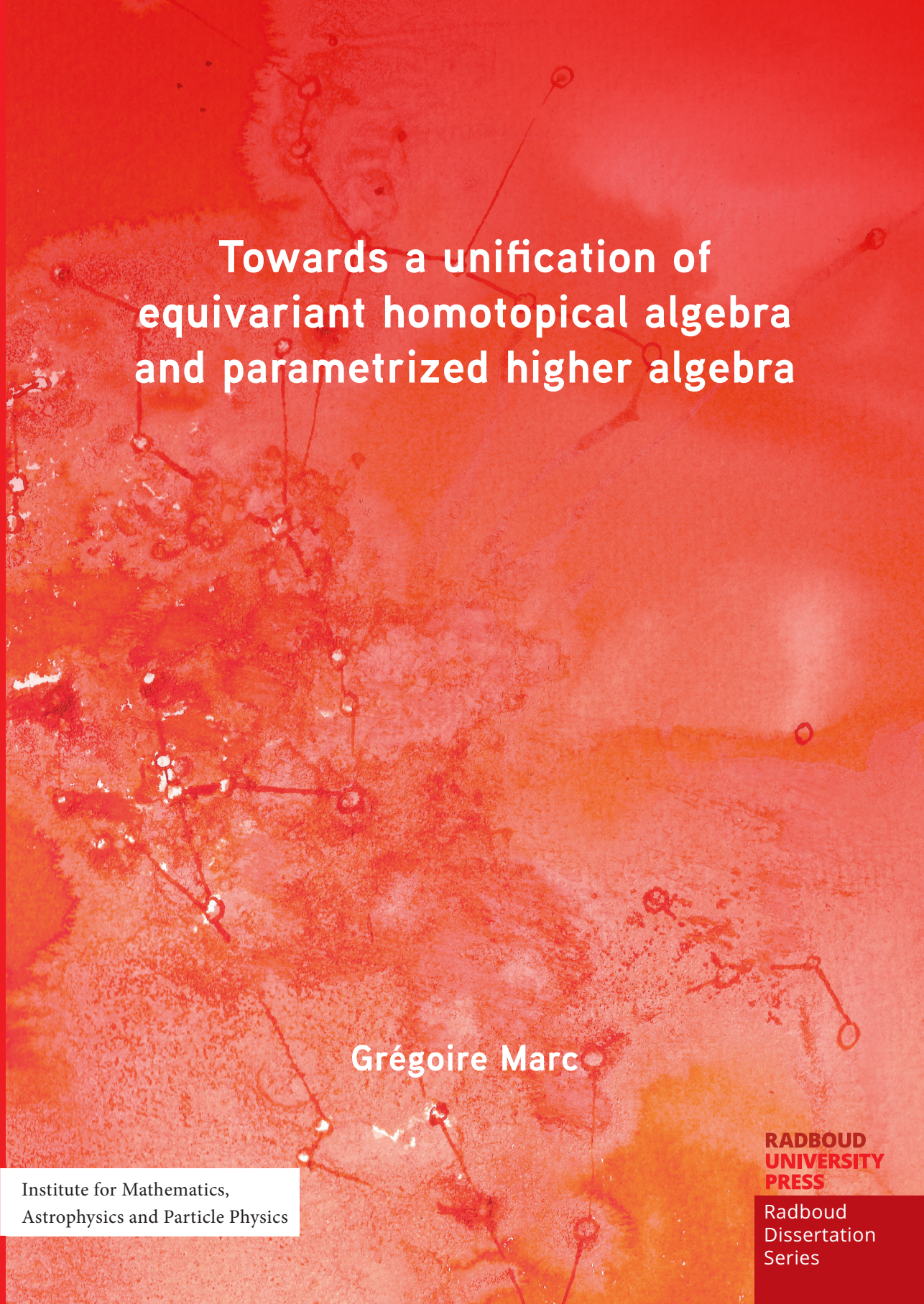




Towards a unification of equivariant homotopical algebra and parametrized higher algebra

Grégoire Marc

Institute for Mathematics,
Astrophysics and Particle Physics

**RADBOLD
UNIVERSITY
PRESS**

Radboud
Dissertation
Series

Towards a unification of equivariant homotopical algebra and parametrized higher algebra

Grégoire Marc

**Towards a unification of equivariant homotopical algebra and parametrized
higher algebra**

Grégoire Marc

Radboud Dissertation Series

ISSN: 2950-2772 (Online); 2950-2780 (Print)

Published by RADBOUD UNIVERSITY PRESS
Postbus 9100, 6500 HA Nijmegen, The Netherlands
www.radbouduniversitypress.nl

Design: Clotilde Marc
Cover: Clotilde Marc
Printing: DPN Rikken/Pumbo

ISBN: 9789465152844
DOI: 10.54195/9789465152844
Free download at: <https://doi.org/10.54195/9789465152844>

© 2026 Grégoire Marc

**RADBOUD
UNIVERSITY
PRESS**

This is an Open Access book published under the terms of Creative Commons Attribution-Noncommercial-NoDerivatives International license (CC BY-NC-ND 4.0). This license allows reusers to copy and distribute the material in any medium or format in unadapted form only, for noncommercial purposes only, and only so long as attribution is given to the creator, see <http://creativecommons.org/licenses/by-nc-nd/4.0/>.

Towards a unification of equivariant homotopical algebra and parametrized higher algebra

Proefschrift

ter verkrijging van de graad van doctor
aan de Radboud Universiteit Nijmegen
op gezag van de rector magnificus prof. dr. J.M. Sanders,
volgens besluit van het college voor promoties
in het openbaar te verdedigen op

dinsdag 18 augustus 2026
om 16:30 uur precies

door

Grégoire Yves Pierre Marc

geboren op 5 september 1998
te Étampes (Frankrijk)

Promotor:

prof. dr. B.J.J. Moonen

Copromotor:

dr. M. Kędziorek

Manuscriptcommissie:

prof. dr. H.T. Koelink

prof. dr. J.E. Bergner (University of Virginia, Verenigde Staten)

prof. dr. M. Hill (University of Minnesota, Verenigde Staten)

prof. dr. I. Moerdijk (Universiteit Utrecht)

prof. dr. A.M. Osorno (Reed College, Verenigde Staten)

Towards a unification of equivariant homotopical algebra and parametrized higher algebra

Dissertation

to obtain the degree of doctor
from Radboud University Nijmegen
on the authority of the Rector Magnificus prof. dr. J.M. Sanders,
according to the decision of the Doctorate Board
to be defended in public on

