Bun}_{m\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\eta}}}$ is representable and its image contains the locus $B \subset \mathsf{Bun}_{m\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\eta}}}$ consisting of V such that $\mathsf{Hom}(V,E) \neq 0$. Hence, using that $\mathtt{dim}\,\mathsf{Aut}(E) \geq 1$,

$$\begin{aligned}
\mathtt{codim}\,B \geq{}& \mathtt{dim}\,\mathsf{Bun}_{m\widetilde{\boldsymbol{\alpha}}+\widetilde{\boldsymbol{\eta}}} - \mathtt{dim}\,\mathcal{E} \\
\geq{}& -\langle m\boldsymbol{\alpha}+\boldsymbol{\eta}, m\boldsymbol{\alpha}+\boldsymbol{\eta}\rangle - g_C \\
&- (-\langle m\boldsymbol{\alpha}+\boldsymbol{\eta}-\boldsymbol{\beta}, m\boldsymbol{\alpha}+\boldsymbol{\eta}-\boldsymbol{\beta}\rangle - \langle\boldsymbol{\beta}, m\boldsymbol{\alpha}+\boldsymbol{\eta}-\boldsymbol{\beta}\rangle - 1 - g_C) \\
={}& -\langle m\boldsymbol{\alpha}+\boldsymbol{\eta}, \boldsymbol{\beta}\rangle + 1 = -\langle\boldsymbol{\eta},\boldsymbol{\beta}\rangle + 1,
\end{aligned}$$

which is positive precisely when $\langle\boldsymbol{\eta},\boldsymbol{\beta}\rangle \leq 0$. ⬠

To prove semiampleness, we only need to apply the above proposition to the case $\widetilde{\boldsymbol{\eta}} = 0$. However in Section 4.4, in order to be able to separate points using the sections σ_V, we will use this proposition when $\widetilde{\boldsymbol{\eta}} = \pm\widetilde{\boldsymbol{\delta}}$, where $\widetilde{\boldsymbol{\delta}}$ is the algebraic invariant of a degree 1 torsion sheaf supported at a non-stacky point

Proposition 4.4.2 Consider the situation of Proposition 4.4.1, and assume in addition that $\langle\boldsymbol{\eta},\boldsymbol{\gamma}\rangle \leq 0$ for every positive numerical invariant $\boldsymbol{\gamma}$. Then the constant κ from Proposition 4.4.1 can be chosen to be

$$\kappa_{\boldsymbol{\beta}} := \mathsf{max}((g_{\mathcal{C}} - 1)(\mathtt{rank}\,\boldsymbol{\beta})^2, 0).$$

Note that the condition on $\boldsymbol{\eta}$ is satisfied for $\boldsymbol{\eta} = 0$ and $\boldsymbol{\eta} = \boldsymbol{\delta}$ is the numerical class of a skyscraper sheaf at a non-stacky point. When $\boldsymbol{\eta} = -\boldsymbol{\delta}$, we can choose the bound

$$\kappa^+_{\boldsymbol{\beta}} := \mathsf{max}\left(\left(g_{\mathcal{C}} - 1 + \frac{1}{\mathtt{rank}\,\boldsymbol{\beta}}\right)(\mathtt{rank}\,\boldsymbol{\beta})^2, 0\right).$$

Proof. We need to ensure that the quantity in (4.5) is positive. It suffices to take

$$\kappa \geq \mathsf{max}_{\gamma}\left(\frac{\langle\gamma,\gamma\rangle - \langle\boldsymbol{\eta},\gamma\rangle}{\langle\boldsymbol{\alpha},\gamma\rangle}\right),$$

where $\boldsymbol{\gamma}$ runs over the finite list of numerical invariants $\boldsymbol{\gamma}$ of subbundles of E, satisfying $\langle \boldsymbol{\alpha}, \boldsymbol{\gamma} \rangle < 0$. Since $\langle \boldsymbol{\alpha}, \boldsymbol{\gamma} \rangle \leq -1$ and by assumption $\langle \boldsymbol{\eta}, \boldsymbol{\gamma} \rangle \leq 0$, it suffices to take

$$\kappa \geq \max_{\boldsymbol{\gamma}}(-\langle \boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle),$$

where $\boldsymbol{\gamma}$ runs over the invariants of subbundles of E. By Proposition 3.1.10, we have

$$-\langle \boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle \leq (g_{\mathcal{C}} - 1)(\operatorname{rank} \boldsymbol{\gamma})^2$$

and as we need $\kappa \geq 0$, we conclude the claimed bound. For the case $\boldsymbol{\eta} = -\boldsymbol{\delta}$ we follow the same argument, but note that $\langle \boldsymbol{\eta}, \boldsymbol{\gamma} \rangle = \operatorname{rank} \boldsymbol{\gamma}$. ⬡

Corollary 4.4.3 Let $\widetilde{\boldsymbol{\alpha}} \in \mathsf{K}_0(\mathcal{C})$ be a positive algebraic invariant, and let $\boldsymbol{\beta}$ be a positive numerical invariant such that $\langle \widetilde{\boldsymbol{\alpha}}, \boldsymbol{\beta} \rangle = 0$. Let $\widetilde{\boldsymbol{\eta}} \in \mathsf{K}_0(\mathcal{C})$ be an effective algebraic invariant. Then there exists $m \gg 0$ such that for any $E \in \mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}(k)$ satisfying $\langle \boldsymbol{\eta}, E_i \rangle \leq 0$ for every stable subquotient E_i of E, a generic vector bundle V with algebraic invariant $m\widetilde{\boldsymbol{\alpha}} + \widetilde{\boldsymbol{\eta}}$ satisfies

$$\mathsf{Hom}(V, E) = 0.$$

Proof. The proof inductively considers the Jordan-Hölder filtration $0 \subsetneq E^{(1)} \subset \cdots \subsetneq E^{(r)} = E$ of E whose subquotients $E_i = E^{(i)}/E^{(i-1)}$ are $\boldsymbol{\alpha}$-stable of the same $\boldsymbol{\alpha}$-slope. By applying Proposition 4.4.1 to each E_i we deduce for $m \gg 0$ that a general vector bundle V with algebraic invariant $m\widetilde{\boldsymbol{\alpha}} + \widetilde{\boldsymbol{\eta}}$ satisfies $\mathsf{Hom}(V, E_i) = 0$ for each $i = 1, \ldots, r$. By inductively applying $\mathsf{Hom}(V, \_)$ to the exact sequences $0 \to E^{(i-1)} \to E^{(i)} \to E_i \to 0$ we obtain $\mathsf{Hom}(V, E) = 0$. ⬡

Remark 4.4.4 In fact, the same effective bound for m giving Hom-vanishing for stables given in Proposition 4.4.2 also work for the Hom-vanishing of semistable vector bundles, as the proof of Corollary 4.4.3 involves applying Proposition 4.4.1 to subinvariants.

Thus semistability can be characterized in terms of a Hom-vanishing condition as follows. This was originally noticed for Higgs bundles over classical curves by Faltings [Fal93, Theorem I.2].

> **Proposition 4.4.5** Let $\widetilde{\boldsymbol{\alpha}} \in \mathsf{K}_0(\mathcal{C})$ be a positive algebraic invariant, and let $\boldsymbol{\beta}$ be a positive numerical invariant such that $\langle \widetilde{\boldsymbol{\alpha}}, \boldsymbol{\beta} \rangle = 0$. Then $E \in \mathsf{Bun}_{\boldsymbol{\beta}}(k)$ is $\boldsymbol{\alpha}$-semistable if and only if there is a vector bundle V with algebraic invariant $m\widetilde{\boldsymbol{\alpha}}$ for some $m > 0$ such that
>
> $$\mathtt{Hom}(V, E) = \mathtt{Ext}^1(V, E) = 0.$$

Proof. By assumption $\langle m\widetilde{\boldsymbol{\alpha}}, \boldsymbol{\beta} \rangle = 0$, so $\mathtt{dim}\,\mathtt{Hom}(V, E) = \mathtt{dim}\,\mathtt{Ext}^1(V, E)$. Thus the forward direction follows from Corollary 4.4.3. Conversely suppose V has invariant $m\widetilde{\boldsymbol{\alpha}}$ and satisfies $\mathtt{Hom}(V, E) = 0$. To show E is $\boldsymbol{\alpha}$-semistable we consider a subbundle $E' \subset E$ with quotient E'' and apply $\mathtt{Hom}(V, \_)$ to the exact sequence $0 \to E' \to E \to E'' \to 0$ to deduce $\mathtt{Hom}(V, E') = 0$. This implies

$$\langle m\boldsymbol{\alpha}, [E'] \rangle = -\,\mathtt{dim}\,\mathtt{Ext}^1\big(V, E'\big) \leq 0 = \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle,$$

from which we obtain $\mu_{\boldsymbol{\alpha}}(E') \leq \mu_{\boldsymbol{\alpha}}(E)$. ⬡

Ext-vanishing and separating stable bundles

Here we prove the key results that enable us to deduce ampleness of the determinantal line bundle. First, we show that Serre duality sends semistable vector bundles to semistable vector bundles in order to translate Hom-vanishing results into Ext-vanishing results. Throughout $\widetilde{\boldsymbol{\delta}} = (\boldsymbol{\delta}, \mathcal{O}_{\mathcal{C}}(x))$ denotes the numerical invariant of a degree 1 torsion sheaf supported at a non-stacky point $x \in \mathcal{C}$, that is $\boldsymbol{\delta} = (0, 1, \underline{0})$.