Tuesday, August 18, 2026
at 4.30 pm

by

Grégoire Yves Pierre Marc

born on September 5, 1998
in Étampes (France)

PhD supervisor:

prof. dr. B.J.J. Moonen

PhD co-supervisor:

dr. M. Kędziorek

Manuscript Committee:

prof. dr. H.T. Koelink

prof. dr. J.E. Bergner (University of Virginia, United states)

prof. dr. M. Hill (University of Minnesota, United states)

prof. dr. I. Moerdijk (Utrecht University)

prof. dr. A.M. Osorno (Reed College, United states)

À Céline

Contents

1	Introduction	11
1.1	From algebraic topology to higher category theory	11
1.2	Homotopical and higher algebra	13
1.3	Equivariant homotopy theory via parametrized ∞ -categories .	15
1.4	Equivariant and parametrized algebra	17
1.5	Overview and results	19
1.5.1	A higher Mackey functor description of algebras over an N_∞ -operad	19
1.5.2	On sifted homotopy colimits of algebras over an N_∞ -operad	20
1.5.3	Homotopy theory of simplicial parametrized operads .	22
1.5.4	A program to compare all models for equivariant operads	23
1.6	Notations and conventions	23
1.7	Structure of the thesis	24
2	A higher Mackey functor description of algebras over an N_∞-operad	25
2.1	Introduction	25
2.1.1	General motivations and related work	27
2.1.2	Structure of the chapter	28
2.2	Preliminaries on N_∞ -operads	29
2.2.1	N_∞ -operads and G -indexing systems	29
2.2.2	The ∞ -categories of $\mathcal{O}$ -algebras for an N_∞ -operad $\mathcal{O}$.	30
2.3	Normed symmetric monoidal categories	32
2.3.1	Preliminaries on $\mathcal{I}$ -normed symmetric monoidal categories	32
2.3.2	Rubin's combinatorial models of N_∞ -operads	36
2.4	Mackey functors vs $\mathcal{E}_{\mathcal{I}}$ -algebras	42
2.4.1	Orbital subcategories and the $\mathcal{I}$ -effective Burnside category	42
2.4.2	Unfurling for 2-categories and orbital subcategories .	44
2.4.3	From $\mathcal{A}_{\mathcal{I}}$ to $\mathcal{ISMS}\mathbf{t}$	47
2.4.4	The proof of the main Theorem	53
2.5	Appendix: Operads in G -categories	58

3	On sifted homotopy colimits of algebras over an N_∞-operad	63
3.1	Introduction	63
3.2	Counter example	64
3.3	Cellular extensions of strong fixed cofibrant operads	65
4	Homotopy theory of simplicial parametrized operads	75
4.1	Introduction	75
4.1.1	Main results	77
4.1.2	Structure of the chapter	78
4.1.3	Related work	78
4.2	$\mathbb{M}$ -operads	78
4.2.1	$\mathbf{F}$ -multicategories	78
4.2.2	$\mathbb{M}$ -operads	81
4.2.3	Coloured $\mathbb{M}$ -operads	84
4.2.4	Parametrized operads	87
4.3	Convolution product for double categories	89
4.3.1	Monoidal structure on spans	89
4.3.2	Left Kan extension along morphisms of cocartesian fibrations	91
4.3.3	Convolution product for double categories	92
4.3.4	Composition product for $\mathbf{F}$ -multicategories	97
4.3.5	Base change adjunctions	100
4.3.6	A Monad for $\mathbb{M}$ -operads	106
4.4	Homotopy theory of simplicial coloured $\mathbb{M}$ -operads	108
4.4.1	Model structure on simplicial $\mathbb{M}$ -operads	108
4.4.2	Model structure on simplicial coloured $\mathbb{M}$ -operads	109
5	A program to compare all models for equivariant operads	123
5.1	The coloured operadic Elmendorf conjecture	123
5.1.1	The Bonventre-Pereira model structure	123
5.1.2	The operadic fixed points adjunction	125
5.1.3	Cellular extension of $\mathbb{M}$ -operads	128
5.1.4	The coloured operadic Elmendorf conjecture	130
5.2	Complete $\mathbb{M}$ -Segal spaces	132
	Bibliography	143

Chapter 1

Introduction

1.1 From algebraic topology to higher category theory

One of the central problems in topology is the classification of topological spaces. As a full classification of topological spaces up to homeomorphism is currently out of reach, one usually attempts to classify spaces up to a weaker notion of equivalences, using *invariants*.

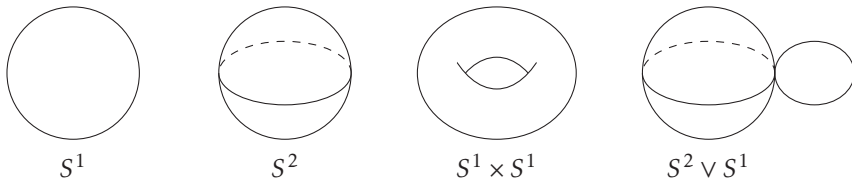


Figure 1.1: Examples of topological spaces.

For instance, the structure of paths between points in a space often encodes significant information about the space. In particular, one of the simplest topological invariants is the *set of path-components*, denoted π_0 , which can be used to distinguish very basic spaces.

Example 1.1.1. One can prove that the disc D^2 and the real line $\mathbb{R}$ are not homeomorphic, since the disc with a point removed is path-connected, whereas the real line with a point removed has two path-components.

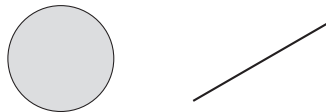


Figure 1.2: The disc D^2 and the line $\mathbb{R}$.

In many situations, invariants are most conveniently expressed in algebraic terms. One of the most basic examples is the *fundamental group*, denoted π_1 , which encodes information about loops (based at a given point) up to *homotopy*. Here a homotopy between two loops is a continuous deformation from one to the other that is keeping the basepoint fixed.

Example 1.1.2. The fundamental group can be used to prove that the circle S^1 and the sphere S^2 are not homeomorphic, since $\pi_1(S^1) \simeq \mathbb{Z}$ whereas $\pi_1(S^2)$ is trivial.

However, the set of path-components and the fundamental group do not suffice to distinguish most spaces, and one is naturally led to consider higher-dimensional invariants.

Example 1.1.3. The sphere S^2 and the point $*$ are not homeomorphic, yet they share the same invariants π_0 and π_1 .

These invariants admit a natural generalisation in the form of the *homotopy groups*, which play a central role in classical homotopy theory. Using these invariants, one obtains a notion of equivalences between topological spaces that is weaker than homeomorphism.

Definition 1.1.4. A map $f: X \rightarrow Y$ is called a *weak homotopy equivalence* if for every $x \in X$, the induced homomorphism

$$\pi_n(f, x): \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$$

is an isomorphism for all $n \geq 0$ (for $n = 0$, this is just a map of sets).

To study topological spaces up to weak homotopy equivalences, one considers the *homotopy category* $\text{Ho}(\text{Top})$, obtained by formally inverting weak homotopy equivalences in the category Top of topological spaces. Although such localisations do not generally admit an explicit description, the category Top carries a *model structure* in the sense of Quillen whose weak equivalences are precisely the weak homotopy equivalences. In particular, the homotopy category can be described as the category whose objects are CW-complexes and whose morphisms are homotopy classes of continuous maps.

In order to classify topological spaces up to weak homotopy equivalence, also called *homotopy types*, one may begin by studying *n-homotopy types*.

Definition 1.1.5. A topological space X in $\text{Ho}(\text{Top})$ is called an *n-homotopy type* if for all $k \geq n + 1$ and all $x \in X$, the group $\pi_k(X, x)$ is trivial.

Denoting by $\text{Ho}(\text{Top})_n$ the full subcategory of $\text{Ho}(\text{Top})$ spanned by *n-homotopy types*, one can show that 0-types are classified by sets, as the functor

$$\pi_0: \text{Ho}(\text{Top})_0 \rightarrow \text{Set}$$

is an equivalence. For 1-homotopy types, one considers instead the *fundamental groupoid* $\Pi_1(X)$, which encodes not only loops based at a point, as in π_1 , but also homotopy classes of paths between arbitrary points of X . One

then proves that the fundamental groupoid functor induces an equivalence between $\mathrm{Ho}(\mathrm{Top})_1$ and the category of groupoids and isomorphism classes of functors between them.

Motivated by the observation that 1-types can be modelled by groupoids, Grothendieck proposed, in a letter to Quillen, to base homotopy theory on a category of ∞ -groupoids whose homotopy category would recover all homotopy types, see [Gro83] for more details. In this framework, n -homotopy types would correspond to n -groupoids. Although still unproven in the precise form stated by Grothendieck, this idea, known as the *homotopy hypothesis*, strongly influenced the development of higher category theory.

Inspired by this perspective, Joyal [Joy08] and Lurie [Lur09] developed a theory of ∞ -categories based on the notion of *quasi-categories*, introduced earlier by Boardman and Vogt [BV73]. In this setting, *Kan complexes* play the role of ∞ -groupoids that are known, by a theorem of Kan [Hov99, Theorem 3.6.7], to model homotopy types. One important feature of this theory is that the localisation of a category $\mathcal{C}$ at a class of maps W can be naturally enhanced into an ∞ -category $\mathcal{C}[W^{-1}]$. In particular, $\mathrm{Ho}(\mathrm{Top})$ can be upgraded into an ∞ -category as follows.

Definition 1.1.6. The localisation of Top at the class of weak homotopy equivalences W yields the ∞ -category of ∞ -groupoids, denoted by $\mathcal{S} := \mathrm{Top}[W^{-1}]$.

The ∞ -category $\mathcal{S}$ plays a central role in higher category theory, analogous to the role of the category of sets in classical category theory.

Other models for ∞ -categories have been developed and shown to be equivalent, including simplicial categories, Segal categories, and complete Segal spaces [Ber07a, Ber07b, Rez01, JT07]. Today, the theory of ∞ -categories plays a central role in homotopy theory, where it is widely regarded as providing its conceptual foundations.

1.2 Homotopical and higher algebra

Given a space X , one may consider equipping it with various types of algebraic structures. Such a structure is typically specified by a collection of operations satisfying certain relations. However, when one wishes to study *homotopical algebraic structures*, the situation becomes more subtle, as relations are no longer expressed by strict equalities but are instead replaced by specified *homotopies*.

Example 1.2.1. An associative algebra structure (without a unit) on a space X consists of an operation

$$\times: X \times X \rightarrow X$$

satisfying the associativity relation

$$x \times (y \times z) = (x \times y) \times z$$

for all $x, y, z \in X$. In the homotopical setting, this equality is replaced by the choice of a homotopy, which in particular provides a path

$$\alpha_{x,y,z}: x \times (y \times z) \rightarrow (x \times y) \times z$$

for all $x, y, z \in X$. Moreover, higher homotopies must be specified to ensure coherence between the chosen homotopies. In particular, part of the structure consists of a homotopy filling the disc whose boundary is the loop obtained by concatenating the paths appearing in the following diagram.

$$\begin{array}{ccc}
 & ((x \times y) \times z) \times w & \\
 \alpha_{x,y,z} \times \text{id}_w \swarrow & & \searrow \alpha_{x \times y, z, w} \\
 (x \times (y \times z)) \times w & & (x \times y) \times (z \times w) \\
 \alpha_{x,y \times z, w} \downarrow & & \downarrow \alpha_{x,y, z \times w} \\
 x \times ((y \times z) \times w) & \xrightarrow{\text{id}_x \times \alpha_{y,z,w}} & x \times (y \times (z \times w))
 \end{array}$$

Similar higher homotopies must be provided, which can be organised and described in terms of Stasheff polytopes [Sta63].

Because of the complexity of these higher homotopical coherence conditions, giving an explicit description of homotopical algebraic structures typically requires a large amount of data. To address this issue, May introduced in [May72] the notion of *operads*. His original motivation was the study of infinite loop spaces, which carry the structure of a homotopy associative and commutative algebra, also known as an E_∞ -algebra.

Roughly speaking, a (topological) operad $\mathcal{O}$ consists of a space $\mathcal{O}_n$ for each $n \geq 0$, equipped with an action of the symmetric group Σ_n , together with composition maps

$$\circ: \mathcal{O}_n \times \mathcal{O}_{k_1} \times \cdots \times \mathcal{O}_{k_n} \longrightarrow \mathcal{O}_{k_1 + \cdots + k_n}$$

and a unit element $1_{\mathcal{O}} \in \mathcal{O}_1$, satisfying associativity, unitality, and equivariance conditions. The space $\mathcal{O}_n$ may be thought of as parametrizing formal operations of arity n . Given an operad $\mathcal{O}$, one defines the notion of an $\mathcal{O}$ -algebra as a space X such that to every operation $\mu \in \mathcal{O}_n$ one assigns, in a continuous and compatible way, an operation

$$\mu_X: X^n \rightarrow X,$$

in a manner that respects the operadic composition.