> **Proposition 4.4.6** Let $\boldsymbol{\alpha}$ be a generating numerical invariant and let E be a vector bundle on $\mathcal{C}$ such that $\langle \boldsymbol{\alpha}, E \rangle = 0$. Then E is $\boldsymbol{\alpha}$-semistable if and only if the Serre dual $\mathsf{SD}(E) := \mathcal{H}om(E, \omega_{\mathcal{C}})$ is $\boldsymbol{\alpha}^\vee$-semistable

Proof. Assume that E is $\boldsymbol{\alpha}$-semistable. Given a subsheaf $F \subset \mathsf{SD}(E)$, we apply Serre duality (Theorem 1.3.6) to get the following equality:

$$\langle \boldsymbol{\alpha}^\vee, F \rangle = -\langle F, \boldsymbol{\alpha}^\vee \otimes \boldsymbol{\omega} \rangle = -\langle \boldsymbol{\alpha}, \mathsf{SD}(F) \rangle.$$

But $\mathsf{SD}(F)$ is a quotient of the $\boldsymbol{\alpha}$-semistable sheaf E, hence $-\langle \boldsymbol{\alpha}, \mathsf{SD}(F) \rangle \leq 0$, as desired. For the converse, we just replace E with $\mathsf{SD}(E)$ and $\boldsymbol{\alpha}$ with $\mathsf{SD}(\boldsymbol{\alpha})$ and notice that $\mathsf{SD}(\mathsf{SD}(E)) \cong E$, hence the argument above applies. ⬡

Lemma 4.4.7 Let $\widetilde{\boldsymbol{\alpha}} \in \mathsf{K}_0(\mathcal{C})$ be a positive algebraic invariant, and let $\boldsymbol{\beta}$ be a positive numerical invariant such that $\langle \widetilde{\boldsymbol{\alpha}}, \boldsymbol{\beta} \rangle = 0$. Then for every $m > \kappa_{\boldsymbol{\beta}}$ and for every $E \in \mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}(k)$, a general vector bundle V with invariant $m\widetilde{\boldsymbol{\alpha}} - \widetilde{\boldsymbol{\delta}}$ satisfies

$$\mathtt{Ext}^1(V, E) = 0.$$

Proof. Note that $\mathtt{Ext}^1(V, E) = \mathtt{Hom}(V^\vee, \mathsf{SD}(E)))^\vee$ and that $V^\vee$ has invariant $m\widetilde{\boldsymbol{\alpha}}^\vee + \widetilde{\boldsymbol{\delta}}$. In addition:

(a) $\langle \boldsymbol{\delta}, G \rangle = -\,\mathtt{rank}(G) \leq 0$ for every bundle G.

(b) if E is $\boldsymbol{\alpha}$-semistable, then $\mathsf{SD}(E)$ is $\alpha^\vee$-semistable (see Proposition 4.4.6);

(c) if $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = 0$, then also $\langle \boldsymbol{\alpha}^\vee, \mathsf{SD}(\boldsymbol{\beta}) \rangle = 0$.

These three conditions ensure that we can apply Corollary 4.4.3 (and Remark 4.4.4) to the $\boldsymbol{\alpha}^\vee$-semistable sheaf $\mathsf{SD}(E)$ and conclude the argument. ⬠

Before we can show that the determinantal line bundle has enough sections to separate most points, we first need a lemma which is a step towards producing the vector bundle defining the section we want: rather than constructing a vector bundle with algebraic invariant $m\widetilde{\boldsymbol{\alpha}}$ we construct a vector bundle with invariant $m\widetilde{\boldsymbol{\alpha}} - \widetilde{\boldsymbol{\delta}}$. We will later extend V to construct the section needed to separate points.

Lemma 4.4.8 Let $E_0, \ldots, E_\ell$ be $\boldsymbol{\alpha}$-stable bundles whose numerical invariants $\boldsymbol{\beta}_i$ satisfy $\langle \widetilde{\boldsymbol{\alpha}}, \boldsymbol{\beta}_i \rangle = 0$ and such that $E_0 \not\cong E_i$ for every $i = 1, \ldots, \ell$. Then for $m > \mathtt{max}_{0 \leq i \leq \ell}\, \kappa^+_{2\boldsymbol{\beta}_i}$, a generic vector bundle V with invariant $m\widetilde{\boldsymbol{\alpha}} - \widetilde{\boldsymbol{\delta}}$ has the following properties:

(i) $\mathtt{Ext}^1(V, E_i) = 0$ for all $i = 0, \ldots, \ell$;

(ii) for all $i = 0, \ldots, \ell$ any non-zero homomorphism $V \to E_i$ is surjective;

(iii) for all $i = 1, \ldots, \ell$ and non-zero homomorphisms $f_0 \colon V \to E_0$ and $f_i \colon V \to E_i$, the homomorphism $g_i = (f_0, f_i) \colon V \to E_0 \oplus E_i$ is surjective.

Proof. Part (i) follows from Lemma 4.4.7. For Part (ii), we can apply Proposition 4.4.1

(ii) to the $\boldsymbol{\alpha}$-stable bundles $E_0, \dots, E_\ell$ to deduce that, for $m > \kappa_{\boldsymbol{\beta}_i}$ a generic vector bundle V with invariant $m\widetilde{\boldsymbol{\alpha}} - \widetilde{\boldsymbol{\delta}}$ every non zero map $V \to E_i$ is surjective.

For Part (iii), we apply Proposition 4.4.1 (i) to $E_0 \oplus E_i$ to deduce that the image of g_i, which we denote G_i, has necessarily the same $\boldsymbol{\alpha}$-slope as $E_0 \oplus E_i$. Note that, by Remark 1.3.26 and the fact that $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}_0 \rangle = 0 = \langle \boldsymbol{\alpha}, \boldsymbol{\beta}_i \rangle$, the slopes coincide $\mu_{\boldsymbol{\alpha}}(E_0) = \mu_\alpha(E_i) = \mu_\alpha(E_0 \oplus E_i)$. Since the map g_i is non-zero, this implies that if g_i is not surjective, then G_i is either isomorphic to either E_0 or E_i. Since both f_0 and f_i are surjective, it follows that also the projections $G_i \to E_0$ and $G_i \to E_i$ are surjective. Since $E_0 \not\cong E_i$, the conditions $G_i \cong E_i$ or $G_i \cong E_0$ are impossible to achieve, thus the map f is surjective. ⬡

Remark 4.4.9 Note that the factor 2 appears in the bound $\mathtt{max}_i\, \kappa^+_{2\boldsymbol{\beta}_i}$. This is because we apply Proposition 4.4.1 to $E_0 \oplus E_i$, which has invariant $\boldsymbol{\beta}_0 + \boldsymbol{\beta}_i \leq \mathtt{max}_i\, 2\boldsymbol{\beta}_i$.

To separate certain polystable vector bundles, we now construct a vector bundle H with algebraic invariant $m\widetilde{\boldsymbol{\alpha}}$ as a Hecke extension of a skyscraper sheaf at a non-stacky point $x \in \mathcal{C}$ by a vector bundle V with algebraic invariant $m\widetilde{\boldsymbol{\alpha}} - \widetilde{\boldsymbol{\delta}}$ as outlined before Lemma 4.4.8.

Proposition 4.4.10 Let $E = E_1 \oplus \cdots \oplus E_\ell$ and $F = F_1 \oplus \cdots \oplus F_{\ell'}$ be $\boldsymbol{\alpha}$-polystable vector bundles on $\mathcal{C}$ with numerical invariant $\boldsymbol{\beta}$, where the E_i and F_j are the stable summands. If $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = 0$ and none of the E_i are isomorphic to F_1, then for $m > \kappa^+_{2\boldsymbol{\beta}}$ there exists a vector bundle H with algebraic invariant $m\widetilde{\boldsymbol{\alpha}}$ such that

$$\mathtt{Hom}(H, E) = 0 \quad \text{and} \quad \mathtt{Hom}(H, F) \neq 0. \tag{4.6}$$

Hence there is a section of the determinantal line bundle $\mathcal{L}^{\otimes m}_{\widetilde{\boldsymbol{\alpha}}}$, separating E and F.

Proof. Let $\boldsymbol{\beta}_i$ be the numerical invariants of E_i. Since up to a constant, the $\boldsymbol{\alpha}$-slope of any vector bundle G with invariant $\boldsymbol{\gamma}$ is $\langle \boldsymbol{\alpha}, \boldsymbol{\gamma} \rangle / \ \mathtt{rank}(\boldsymbol{\gamma})$ and we assumed $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = 0$, we conclude $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}_i \rangle = 0$ for all i.

Since $\widetilde{\boldsymbol{\delta}}$ is the invariant of a skyscraper sheaf supported at a non-stacky point, we have $\langle \boldsymbol{\delta}, \boldsymbol{\beta}_i \rangle = -\,\mathtt{rank}(\boldsymbol{\beta}_i) < 0$, so we may apply Lemma 4.4.8 to the $\boldsymbol{\alpha}$-stable vector bundles $E_0 := F_1$ and $E_1, \dots, E_\ell$. We now fix a vector bundle V with invariant $m\widetilde{\boldsymbol{\alpha}} - \widetilde{\boldsymbol{\delta}}$ with the properties stated in this lemma. The rest of the proof consists of constructing a suitable extension of $\mathcal{O}_x$ by V, which will automatically have invariant $m\widetilde{\boldsymbol{\alpha}}$.

Fix a non-zero surjection $\phi\colon V \twoheadrightarrow E_0$ and let K denote the kernel, so that

$$0 \longrightarrow K \longrightarrow V \xrightarrow{\phi} E_0 \longrightarrow 0$$

is an exact sequence. Let H be any non-split extension of $\mathcal{O}_x$ by V lying in the subspace

$$\mathtt{Ext}^1(\mathcal{O}_x, K) \subset \mathtt{Ext}^1(\mathcal{O}_x, V).$$

Then H is a vector bundle, as this is a non-split extension of a torsion sheaf by a vector bundle, and we claim that there exists a non-zero morphism $H \to E_0$. To see this, consider the corresponding extension G of $\mathcal{O}_x$ by K, fitting into the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, hook] & G \arrow[r] \arrow[d, hook, "a"] & \mathcal{O}_x \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & \mathcal{O}_x \arrow[r] & 0 \\
& E_0 & \mathtt{coker}\, a & &
\end{tikzcd} \tag{4.7}$$

By the snake lemma we get an isomorphism $E_0 \simeq \mathtt{coker}(a)$, so we get a non-zero morphism $H \to \mathtt{coker}(a) \to E_0$.

We will now show that for a general $H \in \mathtt{Ext}^1(\mathcal{O}_x, K)$ there are no non-zero morphisms $H \to E_i$ for $1 \le i \le \ell$. Fix $H \in \mathtt{Ext}^1(\mathcal{O}_x, K)$ and let $\tilde{\psi}_i$ be a non-zero morphism $\tilde{\psi}_i\colon H \to E_i$. The composition $V \to H \to E_i$, denoted by ψ_i, must be non-zero, otherwise we would obtain a non-zero morphism $\mathcal{O}_x \to E_i$. We obtain a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\iota"] \arrow[d, dotted, "\psi_i"] & V \arrow[r, "\phi"] \arrow[d, "(\phi, \psi_i)"] & E_0 \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & E_i \arrow[r] & E_0 \oplus E_i \arrow[r] & E_0 \arrow[r] & 0
\end{tikzcd}$$

By the assumption on V, the map (ϕ, ψ_i) is surjective, so by the snake lemma we see that ψ_i restricts to a surjective morphism $\psi_i \colon K \twoheadrightarrow E_i$. This induces a map on Ext groups $\mathtt{Ext}^1(\mathcal{O}_x, K) \twoheadrightarrow \mathtt{Ext}^1(\mathcal{O}_x, E_i)$, so we obtain an extension of $\mathcal{O}_x$ by E_i, fitting into the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] \arrow[d, two heads, "\psi_i"] & H \arrow[r] \arrow[d, two heads, "b"] \arrow[dl, "\tilde{\psi}_i"] & \mathcal{O}_x \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & E_i \arrow[r] & H_i \arrow[r] & \mathcal{O}_x \arrow[r] & 0 \end{tikzcd}$$

Now we notice that $\tilde{\psi}_i \circ b^{-1}$ is a well-defined splitting, so H must lie in the kernel of $\mathtt{Ext}^1(\mathcal{O}_x, K) \twoheadrightarrow \mathtt{Ext}^1(\mathcal{O}_x, E_i)$.

Let $\mathtt{Gr}(\mathtt{Ext}^1(\mathcal{O}_x, K), r_i)$ be the Grassmanian of r_i dimensional quotients, where r_i is the rank of E_i. For any element J in $\mathtt{Ext}^1(\mathcal{O}_x, K)$, we define the Schubert variety $S_{J,i}$ by

$$S_{J,i} \coloneqq \{f \colon \mathtt{Ext}^1(\mathcal{O}_x, K) \twoheadrightarrow W \text{ s.t. } f(J) = 0\} \subset \mathtt{Gr}(\mathtt{Ext}^1(\mathcal{O}_x, K), r_i).$$

We have shown that for a non-zero morphism $\tilde{\phi}_i \colon H \to E_i$, the induced quotient $\mathtt{Ext}^1(\mathcal{O}_x, K) \twoheadrightarrow \mathtt{Ext}^1(\mathcal{O}_x, E_i)$ lies in $S_{H,i}$. The quotient $\mathtt{Ext}^1(\mathcal{O}_x, K) \twoheadrightarrow \mathtt{Ext}^1(\mathcal{O}_x, E_i)$ only depends on the restriction $\phi_i \colon V \twoheadrightarrow E_i$, so we can consider the maps

$$q_i : \mathbb{P}(\mathtt{Hom}(V, E_i)) \to \mathtt{Gr}(\mathtt{Ext}^1(\mathcal{O}_x, K), r_i) \text{ for each } 1 \leq i \leq \ell.$$

Now notice that $\mathtt{dim}\,\mathtt{image}(q_i) \leq r_i - 1$, whereas the codimension of $S_{H,i}$ is r_i. By Kleiman's theorem, for a general $g \in \mathtt{GL}(\mathtt{Ext}^1(\mathcal{O}_x, K))$ we have that $\mathtt{image}\, q_i \cap g \cdot S_{H,i} = \emptyset$. Since $g \cdot S_{H,i} = S_{gH,i}$ we see that for a general $H \in \mathtt{Ext}^1(\mathcal{O}_x, K)$ the intersection $\mathtt{image}\, q_i \cap S_{H,i} = \emptyset$. It follows that for the general extension H we have $\mathtt{Hom}(H, E_i) = 0$ for $1 \leq i \leq \ell$. We conclude that that there exists an extension H of $\mathcal{O}_x$ by V, such that $\mathtt{Hom}(H, F) \supset \mathtt{Hom}(H, F_1) \neq 0$ and $\mathtt{Hom}(H, E) = \bigoplus_{1 \leq i \leq l} \mathtt{Hom}(H, E_i) = 0.$ ◇

4.5 Ampleness of the determinantal line bundle

Throughout this section, we fix numerical invariants $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with $\boldsymbol{\alpha}$ generating, such that $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = 0$. The following lemma shows that this last assumption can be made without loss of generality.

Lemma 4.5.1 Let $\boldsymbol{\alpha}$ be a generating numerical invariant. For any positive numerical invariant $\boldsymbol{\beta}$, there exists a generating numerical invariant $\boldsymbol{\alpha}'$ such that $\langle \boldsymbol{\alpha}', \boldsymbol{\beta} \rangle = 0$ and that the notions of (semi)stability with respect to $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ coincide.

Proof. Assume that $A := \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle \neq 0$, and let $\boldsymbol{\eta} = [\mathcal{O}(q)] \in \mathsf{K}_0^{\mathsf{num}}(\mathcal{C})$ for a non-stacky point $q \in \mathcal{C}$. Pick $r \in \mathbb{Z}$ so that $B = \langle \boldsymbol{\alpha} \otimes \boldsymbol{\eta}^{\otimes r}, \boldsymbol{\beta} \rangle$ has the opposite sign from A. Then it is straightforward to check that the numerical invariant

$$\boldsymbol{\alpha}' := |B|\boldsymbol{\alpha} + |A|\boldsymbol{\alpha} \otimes \boldsymbol{\eta}^{\otimes r}$$

is orthogonal to $\boldsymbol{\beta}$ and additionally $\boldsymbol{\alpha}'$ is generating, as it is a positive linear combination of a generating invariant and an effective invariant. The equivalence of the corresponding notions of (semi)stability follows from the fact that $\mathbf{deg}_{\boldsymbol{\alpha}'} = (|A| + |B|)\,\mathbf{deg}_{\boldsymbol{\alpha}}$, as this degree is additive and preserved by tensoring by the line bundle $\mathcal{O}(rq)$, which it is the pullback of a line bundle on C by virtue of q being a non-stacky point. ◯

Global generation

Theorem 4.5.2 Let k be an arbitrary field, and assume $\langle \widetilde{\boldsymbol{\alpha}}, \boldsymbol{\beta} \rangle = 0$ with $\widetilde{\boldsymbol{\alpha}}$ a generating algebraic invariant. Then the line bundle $\mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}$ on the stack $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is semiample. More precisely, for every positive integer m with

$$m > (g_{\mathcal{C}} - 1)(\mathtt{rank}\,\boldsymbol{\beta})^2,$$

$\mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}^{\otimes m}$ is basepoint-free. If additionally k has characteristic zero, then the line bundle $\mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}$ descends to a semiample line bundle $L_{\widetilde{\boldsymbol{\alpha}}}$ on the good moduli space $B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$.

Proof. We can assume without loss of generality that k is algebraically closed, as it suffices to know that the base change to an algebraic closure is semiample (see [Vak24, Exercise 19.2.I]).

Fix a positive natural number m such that $m > (g_{\mathcal{C}} - 1)(\mathtt{rank}\,\boldsymbol{\beta})^2$; this gives an effective bound for Hom-vanishing by Proposition 4.4.2 and Remark 4.4.4. For a point of $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ corresponding to an $\boldsymbol{\alpha}$-semistable vector bundle F on $\mathcal{C}$ with numerical invariants $\boldsymbol{\beta}$, we know that a general bundle V with algebraic invariant $m\widetilde{\boldsymbol{\alpha}}$ satisfies $\mathtt{Ext}^1(V, F) = 0$ by Corollary 4.4.3. In particular, we can find such a vector bundle V so that the associated section σ_V of $\mathcal{L}_V \cong \mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}^{\otimes m}$ is non-zero at this point by (4.3). Since $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is quasi-compact, the non-vanishing loci of finitely many such sections cover $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ and so $\mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}^{\otimes m}$ is basepoint-free.

For the final claim, we let $\sigma_0, \ldots, \sigma_n$ be global sections that generate $\mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}^{\otimes m}$ and thus induce a morphism $\phi_m\colon \mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}} \to \mathbb{P}^n$ such that $\mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}^{\otimes m} \cong \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$. Since the good moduli space map $f\colon \mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}} \to B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is initial amongst morphisms to schemes, ϕ_m must factor via f and so there is an induced morphism $\varphi_m\colon B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}} \to \mathbb{P}^n$ such that $L_m \coloneqq \varphi_m^*\mathcal{O}_{\mathbb{P}^n}(1)$ pulls back along f to $\mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}^{\otimes m}$. Then $L_{\widetilde{\boldsymbol{\alpha}}} \coloneqq L_{m+1} \otimes L_m^{-1}$ pulls back along f to $\mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}$. ⬠

Ampleness and projectivity

We are ready to prove the main theorem of the chapter.

> **Theorem 4.5.3** Let k be a field of characteristic zero and assume $\langle \widetilde{\boldsymbol{\alpha}}, \boldsymbol{\beta} \rangle = 0$, where $\widetilde{\boldsymbol{\alpha}}$ is a generating algebraic invariant. Then the line bundle $L_{\widetilde{\boldsymbol{\alpha}}}$ on $B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is ample and $B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is projective.

Proof. Since $B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is proper (Corollary 4.2.5), by the cohomological criterion for ampleness [Stacks, Tag 0D38] and flat base change, we can assume without loss of generality that k is algebraically closed.

As in Theorem 4.5.2, we know that a sufficiently large power m of the determinantal line bundle on $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is globally generated by finitely many sections which determine a morphism $\phi\colon \mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}} \to \mathbb{P}^n$ that factors via the good moduli space map $f\colon \mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}} \to B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ and a morphism $\varphi\colon B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}} \to \mathbb{P}^n$. Since $B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is proper, φ is a proper morphism and to conclude the proof it is then enough to show that φ is finite.

To show that φ is finite, it suffices to show that the fibres of φ are finite by [Stacks, Tag 0A4X]. Since $B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is of finite type, it is enough to check that fibers over k-points are finite, (see [GW20, Remark 12.16]).