Example 1.2.2. The topological operad Com is defined by $\text{Com}_n = \{*\}$ for all n and encodes strictly commutative and associative algebras. Its homotopical analogue is given by E_∞ -operads. An E_∞ -operad is an operad $\mathcal{O}$ such that each space $\mathcal{O}_n$ is contractible and carries a free Σ_n -action. The freeness condition is a technical requirement that allows one to model commutativity up to coherent homotopy rather than strictly.

The theory of topological operads admits generalisations to other categories, such as simplicial sets, which also model homotopy types. Another important extension is the theory of *coloured operads* introduced by Boardman and Vogt in [BV73]. While a classical operad encodes operations on a single object, a coloured operad allows one to describe operations involving several objects simultaneously.

Example 1.2.3. A commutative ring with a module can be encoded as an algebra over a two-coloured operad, with one colour for the ring and one for the module.

Topological or simplicial (coloured) operads are themselves homotopical objects and can therefore be studied using homotopical methods. In particular, one can define a suitable notion of weak equivalences between operads and thereby obtain an ∞ -category $\infty\text{-Op}$ of ∞ -operads. As in the case of ∞ -categories, several models for this theory have been developed, including dendroidal sets [CM11], dendroidal spaces [CM13a], and ∞ -operads [Lur17]. These approaches provide different perspectives on the theory and have been compared, see for instance [Bar18], [CM13b, CM13a], and [CHH18].

The theory of ∞ -operads lies at the foundations of *higher algebra* [Lur17]. Moreover, classical homotopical algebra formalised with topological operads can be compared to algebras over the associated ∞ -operads [PS18a, Proposition 7.11]. This includes in particular the theory of E_∞ -algebras.

1.3 Equivariant homotopy theory via parametrized ∞ -categories

Let G be a finite group. Consider the category Top^G of topological spaces equipped with a G -action and continuous maps that preserve the G -action (called *equivariant maps*). The notion of homotopies between continuous maps admits a natural equivariant generalisation in the form of *G -homotopies*, defined as deformations of equivariant continuous maps. This leads in particular to the notion of *G -homotopy equivalences*. One may also consider *G -CW-complexes*, that is, CW-complexes equipped with a G -action acting by cellular maps and such that G acts trivially on each individual cell. As in the non-equivariant setting, the class of G -CW-complexes contains many interesting G -spaces, including, for instance, G -manifolds [Ill73, Theorem 2.1]. Moreover, Whitehead's Theorem admits the following equivariant generalisation.

Theorem 1.3.1 ([Mat71]). *A G -map $f: X \rightarrow Y$ between G -CW-complexes is a G -homotopy equivalence if and only if, for every subgroup $H \subseteq G$, the induced map on fixed points*

$$f^H: X^H \rightarrow Y^H$$

is a weak equivalence.

This result motivates the following definition.

Definition 1.3.2. A G -map $f: X \rightarrow Y$ in Top^G is called a *G -weak homotopy equivalence* if, for every subgroup $H \subseteq G$, the map

$$f^H: X^H \rightarrow Y^H$$

is a weak homotopy equivalence.

Example 1.3.3. A G -map $f: X \rightarrow Y$ that is a classical weak equivalence need not be a G -weak equivalence. For instance, the infinite-dimensional sphere S^∞ is contractible, but it admits a free C_2 -action. As a consequence, the unique equivariant map $S^\infty \rightarrow *$ is a weak equivalence of the underlying spaces, but not a C_2 -weak equivalence, as $(S^\infty)^{C_2} = \emptyset$ and $*^{C_2} = *$.

The category of G -spaces admits a model structure in the sense of Quillen whose class of weak equivalences is precisely the class of G -weak homotopy equivalences, denoted W_G . The resulting ∞ -category

$$\mathcal{S}^G := \text{Top}^G[W_G^{-1}]$$

may be described as the ∞ -category of G -CW-complexes, G -equivariant maps, and G -homotopies. The ∞ -category $\mathcal{S}^G$, which may be viewed as the ∞ -category of *equivariant ∞ -groupoids*, admits a second important characterisation.

Denote by $\mathcal{O}_G$ the *orbit category of G* , that is, the full subcategory of finite G -sets and equivariant maps spanned by transitive G -sets. Given a topological space X equipped with a continuous G -action, one obtains a functor

$$\phi(X): \mathcal{O}_G^{\text{op}} \rightarrow \text{Top}$$

which sends an orbit G/H to the fixed-point space X^H . The functoriality is induced by inclusions and conjugation maps between fixed-point subspaces. This construction yields an equivalence of ∞ -categories.

Theorem 1.3.4 (Elmendorf). *The functor*

$$\phi: \text{Top}^G \rightarrow \text{Fun}(\mathcal{O}_G^{\text{op}}, \text{Top})$$

induces an equivalence of ∞ -categories

$$\mathcal{S}^G \xrightarrow{\simeq} \text{Fun}(\mathcal{O}_G^{\text{op}}, \mathcal{S}).$$

Elmendorf's theorem (originally proven at the level of homotopy categories [Elm83]) plays a central role in equivariant homotopy theory, as it provides a categorical interpretation of equivariant spaces. While topological spaces may be regarded as ∞ -groupoids, equivariant topological spaces can be described as presheaves of ∞ -groupoids on the orbit category $\mathcal{O}_G$. In particular, the ∞ -category of equivariant spaces is an ∞ -*topos* in the sense of Lurie [Lur09].

This perspective offers several possible formalisms for a theory of G - ∞ -categories, including the theory of *parametrized ∞ -categories* developed by Barwick, Dotto, Glasman, Nardin, and Shah [BDG⁺16], the theory of *internal*

∞ -categories of Martini and Wolf [Mar22a, Mar22b, MW24], and the theory of formal categories via ∞ -equipments of Ruit [Rui25, Rui24]. In each of these approaches, the orbit category $\mathcal{O}_G$ plays a fundamental role, effectively replacing the group G itself. This correspondence between equivariant homotopy theory and parametrized categorical structures naturally leads to the following question.

Question 1.3.5. Given a homotopical theory based on equivariant objects, is it possible to construct a purely categorical avatar of this theory in the parametrized setting ?

In this thesis, we provide partial answers to this question for several theories related to equivariant operads and their algebras.