The k-points of the good moduli space $B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ correspond to the closed points of the stack $\mathsf{Bun}_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$, which are precisely the $\boldsymbol{\alpha}$-polystable vector bundles on $\mathcal{C}$ with invariant $\boldsymbol{\beta}$. Let E and F be two polystable bundles in the same fiber, then we claim that they must have isomorphic stable summands. Suppose for a contradiction that they do not. By Proposition 4.4.10 for every $m > \kappa_{2\boldsymbol{\beta}}^{+}$ there exists a vector bundle H with algebraic invariants $m\widetilde{\boldsymbol{\alpha}}$ such that

$$\mathtt{Hom}(H, E) = 0 \quad \text{and} \quad \mathtt{Hom}(H, F) \neq 0.$$

The vector bundle H determines a section σ_H of $\mathcal{L}_{\widetilde{\boldsymbol{\alpha}}}^{\otimes m}$ that separates these points: $\sigma_H(E) \neq 0$ and $\sigma_H(F) = 0$. It follows that E and F do not lie in the same fiber, which is a contradiction. Since there are finitely many polystable bundles for each fixed set of stable summands we conclude that the fibers of φ are finite.

Since $\varphi\colon B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}} \to \mathbb{P}^n$ is a finite morphism of proper schemes, we can conclude that the ample line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ pulls back to an ample line bundle, which is a power of $L_{\widetilde{\boldsymbol{\alpha}}}$, and thus $L_{\widetilde{\boldsymbol{\alpha}}}$ is ample and $B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ is projective. ⬡

We also obtain an explicit bound for when the determinantal line bundle induces a finite map to projective space.

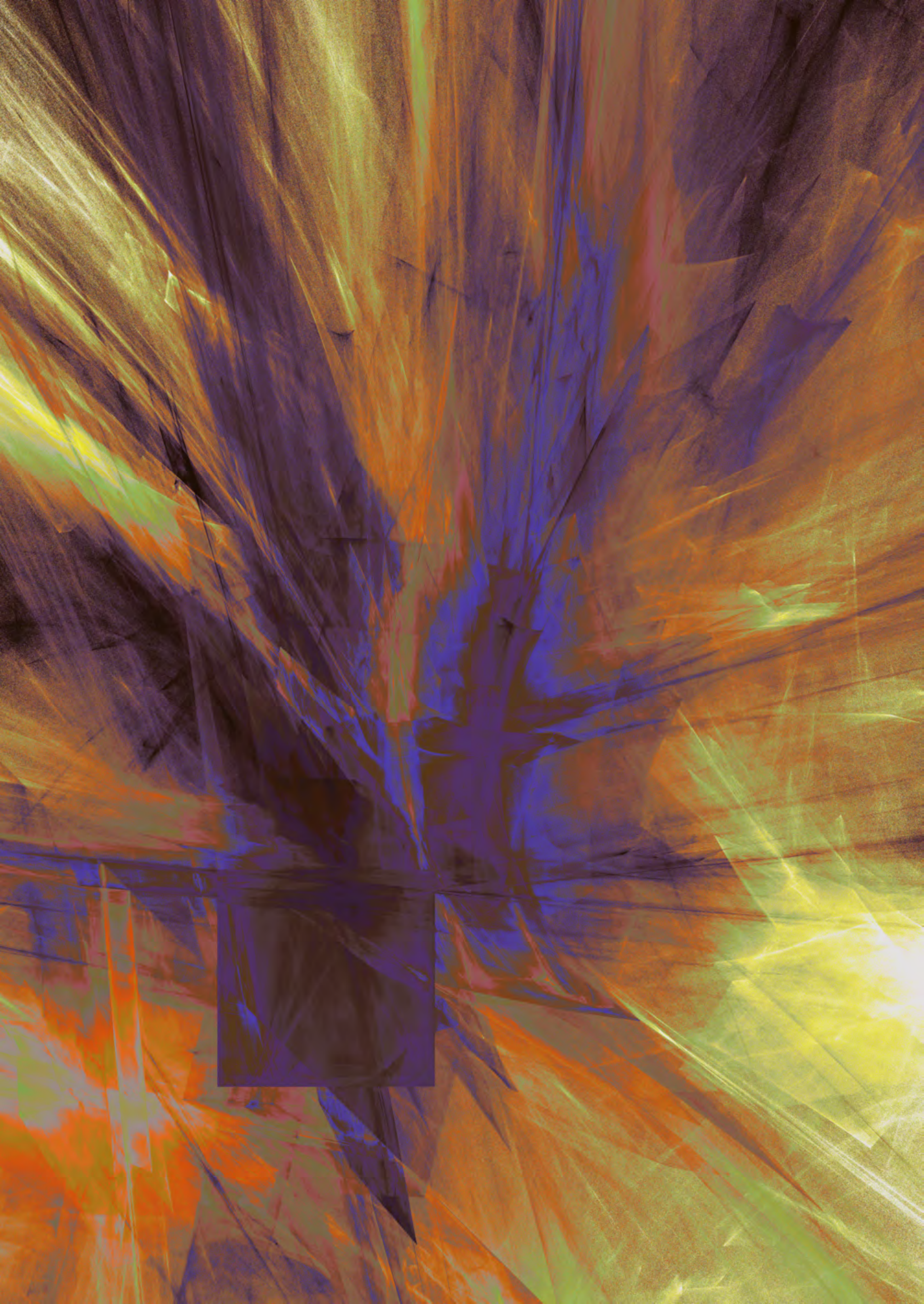
Corollary 4.5.4 For every positive integer m satisfying the inequality

$$m > \kappa_{2\boldsymbol{\beta}}^{+} = 4\left(g_{\mathcal{C}} - 1 + \frac{1}{2\,\mathtt{rank}\,\boldsymbol{\beta}}\right)(\mathtt{rank}\,\boldsymbol{\beta})^2,$$

the line bundle $L_{\widetilde{\boldsymbol{\alpha}}}^{\otimes m}$ induces a finite morphism from $B_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\text{-ss}}$ to a projective space.

Proof. This follows from the proof of Proposition 4.4.10 combined with Remark 4.4.9. ⬡

This bound does not seem to be optimal. For example, when $\mathcal{C} = C$ is a classical curve $m \geq \mathtt{rank}\,\boldsymbol{\beta}^2 + \mathtt{rank}\,\boldsymbol{\beta}$ would suffice [EP04]. This suggests that the factor 4 could probably be removed with a more careful analysis of the bounds, but more interestingly, the dependence on the genus could potentially be removed.



APPENDIX A

The flattening stratification

> It should be pointed out that the
> fancy definitions given cohomology
> recently—via standard resolutions,
> derived functors, especially in the
> category of all sheaves
> —which look very uncomputable—
> are just technical devices to simplify
> somebody's general theory.
>
> *Lectures on curves on an algebraic surface*
> David Mumford

In this chapter we will recall the basic theory of weighted projective stacks and provide a "global" flattening stratification of families of coherent sheaves of weighted projective stacks in the spirit of [Mum66, Lecture 8]. In [OS03] the authors conjecture that a "global" construction exits for all projective stacks. This result gives such a construction precisely for the cyclotomic stacks of [AH11], which include all stacky curves.

Question A.0.1 Can the techniques of this chapter be generalized to the twisted Grassmanians defined in [FL21] or the generalized weighted projective spaces of [BOW24]? This would provide a "global" construction of the universal flattening stratification for very general classes of stacks.

As an application of the flattening stratification for weighted projective spaces we obtain a universal flattening stratification for stacky curves, parameterized by the discrete invariants.

A.1 Weighted projective stacks

Let us recall the definition of a weighted projective stack.

Definition A.1.1 Let $\underline{w} := 0 < w_0 \leq w_1 \leq \cdots \leq w_r$ be a collection of positive integers. Let $A = k[x_0, \ldots, x_r]$ be a $\mathbb{Z}$-graded ring, where x_i is homogeneous of degree w_i. This graded ring corresponds to a $\mathbb{G}_m$-action $\mathbb{G}_m \times \mathbf{Spec}(A) \to \mathbf{Spec}(A)$ and we denote the **weighted projective stack** $\mathcal{P}(\underline{w}) := [(\mathbf{Spec}(A) - \{0\})/\mathbb{G}_m]$. We will denote the affine space $\mathbf{Spec}(A)$ by $\mathbb{A}(\underline{w})$. For a scheme S we set $\mathcal{P}(\underline{w})_S = \mathcal{P}(\underline{w}) \times_k S$ and $\mathbb{A}(\underline{w})_S = \mathbb{A}(\underline{w}) \times_k S$.

A weighted projective stack is a smooth tame Artin stack, with a projective coarse space. These facts are purely computational and can be seen from the following proposition.

Proposition A.1.2 Let $\underline{w}$ be as above and let N be the least common multiple of $\underline{w}$. Consider the graded subring $A^{[N]} \subset A$ generated by homogeneous elements of degree divisible by N. Let $A^{(N)}$ be the graded ring defined by $(A^{(N)})_i = (A^{[N]})_{i \cdot N}$.

The graded rings $A^{(N)}$ and $A^{[N]}$ are generated in degree 1 and N respectively. Let $0 \in \mathbf{Spec}\left(A^{(N)}\right)$ and $0 \in \mathbf{Spec}\left(A^{[N]}\right)$ denote the points defined by the ideals generated by all homogeneous elements. We have natural maps

$$\begin{aligned} \mathcal{P}(\underline{w}) \to \left[(\mathbf{Spec}\left(A^{[N]}\right) - \{0\})/\mathbb{G}_m\right] \to \\ \left[(\mathbf{Spec}\left(A^{(N)}\right) - \{0\})/\mathbb{G}_m\right] =: \mathbb{P}(\underline{w}). \end{aligned}$$

The second arrow is a μ_N-gerbe and the composition $\pi\colon \mathcal{P}(\underline{w}) \to \mathbb{P}(\underline{w})$ is the coarse space morphism.

The projective schemes $\mathbb{P}(\underline{w})$ are called **weighted projective spaces**.