1.4 Equivariant and parametrized algebra

One of the core ideas of equivariant commutative algebra is that combining commutative algebra with equivariance naturally leads to additional structures, called *transfers*. Consider, for instance, a commutative monoid $(X, +)$ equipped with a G -action compatible with its monoid structure. For every subgroup $H \subseteq G$, one may form the fixed-point set X^H , which inherits the structure of a commutative monoid. Moreover, the restriction maps between these fixed-point sets are monoid homomorphisms.

Given two subgroups $K \subseteq H$ of G , one also obtains a monoid map $\mathrm{Tr}_K^H : X^K \rightarrow X^H$, called the *transfer*, defined by

$$\mathrm{Tr}_K^H(x) = h_1 \cdot x + \cdots + h_{[H:K]} \cdot x,$$

where $h_1, \dots, h_{[H:K]}$ are representatives of the cosets in H/K . The data of these commutative monoids of fixed points, together with the restriction, conjugation and transfer maps, assemble into a *Mackey functor*, as introduced by Dress in [Dre71] in the case of abelian groups.

Example 1.4.1. For $G = C_p$, we unpack the structure of a C_p -Mackey functor M . It consists of two abelian groups $M(C_p/C_p)$ and $M(C_p/e)$ together with a restriction map, a transfer map, and a C_p -action on $M(C_p/e)$ given by conjugation maps. The composition $\mathrm{Res} \circ \mathrm{Tr}$ is given by a Mackey formula, see [Lew88]. This structure can be represented by a *Lewis diagram*.

$$\begin{array}{c} M(C_p/C_p) \\ \begin{array}{c} \uparrow \quad \downarrow \\ \mathrm{Tr} \quad \mathrm{Res} \end{array} \\ M(C_p/e) \\ \begin{array}{c} \hookrightarrow \\ \mathrm{Conj} \end{array} \end{array}$$

The existence of these transfers relies on the strict commutativity of the multiplication on X . If instead X were a space equipped only with an E_∞ -structure, such transfers would no longer exist in general. This observation

suggests one of the guiding principles of equivariant homotopical algebra: transfers should be regarded as additional structures.

As discussed above, equivariant homotopy theory admits two complementary points of view: that of equivariant topology and that of parametrized ∞ -categories. Each of these perspectives gives rise to a theory of equivariant operads and algebras centred around the notion of transfers.

From the viewpoint of equivariant topology, one starts with the classical theory of simplicial (or topological) operads equipped with a G -action. Within this framework, Blumberg and Hill introduced in [BH15] the notion of N_∞ -operads, which generalise May's E_∞ -operads to the equivariant setting. Unlike the classical case, the homotopy category of N_∞ -operads is no longer contractible, but instead forms a poset. This poset can be described in terms of G -transfer systems, as defined in [BBR21] and [Rub19] (independently), by work of Bonventre and Pereira [BP21] and Rubin [Rub21].

Intuitively, an N_∞ -operad encodes a classical E_∞ -structure together with a compatible collection of transfers. The G -transfer system associated with an N_∞ -operad $\mathcal{O}$ records precisely the inclusions of subgroups of G for which $\mathcal{O}$ encodes the corresponding transfers. The combinatorics of these G -transfer systems is intricate and difficult to compute explicitly for a given group. Their study has become an active area of research in recent years, as illustrated for instance by work of Balchin, Barnes, and Roitzheim [BBR21], or Balchin, MacBrough, and Ormsby [BMO25].

While all N_∞ -operads are equivalent in the *naive homotopy theory of G -operads*, that is, the homotopy theory of operads [BM03] applied to equivariant spaces, they can be distinguished in the homotopy theory of G -simplicial operads associated with *graph weak equivalences*, as defined by Blumberg and Hill in [BH15]. Moreover, Bonventre and Pereira showed that these graph weak equivalences are part of a model structure on the category of G -simplicial operads [BP22c].

The perspective of G -simplicial (or topological) operads and N_∞ -operads thus provides a framework for equivariant operads and algebras that is well suited for constructing explicit models, due to its close connection with topology. However, this approach does not provide foundational tools for equivariant homotopical algebra, as it is not formulated in a fully categorical language. To obtain such foundations, one needs a theory of *parametrized higher algebra*.

As emphasised above, transfers are expected to play a central role in any satisfactory theory of equivariant higher algebra. In particular, such a theory should be based on suitable notions of *higher Mackey functors*. This idea first appeared in work of Guillou and May [GM24], where they showed that the model category of *genuine G -spectra* can be related to *spectrally enriched Mackey functors* via a zig-zag of Quillen equivalences. Motivated by this result, Barwick developed in [Bar17] the theory of *higher categories of spans*, which provides a general framework for various notions of higher Mackey functors, including *spectral Mackey functors*. Building on Barwick's work and the formalism of parametrized ∞ -categories, Nardin introduced in [Nar16] notions of equivariant semi-additivity, additivity, and stability, for which different categories of Mackey functors arise as universal theories. He also

gave an alternative description of the universal semi-additive theory in terms of *parametrized E_∞ -algebras*, extending the ∞ -categorical theory of E_∞ -algebras to the parametrized setting.

The notion of parametrized semi-additivity was later generalised to *T -parametrized semi-additivity* for any G -transfer system T by Cnossen, Linskens, and Lenz in [CLL25b]. In the same work, they described the corresponding universal theory in terms of *T -parametrized E_∞ -algebras*. Subsequently, in [CLL25c], they showed that *T -parametrized E_∞ -algebras* can also be described in terms of *T -incomplete Mackey functors*, namely Mackey functors admitting only those transfers encoded by the G -transfer system T . This theory of incomplete parametrized E_∞ -algebras also admits an operadic formulation. In [NS22], Nardin and Shah introduced the notion of *parametrized operads*, and Stewart studied incomplete parametrized E_∞ -algebras from this operadic perspective in [Ste25b, Ste25c]. These developments lead to the following question, which is a particular case of Question 1.3.5 and serves as a guiding theme for this thesis.

Question 1.4.2. How can equivariant homotopical algebra and parametrized higher algebra be unified?

Some results in the literature already suggest possible approaches, such as the work of Bonventre [Bon19], which constructs functors between different categories of operads and their associated categories of algebras. However, these functors do not yet admit a homotopical interpretation.