Definition A.1.3 There is a natural map $\mathcal{P}(\underline{w})_S \to B\mathbb{G}_m$ coming from the quotient structure. The induced line bundle is Serre's twisting sheaf and is denoted by $\mathcal{O}_{\mathcal{P}(\underline{w})_S}(-1)$. For a coherent sheaf on $\mathcal{P}(\underline{w})_S$, we set $\mathcal{F}(i) := \mathcal{F} \otimes \mathcal{O}_{\mathcal{P}(\underline{w})_S}(-1)^{\otimes -i}$.

We see that by construction $\pi^*\mathcal{O}_{\mathbb{P}(\underline{w})}(1) = \mathcal{O}_{\mathcal{P}(\underline{w})}(N)$.

Definition A.1.4 For a coherent sheaf $\mathcal{F}$ on $\mathcal{P}(\underline{w})$, we define the **Hilbert function** of $\mathcal{F}$ to be $HF_{\mathcal{F}}(m) := \mathtt{dim}\, H^0(\mathcal{P}(\underline{w}), \mathcal{F}(m))$.

We would like to say that the Hilbert function behaves like a polynomial for large values of m, which is false, but it is almost true.

Definition/Proposition A.1.5 We define an N**-almost-polynomial** of degree d to be a function $P\colon \mathbb{Z} \to \mathbb{Z}$, such that there exist polynomials P_a of degree d with the same leading coefficients, for each integer $0 \leq a < N$, satisfying $P(Nm + a) = P_a(m)$.

Let N be the least common multiple of $\underline{w}$, and let $\mathcal{F}$ be a coherent sheaf on $\mathcal{P}(\underline{w})$. There exists an N-almost-polynomial P and integer m_0, such that for all $m \geq m_0$ we have $HF_{\mathcal{F}}(m) = P(m)$. We call this N-almost-polynomial the **Hilbert almost-polynomial** of $\mathcal{F}$ and denote it by $P_{\mathcal{F}}$.

Proof. Let $\pi\colon \mathcal{P}(\underline{w}) \to \mathbb{P}(\underline{w})$ be the coarse space map. Let P_a be the Hilbert polynomial of $\pi_*\mathcal{F}(a)$ with respect to $\mathcal{O}_{\mathbb{P}(\underline{w})}(1)$; then

$$\begin{aligned} HF_{\mathcal{F}}(Nm + a) &= H^0(\mathcal{P}(\underline{w}), \mathcal{F}(a) \otimes (\pi^*\mathcal{O}_{\mathbb{P}(\underline{w})}(1))^{\otimes m}) \\ &= H^0(\mathbb{P}(\underline{w}), \pi_*\mathcal{F}(a) \otimes \mathcal{O}_{\mathbb{P}(\underline{w})}(m)) = P_a(m). \end{aligned}$$

Since $P_0(m) \leq P_a(m) \leq P_0(m+1)$ for all $m \geq m_0$ we see that all the P_a have the same leading coefficient. ⬠

Sheaves

As above we consider $\mathcal{P}(\underline{w}) = [\mathbb{A}(\underline{w}) - \{0\}/\mathbb{G}_m]$. By construction we have a functor τ (often denoted by a tilde in the classical unweighted setting) which sends quasi-coherent graded $\mathcal{O}_S \otimes A$-modules of finite type to coherent $\mathcal{O}_{\mathcal{P}(\underline{w})_S}$-modules. The functor τ sends two graded $\mathcal{O}_S \otimes A$-modules to the same $\mathcal{O}_{\mathcal{P}(\underline{w})}$-module if and only if they define the same graded $\mathcal{O}_S \otimes A[\frac{1}{x_0}, \dots, \frac{1}{x_r}]$-module. This happens precisely when their graded parts agree for all arbitrarily large grades.

As with projective space there is a natural right inverse to τ.

Definition A.1.6 Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{P}(\underline{w})_S$, and denote the projection map by $f\colon \mathcal{P}(\underline{w})_S \to S$. For each integer $m \geq 0$, set $\Gamma_m(\mathcal{F}) := f_*\mathcal{F}(m)$ and let $\Gamma_\bullet(\mathcal{F}) := \bigoplus_{m\geq 0} \Gamma_m(\mathcal{F})$.

It is clear that for any coherent sheaf $\mathcal{F}$ on $\mathcal{P}(\underline{w})_S$, we have $\tau\Gamma_\bullet(\mathcal{F}) = \mathcal{F}$.

Lemma A.1.7 Let $g\colon T \to S$ be a morphism of Noetherian schemes and let $h\colon \mathcal{P}(\underline{w})_T \to \mathcal{P}(\underline{w})_S$ be the corresponding morphism of weighted projective stacks. We have

$$\tau(g^*\Gamma_\bullet(\mathcal{F})) = \tau(\Gamma_\bullet(h^*\mathcal{F})).$$

As a consequence there exists an m_0, such that for $m \geq m_0$

$$g^*\Gamma_m(\mathcal{F}) = \Gamma_m(h^*\mathcal{F}).$$

Proof. Since τ commutes with base change by construction, we have

$$\tau(g^*\Gamma_\bullet(\mathcal{F})) = h^*\tau(\Gamma_\bullet(\mathcal{F})) = h^*\mathcal{F} = \tau(\Gamma_\bullet(h^*\mathcal{F}))$$

and the result follows. ⬠

Cohomology and base change

We now explain how cohomology and Hilbert almost-polynomials behave in families of sheaves, of course the best behavior is for **flat** families of sheaves.

Definition A.1.8 Let S be a scheme over k and $\mathcal{X}$ a stacky curve or weighted projective space. A flat family of sheaves on $\mathcal{X}$ is a coherent sheaf $\mathcal{F}$ on $S \times \mathcal{X}$, flat over S.

Theorem A.1.9 (Cohomology and base change) Let $\mathcal{F}$ be a flat family of sheaves on $p\colon \mathcal{P}(\underline{w})_S \to S$ and let $s \in S$ be a point.

1. If the natural map
$$\phi^i(s) : R^i p_* \mathcal{F} \otimes k(s) \to H^i(\mathcal{X}_s, \mathcal{F}_s)$$
is surjective, then there exists an open subscheme $U \subset S$, containing s, such that for any diagram
$$\begin{array}{ccc} \mathcal{P}(\underline{w})_T & \xrightarrow{h} & \mathcal{P}(\underline{w})_U \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{g} & U \end{array}$$
we have
$$g^* R^i p_* \mathcal{F} \simeq R^i q^*(h^* \mathcal{F}).$$
In particular $\phi^i(s)$ is an isomorphism.
2. If $\phi^i(s)$ is surjective, then ϕ^{i-1} is surjective if and only if $R^i f_* \mathcal{F}$ is free in a neighborhood of s.

Proof. This follows from [Bro12, A.1.4-5]. Since our situation is very concrete, we can also consider $D(x_i) \subset \mathbb{A}(\underline{w})$ and let $U_i := [D(x_i)/\mathbb{G}_m]_S$ be the standard covering of $\mathcal{P}(\underline{w})$. The standard opens are isomorphic to $[\mathbb{A}(\underline{w} - \{w_i\})/\mu_{w_i}]$ and it is easy to show that Čech cohomology computes sheaf cohomology:
$$\check{H}^p((U_i)_i, \mathcal{F}) = H^p(\mathcal{P}(\underline{w}), \mathcal{F}).$$
Then we can proceed as in [Bro12, A.1.4-5]. ⬠

We recall some standard corollaries of cohomology and base change.

Corollary A.1.10 Let $\mathcal{F}$ be a flat family of coherent sheaves on $p\colon \mathcal{P}(\underline{w})_S \to S$, such that $R^i p_* \mathcal{F} = 0$ for all $i > 0$; then $p_* \mathcal{F}$ is locally free.

Proof. Let d be the dimension of $\mathcal{P}(\underline{w})$; then $H^{d+1}(\mathcal{P}_s, \mathcal{F}_s) = 0$ for each point $s \in S$. Then $\phi^{d+1}(s)\colon (R^{d+1}p_*\mathcal{F}) \otimes k(s) \to H^{d+1}(\mathcal{P}_s, \mathcal{F}_s)$ is surjective and $R^d p_* \mathcal{F}$ is locally free by assumption. It follows that $\phi^d(s)$ is also surjective and $R^{d-1}p_*\mathcal{F}$ is again locally free by assumption. Now we iterate this argument to see that $\phi^0(s)$ is surjective. Since $\phi^{-1}(s)$ is always surjective it follows that $R^0 p_* \mathcal{F}$ is free in a neighborhood of s for each $s \in S$. ⬠

Theorem A.1.11 Let $\mathcal{F}$ be a sheaf on $\mathcal{P}(\underline{w})_S$; then $\mathcal{F}$ is flat over S if and only if $\Gamma_m(\mathcal{F})$ is a locally free $\mathcal{O}_S$-module for all $m \gg 0$. It follows that for $\mathcal{F}$ flat over S, the Hilbert almost-polynomial $P_{\mathcal{F}_s}$ for $s \in S$ is locally constant.

Proof. Assume first that $\mathcal{F}$ is flat over S. Let m_0 be such that for all $m \geq m_0$ and $i > 0$ we have $R^i f_* \mathcal{F}(m) = 0$. This can be done because of [AH11, Proposition 2.4.2].

By Corollary A.1.10, it follows that $f_*\mathcal{F}(m)$ is locally free.

If $\Gamma_m(\mathcal{F})$ is locally free for $m \geq m_0$, consider the module $\bigoplus_{m \geq m_0} \Gamma_m(\mathcal{F})$. This module is locally free, hence flat over $\mathcal{O}_S$. Since $\mathcal{F} = \tau\left(\bigoplus_{m \geq m_0} \Gamma_m(\mathcal{F})\right)$, it follows that $\mathcal{F}$ is flat. ⬠

A.2 Flattening stratifications

What is often called a flattening stratification is in general only a quite weak notion of stratification. We will follow [Stacks, Definition 09XZ] and call this a partition.

Definition A.2.1 A **partition** of a scheme S is a collection of schemes $(S_i)_{i \in I}$ together with a bijection $\coprod_{i \in I} S_i \to S$, such that the induced maps $S_i \to S$ are locally closed embeddings.

The following lemma can be viewed as a flattening stratification result for $\mathbb{P}^0_S = S$.