1.5 Overview and results

In this thesis, we propose to investigate Question 1.4.2, provide partial answers, and develop ideas and constructions toward a complete answer. In Chapter 2, we compare N_∞ -algebras with higher Mackey functors, thus answering Question 1.4.2 in the case of commutative algebra. In Chapter 3, we generalise some results of [BM03] and [PS18a] to the equivariant setting. The results of Chapter 3 are used for N_∞ -operads in Chapter 2 in order to prove the main theorem. In Chapter 4, we develop a homotopy theory of *generalised simplicial operads* which, in particular, provides a homotopy theory of simplicial parametrized operads. In Chapter 5, we outline a program that aims at answering Question 1.4.2 in the case of operads by using the tools we developed in Chapter 4.

1.5.1 A higher Mackey functor description of algebras over an N_∞ -operad

The goal of this first chapter is to compare N_∞ -algebras with incomplete higher Mackey functors, thereby providing a complete answer to Question 1.4.2 in the case of commutative algebra. More precisely, we prove the following theorem.

Theorem 1.A (Theorem 2.4.37). *For every group G and every G -transfer system T , let $\mathcal{E}_T$ denote Rubin’s N_∞ -operad associated with T [Rub21]. The ∞ -category $\mathbf{Alg}_{\mathcal{E}_T}$ obtained from $\mathbf{Alg}_{\mathcal{E}_T}(\mathbf{sSet}^G)$ by inverting G -weak homotopy equivalences is equivalent to the ∞ -category of T -incomplete Mackey functors with values in spaces.*

Rubin’s N_∞ -operad $\mathcal{E}_T$ is obtained as the nerve of an operad $\mathcal{SM}_T$ in the category $\mathbf{Cat}^G$ of G -objects in categories. Algebras over $\mathcal{SM}_T$ in $\mathbf{Cat}^G$ can be described as T -normed symmetric monoidal categories [Rub20]. We first construct a functor from the category of T -incomplete spans $\mathcal{A}(\mathcal{O}_G, \mathcal{O}_T)$ to the 2-category $\mathbf{ISMStg}$ of T -normed symmetric monoidal categories, using the *Beck–Chevalley condition*. By composing this functor with the *core functor*, we obtain a fully faithful functor

$$\mathcal{A}(\mathcal{O}_G, \mathcal{O}_T) \rightarrow \mathbf{Alg}_{\mathcal{E}_T}$$

which sends a finite G -set T to the free $\mathcal{E}_T$ -algebra on T . We conclude by showing, using the results of Chapter 3, that the left Kan extension of this functor along the Yoneda embedding

$$\mathcal{A}(\mathcal{O}_G, \mathcal{O}_T) \hookrightarrow \mathbf{Fun}_\times(\mathcal{A}(\mathcal{O}_G, \mathcal{O}_T)^{\mathrm{op}}, \mathcal{S})$$

is an equivalence of ∞ -categories.

1.5.2 On sifted homotopy colimits of algebras over an N_∞ -operad

In Chapter 3, we establish a key technical result concerning sifted homotopy colimits of algebras over an N_∞ -operad $\mathcal{O}$. Unlike E_∞ -operads, N_∞ -operads are not Σ -cofibrant in the sense of [BM03]. This presents a difficulty, as many results concerning E_∞ -operads and their algebras rely on this property, for example [BM03, Proposition 5.7]. In Chapter 3, we seek a generalisation of the following proposition that applies to N_∞ -operads.

Proposition 1.5.1 ([PS18a, Proposition 7.9]). *For every Σ -cofibrant operad $\mathcal{O}$ in $\mathbf{sSet}^G$, that is such that the morphism of collections $I \rightarrow \mathcal{O}$ given by the identity of $\mathcal{O}$ is a cofibration in $\mathbf{Coll}(\mathbf{sSet}^G)$ for the projective model structure, the forgetful functor*

$$U: \mathbf{Alg}_{\mathcal{O}}(\mathbf{sSet}^G) \rightarrow \mathbf{sSet}^G$$

preserves sifted homotopy colimits.

This result makes essential use of the following proposition.

Proposition 1.5.2 ([BM03, Corollary 5.5]). *For every category I with finite coproducts and every Σ -cofibrant operad $\mathcal{O}$ in $\mathbf{sSet}^G$, the forgetful functor*

$$U: \mathbf{Fun}(I, \mathbf{Alg}_{\mathcal{O}}(\mathbf{sSet}^G)) \simeq \mathbf{Alg}_{\mathcal{O}}(\mathbf{Fun}(I, \mathbf{sSet}^G)) \rightarrow \mathbf{Fun}(I, \mathbf{sSet}^G)$$

preserves cofibrant objects.

However, as we show in Section 3.2, this result does not hold for N_∞ -operads.

Theorem 1.B (Proposition 3.2.3). *Let $\mathcal{O}$ be an N_∞ -operad for C_2 with a complete transfer system. The forgetful functor*

$$U: \text{Alg}_{\mathcal{O}}\left(\text{Fun}(\text{Fin}, \text{sSet})^{C_2}\right) \rightarrow \text{Fun}(\text{Fin}, \text{sSet})^{C_2}$$

does not preserve cofibrant objects, where Fin denotes the category of finite sets.

Although N_∞ -operads are not Σ -cofibrant, their symmetric group actions are always free by definition. This motivates the following notion, which serves as a replacement for Σ -cofibrancy.

Definition 1.5.3. An operad $\mathcal{O}$ in sSet^G is *graph cofibrant* if, for every $n \geq 0$, the Σ_n -action on $\mathcal{O}_n$ is free.

To address the failure exhibited in Theorem 1.B, we prove in Chapter 3 the following result.

Theorem 1.C (Proposition 3.3.19). *For every graph cofibrant operad $\mathcal{O}$ in sSet^G and every subgroup H of G , the fixed-point functor*

$$(-)^H: \text{Alg}_{\mathcal{O}}\left(\text{Fun}(I, \text{sSet}^G)\right) \rightarrow \text{Fun}(I, \text{sSet}),$$

defined as the composite

$$\text{Alg}_{\mathcal{O}}\left(\text{Fun}(I, \text{sSet}^G)\right) \xrightarrow{U} \text{Fun}(I, \text{sSet}^G) \xrightarrow{(-)^H} \text{Fun}(I, \text{sSet}),$$

preserves cofibrant objects.

This result implies the following theorem.

Theorem 1.D (Corollary 3.3.21). *For every graph cofibrant operad $\mathcal{O}$ in sSet^G and every subgroup H of G , the H -fixed-point functor*

$$(-)^H: \text{Alg}_{\mathcal{O}}(\text{sSet}^G) \rightarrow \text{sSet}$$

preserves sifted homotopy colimits.

Finally, using Elmendorf's Theorem, Theorem 1.D immediately implies the following result.

Theorem 1.E (Corollary 3.3.22). *For every N_∞ -operad $\mathcal{O}$, if $\mathbf{Alg}_{\mathcal{O}}$ denotes the ∞ -category obtained from $\text{Alg}_{\mathcal{O}}(\text{sSet}^G)$ by inverting G -weak homotopy equivalences, then the forgetful functor*

$$U: \mathbf{Alg}_{\mathcal{O}} \rightarrow \mathcal{S}^G$$

preserves sifted colimits.

1.5.3 Homotopy theory of simplicial parametrized operads

After comparing N_∞ -algebras with their parametrized analogues in Chapter 2, one may ask how to generalise this comparison to other types of algebras. Since algebras are defined as morphisms of operads, it is natural to begin by establishing such comparisons at the level of equivariant operads themselves. In Chapter 3, we develop a general framework for studying the homotopy theory of various notions of operads. In particular, this formalism is designed to compare different models for equivariant operads.

Classically, the structure of an operad is intrinsically based on symmetric groups. Motivated by this observation, we introduce the notion of **F-multicategories** (where **F** is the *finite coproducts completion functor*), which are intended to replace symmetric groups and thereby yield a generalisation of classical operads.

Definition 1.5.4. An **F-multicategory** $\mathbb{M}$ consists of the following data:

1. a small category $\mathbb{M}_0$ of *objects* and a small category $\mathbb{M}_1$ of *multimorphisms*;
2. a *composition functor* $\odot: \mathbb{M}_1 \times_{\mathbf{F}\mathbb{M}_0} \mathbf{F}\mathbb{M}_1 \rightarrow \mathbb{M}_1$ together with an identity functor $1: \mathbb{M}_0 \rightarrow \mathbb{M}_1$;

such that the composition functor is associative and unital with respect to the identity functor.

Every **F-multicategory** $\mathbb{M}$ gives rise to a corresponding notion of simplicial $\mathbb{M}$ -operads.

Definition 1.5.5. A simplicial $\mathbb{M}$ -operad $\mathcal{O}$ consists of the following data:

1. an $\mathbb{M}$ -*collection* $\mathcal{O}$, given by a functor $\mathcal{O}: \mathbb{M}_1^{\text{op}} \rightarrow \mathbf{sSet}$;
2. a composition map

$$\circ: \mathcal{O}(f) \times \mathcal{O}(f_1) \times \cdots \times \mathcal{O}(f_n) \rightarrow \mathcal{O}(f \odot (f_1, \dots, f_n))$$

for all composable multimorphisms $(f, (f_1, \dots, f_n))$ in $\mathbb{M}_1 \times_{\mathbf{F}\mathbb{M}_0} \mathbf{F}\mathbb{M}_1$;

3. an identity element 1_X in $\mathcal{O}(1_X)_0$ for every object X in $\mathbb{M}_0$;

such that the composition maps and identity elements are natural and satisfy associativity and unitality conditions.

Under suitable assumptions on $\mathbb{M}$, one can show that the category of simplicial $\mathbb{M}$ -operads is the category of algebras in the monoidal category of $\mathbb{M}$ -collections

$$\text{Coll}_{\mathbb{M}}(\mathbf{sSet}) := \text{Fun}(\mathbb{M}_1^{\text{op}}, \mathbf{sSet}),$$

endowed with a *composition product* $\circ_{\mathbb{M}}$.

Theorem 1.F (Corollary 4.3.20). *For every **F-multicategory** $\mathbb{M}$ that is target right fibrant in the sense of Definition 4.3.16, the category $\text{Coll}_{\mathbb{M}}(\mathbf{sSet})$ of $\mathbb{M}$ -collections in $\mathbf{sSet}$ is endowed with the structure of a monoidal category for which the category of algebras is precisely the category of simplicial $\mathbb{M}$ -operads.*

Using this generalised composition product, together with the fact that $\text{Coll}_{\mathbb{M}}(\text{sSet})$ admits a projective model structure, we prove the following result.

Theorem 1.G (Theorem 4.4.1). *For every target right fibrant $\mathbf{F}$ -multicategory $\mathbb{M}$, the category $\text{sOp}_{\mathbb{M}}$ admits a model structure that is right lifted from $\text{Coll}_{\mathbb{M}}(\text{sSet})$ along the free-forgetful adjunction*

$$F_{\mathbb{M}}: \text{Coll}_{\mathbb{M}}(\text{sSet}) \rightleftarrows \text{sOp}_{\mathbb{M}}: U_{\mathbb{M}}.$$

One main feature of our notion of generalised operads is that it provides a convenient way to encode coloured operads. For every $\mathbb{M}$ -set of colours A (in the sense of Definition 4.2.18), A -coloured operads are by definition $\mathbb{M}^A$ -operads for a certain $\mathbf{F}$ -multicategory $\mathbb{M}^A$. Using this description of coloured $\mathbb{M}$ -operads, we prove the following theorem.

Theorem 1.H. (Theorem 4.4.28) *For every $\mathbf{F}$ -multicategory $\mathbb{M}$ that is target right fibrant, the category of simplicial coloured $\mathbb{M}$ -operads admits a model structure for which the weak equivalences are the fully faithful essentially surjective morphisms and for which the fibrations are the isofibrations (See Definition 4.4.11 for all the terminology used here).*

We can specialise this theorem to obtain a model structure on the simplicial parametrized operads of Nardin and Shah.

Theorem 1.I. (Example 4.4.29) *For every category $\mathcal{O}$ and every orbital subcategory $\mathcal{T}$ of $\mathcal{O}$, the category $\text{sOp}_{(\mathcal{O}, \mathcal{T})}^{\text{col}}$ of simplicial coloured $(\mathcal{O}, \mathcal{T})$ -parametrized operads admits a model structure for which the weak equivalences are the fully faithful essentially surjective morphisms and for which the fibrations are the isofibrations (respectively defined in Definition 4.4.11).*

1.5.4 A program to compare all models for equivariant operads

In this Chapter 5, we outline a program aimed at applying the techniques developed in Chapter 4 to address Question 1.4.2 in the context of equivariant operads. In Section 5.1, we explain how one might expect to establish an *operadic Elmendorf theorem* comparing simplicial parametrized operads with equivariant simplicial operads. The Section 5.2 is devoted to the definition of the notion of *complete $\mathbb{M}$ -Segal spaces*. In particular, we conjecture the existence of an equivalence between complete $\mathbb{M}$ -Segal spaces and simplicial coloured $\mathbb{M}$ -operads, and we explain how such an equivalence would provide a definitive answer to Question 1.4.2 in the operadic case.