Lemma A.2.2 ([Mum66, Lecture 8 Theorem 1°]) Let S be a Noetherian scheme and F be a coherent sheaf on S. There exists a partition $\coprod S_r \to S$, such that any map $g\colon T \to S$ factors through S_r if and only if g^*F is locally free of rank r.

Theorem A.2.3 (Existence of flattening stratifications) Let S be a Noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on a weighted projective space $\mathcal{P}(\underline{w})_S$ and let $N = \mathtt{lcm}(\underline{w})$. There exists a partition $\coprod_P S_P \to S$ parameterized by sets of N-almost-polynomials P, satisfying the following universal property: for every $g\colon T \to S$ we have that g factors through S_P if and only if $g^*\mathcal{F}$ is flat over S with Hilbert almost-polynomial P.

Proof. By [Nir09, Proposition 1.13], there exists some finite partition $i_0\colon S_0 := \coprod S_i \to S$ such that $\mathcal{F}|_{S_0}$ is flat. As Hilbert almost-polynomials are constant in flat families it follows that only finitely many sets of Hilbert almost-polynomials appear for the fibers $\mathcal{F}_s$.

We claim their exists a uniform m_0, such that for all $m > m_0$ and all $s \in S$ we have that

$$H^i(\mathcal{P}(\underline{w}), \mathcal{F}_s(m)) = 0,$$

for $i > 0$ and $\Gamma_m(\mathcal{F}) \otimes k(s) \simeq H^0(\mathcal{P}(\underline{w}), \mathcal{F}_s(m))$.

To see this, consider the diagram

$$\begin{tikzcd}
\mathcal{P}(\underline{w})_{S_0} \arrow[r, "\widetilde{i_0}"] \arrow[d, "q"] & \mathcal{P}(\underline{w})_S \arrow[d, "p"] \\
S_0 \arrow[r, "i_0"] & S
\end{tikzcd}$$

Apply Lemma A.1.7 to the inclusion map i_0 to obtain a positive integer m_1, such that $i_0^* p_* \mathcal{F}(m) = q_* \widetilde{i_0}^* \mathcal{F}(m)$ for all $m \geq m_1$.

Next apply [AH11, Proposition 2.4.2] to obtain an m_2, such that $R^i q_* \widetilde{i_0}^* \mathcal{F}(m) = 0$ for $i > 0$ and $m \geq m_2$.

By Corollary A.1.10, it follows that for $m_0 = \mathbf{sup}(m_1, m_2)$ we have the desired claim.

Let m_0 be as above and let $g\colon T \to S$ be any map of Noetherian schemes and consider the following diagram.

$$\begin{array}{ccc} \mathcal{P}(\underline{w})_T & \xrightarrow{h} & \mathcal{P}(\underline{w})_S \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle p} \\ T & \xrightarrow{\quad g \quad} & S \end{array}$$

We claim that $h^*\mathcal{F}$ is flat over T if and only if $g^*\Gamma_m(\mathcal{F})$ is locally free for all $m \geq m_0$.

First assume that $h^*\mathcal{F}$ is flat over T and consider the morphisms

$$g^*\Gamma_m(\mathcal{F}) \otimes k(s) \to \Gamma_m(h^*\mathcal{F}) \otimes k(s) \to H^0(\mathcal{P}(\underline{w}), \mathcal{F}_s(m)).$$

For $m \geq m_0$ the composition is an isomorphism by assumption, so the second arrow is surjective and by cohomology and base change it is also an isomorphism. Again by cohomology and base change $\Gamma_m(h^*\mathcal{F})$ is locally free and by Nakayama's lemma we have an isomorphism

$$g^*\Gamma_m(\mathcal{F}) \simeq \Gamma_m(h^*\mathcal{F}).$$

Conversely, if $g^*\Gamma_m(\mathcal{F})$ is flat for all $m \geq m_0$ then $h^*\mathcal{F}$ is flat by Theorem A.1.11.

Let $\coprod S_{m,r} \to S$ be the flattening stratification of $\Gamma_m(\mathcal{F})$ from Lemma A.2.2. We have shown that $h^*\mathcal{F}$ is flat if and only if g factors through $S_{m,P(m)}$ for all $m \geq m_0$ and some N-almost-polynomial. A priori this is an infinite limit, but if we let d be the degree of P, then g factors through $S_{m,P(m)}$ for all $m \geq m_0$ if and only if g factors through $S_{m,P(m)}$ for $m_0 \leq m \leq m_0 + d \cdot N$.

Now we see that

$$S_P = S_{m_0,P(m_0)} \times_S \ldots \times_S S_{m_0+d\cdot N, P(m_0+d\cdot N)}$$

defines the desired flattening stratification. ◯

Hilbert almost-polynomials versus discrete invariants

We will now explain how to relate the discrete invariants of sheaves on a stacky curve to the Hilbert almost-polynomials. This will allow us to construct a flattening stratification parameterized by the discrete invariants.

> **Lemma A.2.4** Let $\mathcal{C}$ be a stacky curve with coarse space C, let p be a stacky point of order e and let $\mathrm{T}_p = \mathcal{O}_{\mathcal{C}}(\frac{1}{e}p)$ be the corresponding tautological sheaf. Then there is a commutative triangle,
>
> $$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "\pi^p"] & \mathcal{P}(\underline{w}) \\ \sqrt[e]{p/C} \arrow[ur, "\iota^p"'] & \end{tikzcd}$$
>
> where the maps to $\mathcal{P}(\underline{w})$ are induced by the tautological sheaves at p, defined as in [AH11, Corollary 2.4.4]. Moreover $\sqrt[e]{p/C} \to \mathcal{P}(\underline{w})$ is an embedding.

Proof. We have $\pi^p_* \mathcal{O}_{\mathcal{C}}(\frac{1}{e}p) = \mathcal{O}_{\sqrt[e]{p/C}}(\frac{1}{e}p)$ and

$$H^0(\mathcal{C}, \mathrm{T}_p^{\otimes m}) = H^0(\sqrt[e]{p/C}, \pi^p_* \mathrm{T}_p^{\otimes m}).$$

It follows that the induced maps to weighted projective space have the same target. It is clear that T_p is ample on $\sqrt[e]{p/C}$, so the corresponding map is an embedding by [AH11, Corollary 2.4.4]. ◇

> **Lemma A.2.5** With the notation as in the lemma above, let $\mathcal{F}$ be a coherent sheaf on $\mathcal{C}$ and let
>
> $$F^p_{\mathcal{F}}(m) \coloneqq H^0(\sqrt[e]{p/C}, \pi^p_* \mathcal{F} \otimes \mathrm{T}_p^{\otimes m}) = H^0(\mathcal{P}(\underline{w}), \iota^p_* \pi^p_* \mathcal{F}(m)),$$
>
> be the Hilbert function of $\pi^p_* \mathcal{F}$ with respect to T_p; then for $m \gg 0$
>
> $$F^p(m) = \mathtt{rank}(\mathcal{F})(1 - g_C) + \left\lfloor \frac{m}{e} \right\rfloor \mathtt{rank}\, \mathcal{F} + d_{p, m \bmod e}(\mathcal{F}).$$

Proof. This is immediate from the naive Riemann-Roch theorem 1.3.8. ◇

It follows that we can completely recover the discrete invariants of a coherent sheaf by considering the Hilbert functions $F^p_{\mathcal{F}}$ for each stacky point p and vice versa. We will denote the Hilbert almost-polynomial at p corresponding to invariants $(n, \underline{d})$ by

$$P^p_{n,\underline{d}}(m) \coloneqq n(1-g_C) + \left\lfloor \frac{m}{e} \right\rfloor \cdot n + d_{p, m \bmod e}. \tag{A.1}$$

Since the Hilbert almost-polynomial is constant in flat families, we get the following immediate corollary.

Corollary A.2.6 Let $\mathcal{F}$ be a flat family of sheaves on $\mathcal{C} \times S$, the twisted degree functions $d_{p,i}(\mathcal{F})\colon s \to d_{p,i}(\mathcal{F}_s)$ and the rank function $\mathtt{rank}(\mathcal{F})\colon s \mapsto \mathtt{rank}(\mathcal{F}_s)$ are locally constant.

Note that the multiplicities are **not** locally constant in flat families. Consider for example a family that degenerates $\mathcal{O}(\frac{1}{e}p)$ into $\mathcal{O} \oplus \mathcal{O}_{\frac{1}{e}p}$. One way to construct such a family is to consider the moduli stack of extensions of $\mathcal{O}_{\frac{1}{e}p}$ by $\mathcal{O}$, which is isomorphic to $\left[\mathtt{Ext}^1\left(\mathcal{O}_{\frac{1}{e}p}, \mathcal{O}\right) / \mathtt{Ext}^0\left(\mathcal{O}_{\frac{1}{e}p}, \mathcal{O}\right)\right]$ hence connected. However the multiplicities are constant when restricted to natural subclasses of families.

Corollary A.2.7 Let $\mathcal{F}$ be a flat family of sheaves on $\mathcal{C} \times S$, such that all the fibers are torsion sheaves or all the fibers are vector bundles; then the multiplicities $m_{p,i}(\mathcal{F})\colon s \mapsto m_{p,i}(\mathcal{F}_s)$ are locally constant.

Proof. This follows from the lemma above, Corollary 1.2.21 and Remark 1.2.25. ⬠

Theorem A.2.8 Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{C} \times S$, with S a Noetherian scheme. Then there exists a partition, called the **flattening stratification**,

$$\coprod_{(n,\underline{d})} S_{n,\underline{d}} \to S,$$

satisfying the following universal property: for every map $g\colon T \to S$ of Noetherian schemes we have that g factors through $S_{n,\underline{d}}$ if and only if $g^*\mathcal{F}$ is a flat family of sheaves with invariants $n, \underline{d}$.

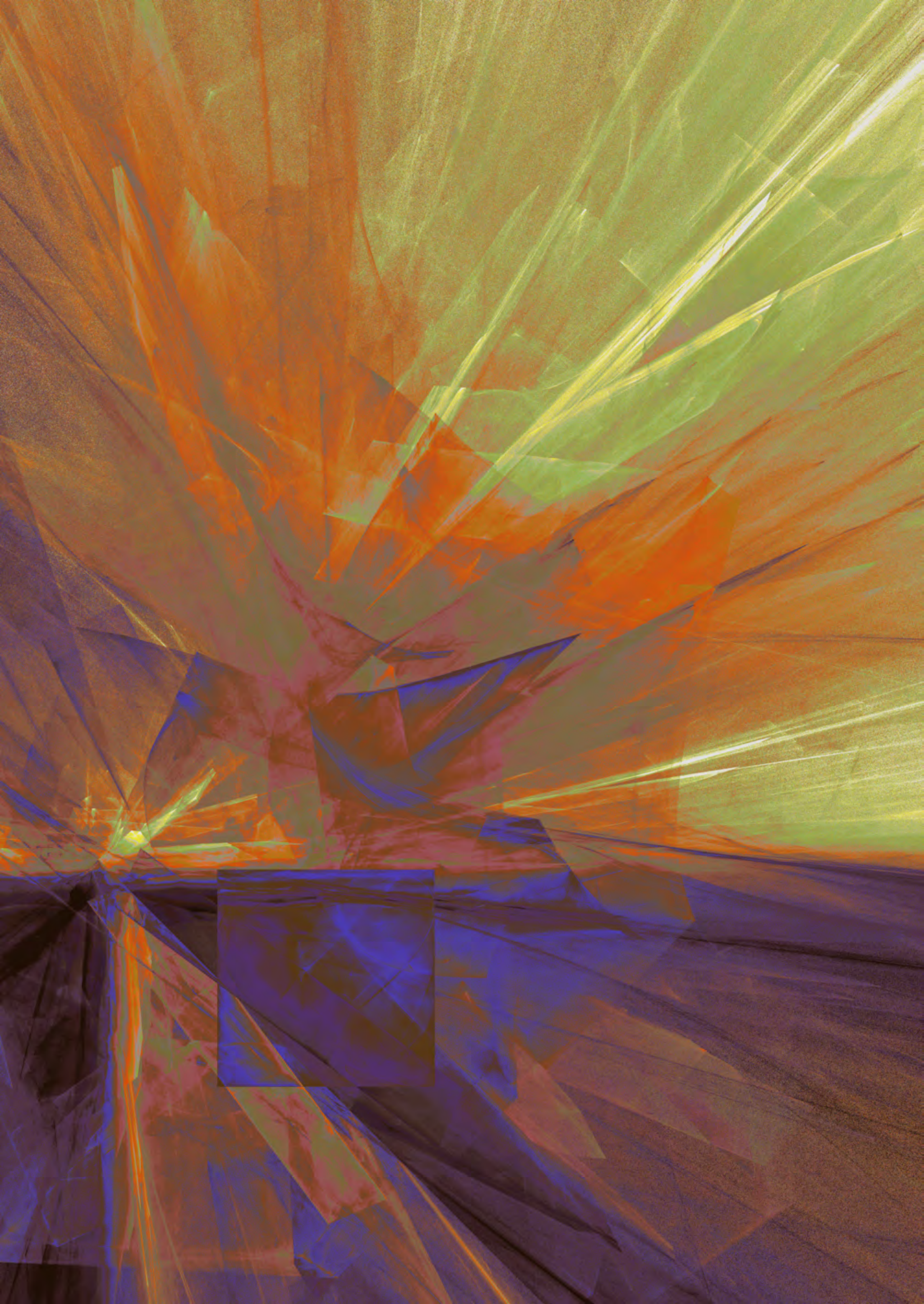
Proof. For each stacky point $p \in \mathcal{C}$ consider the embedding $\iota^p \colon \sqrt[e]{p/C} \to \mathcal{P}(\underline{w})$ as in Lemma A.2.4 and consider the flattening stratification $\coprod_P S_P^p \to S$ of the sheaf $\iota_*^p \pi_*^p \mathcal{F}$. For invariants $n, \underline{d}$, we let $P_{n,\underline{d}}^p$ be the Hilbert almost-polynomial at p as in Equation (A.1). Denote the stacky points of $\mathcal{C}$ by $p_1, \ldots, p_r$. We claim that

$$S_{n,\underline{d}} = S^{p_1}_{P^{p_1}_{n,\underline{d}}} \times_S \cdots \times_S S^{p_r}_{P^{p_r}_{n,\underline{d}}}$$

has the desired properties.

Assume that $g^*\mathcal{F}$ is flat, then also $\pi_*^p g^* \mathcal{F} = g^* \pi_*^p \mathcal{F}$ is flat for each p, and since the twisted degrees d and rank n are constant in flat families it follows that g factors through $\coprod S_{n,\underline{d}}$.

Conversely, whenever g factors through the stratification, we get that $g^* \pi_*^p \mathcal{F} = \pi_*^p g^* \mathcal{F}$ is flat for each p. Since π^p is an isomorphism away from the stacky points that are not p, it follows that $g^*\mathcal{F}$ is flat away from the stacky points that are not p. However this holds for all p, so $g^*\mathcal{F}$ is flat everywhere. ⬡



APPENDIX B

Calculus of motives

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Twan Niesten

In this appendix we will collect various tools for the computation of motives. This appendix is **not** meant as an introduction to motives and we will not prove any fundamentally new results. Instead we will show how a geometric approach can be used **compute** motives. These computations can also be used to describe many other invariants, such as cohomology or Chow groups, via realization functors and the description of Chow groups as groups of homomorphism between motives respectively. For an introduction to motives we recommend [Ayo14].

B.1 Setup

In this appendix we will work with Voevodsky's category of (mixed) motives with rational coefficients $\mathtt{DM}(k, \mathbb{Q})$, which comes with a functor $M\colon \mathtt{Sch}/k \to \mathtt{DM}(k, \mathbb{Q})$, sending a scheme X to its (homological) motive $M(X)$. This functor can be extended to the category of algebraic stacks $M\colon \mathtt{Stck}/k \to \mathtt{DM}(k, \mathbb{Q})$ in various equivalent ways. The most general construction is due to Khan [Kha19, Appendix A]. See also [HP21b, Appendix A] for a slightly simpler approach. The category of motives $\mathtt{DM}(k, \mathbb{Q})$ is a monoidal triangulated category, so it has direct sums $M_1 \oplus M_2$ and tensor products $M_1 \otimes M_2$. Moreover it comes with a translation functor $M \mapsto M[1]$ and a collection of distinguished triangles

$$M \to N \to L \to M[1],$$

or $M \to N \to L \xrightarrow{+}$ for short. For every morphism $M \to N$ there exists a unique **cone** L completing this morphism into a distinguished triangle. Note however that the cone is **not** functorial, i.e. it exists up to a potentially non-unique isomorphism. A triangle is said to be **split** if it is isomorphic to the triangle

$$M_1 \to M_1 \oplus M_2 \to M_2 \xrightarrow{+}$$

We denote by $\mathbb{Q} := M(\mathbf{Spec}(k))$, which is the neutral object for the tensor product. The projection to a point $\pi : \mathbb{P}^1_k \to \mathbf{Spec}(k)$ induces a distinguished triangle

$$\mathbb{Q}\{1\} := \mathbf{cone}(M(\pi))[-1] \to M(\mathbb{P}^1_k) \to \mathbb{Q} \xrightarrow{+},$$

which is split by the inclusion of any rational point into $\mathbb{P}^1_k$. As a consequence we have $M(\mathbb{P}^1_k) = \mathbb{Q} \oplus \mathbb{Q}\{1\}$. By construction $\mathbb{Q}\{1\}$ has a tensor inverse in $\mathtt{DM}(k, \mathbb{Q})$, which we denote by $\mathbb{Q}\{-1\}$. For a positive integer n we set $\mathbb{Q}\{n\} := \mathbb{Q}\{1\}^{\otimes n}$ and $\mathbb{Q}\{-n\} := \mathbb{Q}\{-1\}^{\otimes n}$. For any motive M, we write $M\{n\} := M \otimes \mathbb{Q}\{n\}$ and $M\{n\}$ is said to be a **Tate twist** of M. A motive is said to be **pure Tate** if it can be written in terms of sums and tensor products of $\mathbb{Q}\{\pm 1\}$. A motive is said to be **pure** if it can be written as a Tate twist of a direct summand of the motive of a projective variety. To "compute" a motive in the strongest sense means to write it in terms of sums and tensor products of simpler motives. In particular, saying that a motive is pure (Tate) is a qualitative statement on the complexity of terms that might appear in a computation.

A slightly weaker notion of computation would be to allow for cones and translations as well.

> **Definition B.1.1 thick tensor subcategory** For a set S of motives in $\mathtt{DM}(k, \mathbb{Q})$, the **thick tensor subcategory generated by** S is the smallest full subcategory $\langle S \rangle \subset \mathtt{DM}(k, \mathbb{Q})$ containing S, such that:
>
> - $\langle S \rangle$ is closed under taking tensor products.

- For every distinguished triangle $X \to Y \to Z \xrightarrow{+}$ such that $X, Y \in \langle S \rangle$, we have $Z \in \langle S \rangle$.
- For $X \oplus Y \in \langle S \rangle$, we have $X \in \langle S \rangle$.

Saying that a motive M lies in $\langle S \rangle$ thus means that **there exists** a way to compute M in terms of S in the weak sense mentioned above.

B.2 Computational tools

We now collect some results that let us compute motives. The following properties of motives are basic consequences of the construction [VSF00].

Proposition B.2.1 Let X and Y be algebraic stacks; then we have

1. $M(X \times Y) = M(X) \otimes M(Y)$,
2. $M(X \amalg Y) = M(X) \oplus M(Y)$,
3. $M(X \times \mathbb{A}^n) = M(X)$.

Proposition B.2.2 (Gysin triangles) Let X be a smooth stack and let Z be a smooth substack of codimension c and set $U = X \setminus Z$. There is a distinguished triangle

$$M(U) \to M(X) \to M(Z)\{c\} \xrightarrow{+} .$$

To compute motives in the strong sense it is important to know when a distinguished triangle splits. In general this is very complicated, but for pure motives this is automatic.

Proposition B.2.3 ([HP22, Lemma 4.2]) Let $M \to N \to L \xrightarrow{+}$ be a distinguished triangle, such that M and L are pure, then N is also pure and the triangle splits.

The next couple of results show that one can obtain motivic computations from maps with nice fibers.

Proposition B.2.4 Let $B \to X$ be a smooth morphism, such that the fibers have trivial motive, i.e. $M(B_x) \to M(\mathsf{Spec}(\kappa(x)))$ is an isomorphism, then $M(B) \to M(X)$ is an isomorphism.

Proof. This is proven in first half of the proof of [HP22, Proposition 3.2]. ◻

Proposition B.2.5 (See [AH17]) Let $\Gamma \to X$ be a fibration (locally trivial in the Zariski topology), such that the fiber F is a cellular variety satisfying Poincaré duality, then $M(\Gamma) = M(F) \otimes M(X)$.

The following result applies to small maps [HP21a, Definition 2.1]. See also [CM04] for a version of this statement on the level of chow motives.

Proposition B.2.6 ([HP21a, Theorem 2.11]) Let X and Y be smooth stacks, which are locally of finite type over k, and let $Y \to X$ be a representable surjective proper small map, which is generically a Γ-torsor for a finite group Γ. Then the action of Γ extends to $M(Y)$ and $M(X) = M(Y)^{\Gamma}$.

The most important example of a small map for us is the Grothendieck-Springer resolution. Let G be a reductive group over an algebraically closed field k with Lie algebra $\mathfrak{g}$ and a choice of Borel subgroup $B \subset G$. The set of Borel subalgebras $\mathcal{B} := \{\mathfrak{b} \mid \mathfrak{b} \subset \mathfrak{g} \text{ Borel}\}$ is isomorphic to $\mathcal{B} \simeq (G/B)$. We define $\tilde{\mathfrak{g}} \subset \mathfrak{g} \times \mathcal{B}$ to be the subset containing pairs $(x, \mathfrak{b})$, where $x \in \mathfrak{b}$. The natural projection $\overline{\mathfrak{g}} \to \mathfrak{g}$ is the Grothendieck-Springer resolution and is small. Moreover it is generically a W-torsor for the Weyl group W.

The following proposition is a technical but extremely flexible approximation result, showing that motives can be computed by constructing increasingly good approximations.

Proposition B.2.7 Let X be a smooth stack. Let $X_\bullet$ be an increasing sequence of quasi-compact open substacks covering X. Let $V_\bullet$ be a sequence of vector bundle stacks on $X_\bullet$ together with injections $V_m \to V_{m+1} \times_{X_{m+1}} X_m$. Let $W_\bullet \subset V_\bullet$ be closed substacks preserved by the injections and denote the complements by $U_\bullet := V_\bullet \setminus W_\bullet$. Assume that the codimensions of $W_n \subset V_n$ go to infinity. We call the above data an **exhaustive system**. For any exhaustive system we have $M(X) = \mathbf{hocolim}_m M(U_m)$.

Finally we give an inductive version of Proposition B.2.2 for stratifications.

Proposition B.2.8 Let X be a smooth stack and let $\coprod_{i\in\mathbb{I}} X_i \to X$ be a locally finite stratification with smooth strata. Assume that $\mathbb{I}^{\mathrm{op}}$ is countable and well-founded (i.e. there is no infinitely **increasing** sequence of strata). Then the motive $M(X)$ lies in the localizing thick tensor subcategory $\langle\{M(X_i)\}_{i\in\mathbb{I}}\rangle$. Assume in addition that the motives $M(X_i)$ are pure and X_i has codimension c_i, then

$$M(X) = \bigoplus_{i\in\mathbb{I}} M(X_i)\{c_i\}.$$

Proof. As $\mathbb{I}^{\mathrm{op}}$ is well-founded and countable, it can be extended to a countable ordinal α, so we may assume without loss of generality that $\mathbb{I}^{\mathrm{op}} = \alpha$ is an ordinal. Since the stratification is locally finite the sets $U_\beta := X \setminus \coprod_{\beta\leq\gamma} X_\gamma$ are open for every $\beta \leq \alpha$. By construction we have $U_{\beta+1} \setminus X_\beta = U_\beta$, so by Proposition B.2.2 we get a distinguished triangle

$$M(U_\beta) \to M(U_{\beta+1}) \to M(X_\beta)\{c_\beta\} \xrightarrow{+} .$$

For a limit ordinal $\beta \leq \alpha$ we take any cofinal embedding $\rho\colon \mathbb{N} \to \beta$ and by Proposition B.2.7 we have

$$M(U_\beta) = \mathbf{hocolim}_{n\in\mathbb{N}} M(U_{\rho(n)}).$$

By ordinal induction it follows that $M(U_\beta)$ lies in $\langle\{M(X_\gamma)\}_{\gamma\in\beta}\rangle$ for every $\beta \leq \alpha$. In the case that X_β is pure for every $\beta \leq \alpha$ we notice that the triangles split by inductively applying Proposition B.2.3, and we have

$$M(U_{\beta+1}) = M(U_\beta) \oplus M(X_\beta)\{c_i\}.$$

The result again follows from ordinal induction. ◯

B.3 Examples

We end by giving some basic examples showcasing the potency of these tools. For some state of the art applications, see the series of papers [HP21b], [HP21a], [HP22].

Example B.3.1 We can stratify $\mathbb{P}^n = \mathbb{A}^n \amalg \mathbb{A}^{n-1} \amalg \ldots \amalg \mathbb{A}^1 \amalg \mathbf{Spec}(k)$. Since $M(\mathbb{A}^n) = \mathbb{Q}$ is pure, we get $M(\mathbb{P}^n) = \bigoplus_{i=0}^n \mathbb{Q}\{i\}$.

Over $\mathbb{C}$ the **topological** classifying space $\mathcal{B}\mathbb{G}_m$ is $\mathbb{P}^\infty$ and this is reflected on the level of motives.

Example B.3.2 ([Tot16, Example 8.5]) Consider the vector bundles $[\mathbb{A}^n/\mathbb{G}_m] \to B\mathbb{G}_m$ and closed substacks $B\mathbb{G}_m \subset [\mathbb{A}^n/\mathbb{G}_m]$ with open complement $\mathbb{P}^{n-1}$. This is an exhaustive system, so we have $M(B\mathbb{G}_m) = \mathbf{hocolim}_n\, M(\mathbb{P}^n) = \bigoplus_{i\geq 0} \mathbb{Q}\{i\}$.

We have $\mathbb{A}^1$-homotopy invariance, but also $B\mathbb{G}_a$-homotopy invariance.

Example B.3.3 The map $\mathbf{Spec}(k) \to B\mathbb{G}_a^n$ is smooth and the fiber $\mathbb{A}^n$ has trivial motive. It follows that $M(B\mathbb{G}_a^n) = \mathbb{Q}$.

Example B.3.4 ([HP22, Proposition 3.2]) Let $\mathcal{V} \to X$ be a vector bundle stack, then the fibers are isomorphic to $\mathbb{A}^n \times B\mathbb{G}_a^m$. By the previous example it follows that the fibers have trivial motive and $M(\mathcal{V}) = M(X)$.

We can even make very general statements about motives of classifying spaces of algebraic groups.

Example B.3.5 Let k be a perfect field and G be a linear algebraic group. Let U be its unipotent radical and $G_{\text{red}} := G/U$ be the reductive quotient. The map

$BG \to BG_{\text{red}}$ has fiber BU. Since k is perfect, U is isomorphic to $\mathbb{A}^n$ as a variety, so we have $M(BU) = \mathbb{Q}$ and $M(BG) = M(BG_{\text{red}})$.

The following geometric argument is standard, even though it is usually not stated using stacky language. See for example [CG10, Theorem 3.1.38, Lemma 6.1.6]

Example B.3.6 (Chevalley restriction theorem) Let $k = \bar{k}$ be an algebraically closed field and let G be a reductive group with Lie algebra $\mathfrak{g}$, a choice of Borel subgroup H with lie algebra $\mathfrak{h}$ and Weyl group W. We have $M(BG) = M([\mathfrak{g}/G])$, where G acts by conjugation. The Grothendieck-Springer resolution $[\overline{\mathfrak{g}}/G] \to [\mathfrak{g}/G]$ is a small map, which is generically a W-torsor, so we have an isomorphism $M(BG) = M([\overline{\mathfrak{g}}/G])^W$. The projection $[\overline{\mathfrak{g}}/G] \to [(G/H)/G] \simeq BH$ is a fibration with fiber $\mathfrak{h}$. Let $\mathbb{T} \coloneqq H/[H,H]$, which is isomorphic to the maximal torus of H. The fiber of $BH \to B\mathbb{T}$ is given by $B[H,H]$, which is the unipotent subgroup of H, hence affine. It follows that $M([\overline{\mathfrak{g}}/G]) = M(BH) = M(B\mathbb{T})$ and $M(BG) = M(B\mathbb{T})^W$. In particular $M(BG)$ is pure Tate.

We should remark that with integral coefficients and general fields the story of classifying spaces is much more interesting [Tot16].

Example B.3.7 Let C be a smooth projective curve and let $C^{(n)}$ be the symmetric power, then the map $C^n \to C^{(n)}$ is finite and a fortiori small, so $M(C^{(n)}) = M(C^n)^{S_n}$ and $M(C^{(n)})$ lies in $\langle M(C) \rangle$.

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Curriculum vitea

Lisanne Taams was born on Curaçao on the 28th of October, 1996. During elementary school her parents taught her how to do long division, which ignited her passion for mathematical structures. At the age of 11 she moved to Hoogeveen and started high school, where she learned Euclidean geometry and an appreciation for mathematical proofs. She started her bachelor's in Groningen at 17 and tried to become an independent adult, while learning some mathematics along the way. Realizing that mathematics is best enjoyed together, she joined a committee which organized social events, where people solved mathematical puzzles in teams. At the end of her second year she decided she wanted to be a number theorist and finished her bachelor by taking courses in Leiden, writing her Bachelor's thesis on analytic number theory. Afterwards she was accepted into the ALGANT master's program and studied in Leiden for the first year and in Milan for the second. She decided to write a master's thesis on modular forms, but got distracted by a subtle tangent and she ended up writing her thesis on algebraic geometry instead. This escalated even further into writing this PhD thesis on a very classical topic in algebraic geometry.

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