1.6 Notations and conventions

We fix three Grothendieck universes $\mathbf{U} \in \mathbf{V} \in \mathbf{W}$ that contain the first infinite ordinal. A set is *small* if it is contained in $\mathbf{U}$ and *large* if it is contained in $\mathbf{W}$. We will also use an analogous naming convention for categories. In what

follows, we denote by $\mathbf{Set}$ the category of small sets, by $\mathbf{sSet}$ the category of small simplicial sets, by $\mathbf{Cat}$ the category of small categories, and by $\widehat{\mathbf{Cat}}$ the large category of categories. For every closed symmetric monoidal category $\mathcal{C}$, we denote by $\mathbf{Op}(\mathcal{C})$ the category of operads in $\mathcal{C}$. If $\mathcal{O}$ is an operad in $\mathcal{C}$, then we denote by $\mathcal{O}_n$ its n -th level, that is the associated object of $\mathcal{C}^{\Sigma_n}$. Until the end of this thesis, by ∞ -category we mean quasi-category and we denote by $\mathcal{S}$ the ∞ -category of spaces.

Throughout this thesis, G is a finite group and for every category $\mathcal{C}$, we denote by $\mathcal{C}^G$ the category of G -objects in $\mathcal{C}$. A G -category is an internal category in $\mathbf{Set}^G$ or, equivalently, a G -object in the category of small categories. If $\mathcal{C}$ and $\mathcal{D}$ are G -categories, then we denote by $\mathbf{Fun}_G(\mathcal{C}, \mathcal{D})$ the category of G -functors and G -natural transformations between $\mathcal{C}$ and $\mathcal{D}$ and by $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ the G -category of non-equivariant functors and non-equivariant natural transformations. We denote by $\mathbf{Cat}^G$ the 2-category of small G -categories and by $\mathcal{S}^G$ the ∞ -category of equivariant spaces $\mathbf{Fun}(\mathcal{O}_G^{\mathbf{Op}}, \mathcal{S})$.

As we discussed previously, the ∞ -category of N_∞ -operads is equivalent to the poset of G -transfer systems [BH15, BBR21, Rub21, BP21]. This poset also admits a description in terms of G -indexing systems. Throughout this thesis, depending on the context, we will use the terminology of G -indexing systems or G -transfer systems.

1.7 Structure of the thesis

This thesis contains 4 chapters. The two main chapters, Chapter 2 and Chapter 4, are independent and are based, respectively, on the preprints [Mar24] and [Mar25]. The Chapter 2 depends on Chapter 3, which is a technical chapter. The Chapter 5 is a further directions chapter that depends only on Chapter 4.

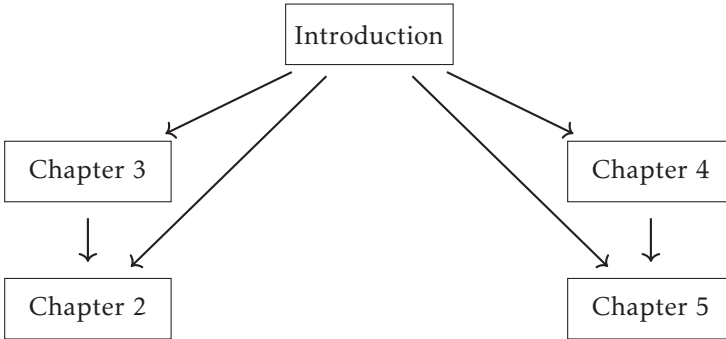


Figure 1.3: Dependence of the chapters.

Chapter 2

A higher Mackey functor description of algebras over an N_∞ -operad

This chapter is based on the preprint [Mar24].

2.1 Introduction

Let G be a finite group and consider a set X endowed with an action of G . For every subgroup H of G , we can consider the set of H -fixed elements of X

$$X^H = \{x \in X \mid h \cdot x = x \text{ for all } h \in H\}.$$

For every pair of subgroups $K \leq H$ of G and every element g of G , there is a restriction map $\text{Res}_K^H : X^H \hookrightarrow X^K$ (the inclusion of H -fixed elements in K -fixed elements) and a conjugation map $c_g : X^H \rightarrow X^{gHg^{-1}}$ that sends x to $g \cdot x$. These data assemble into a presheaf $F_X : \mathcal{O}_G^{\text{op}} \rightarrow \text{Set}$ over the category of transitive G -equivariant sets $\mathcal{O}_G$ that sends the coset G/H to the set X^H . This construction forms a fully faithful functor $F_{(-)} : \text{Set}^G \rightarrow \text{Fun}(\mathcal{O}_G^{\text{op}}, \text{Set})$ that is, in general, not an equivalence of categories. However, if X is a topological space, then, up to weak homotopy equivalence, every topological presheaf $P : \mathcal{O}_G^{\text{op}} \rightarrow \text{Top}$ has the form F_X for some G -equivariant topological space X . Moreover, for the right homotopy theory on G -equivariant spaces, *Elmendorf's Theorem* [Elm83] states that this functor is an equivalence of homotopy theories. Elmendorf's Theorem can be considered to be one of the cornerstones of equivariant homotopy theory as it brings upon it categorical features, either with the theory of *parametrized ∞ -categories* developed by Barwick, Dotto, Glasman, Nardin, and Shah [BDG⁺16, Sha17], or with the more general theory of *categories internal in an ∞ -topos* of Martini and Wolf [Mar22b, MW24, Mar22a].

Suppose now that X is a commutative monoid with a G -action compatible with its monoid structure. The presheaf provided by the previous construction

comes with an extra structure given by the monoid law. For every pair of subgroups $K \leq H$ of G , there is a transfer map $\mathrm{Tr}_K^H : X^K \rightarrow X^H$ defined by

$$\mathrm{Tr}_K^H(x) = h_1 \cdot x + \cdots + h_{[H:K]} \cdot x$$

where $h_1, \dots, h_{[H:K]}$ are representatives of cosets in H/K . These transfers, along with the previous structure of restriction and conjugation maps, assemble into a *Mackey functor* (with value in the category of sets) as defined by Dress in [Dre71]. A Mackey functor $\mathcal{M}_X : A_G^{\mathrm{op}} \rightarrow \mathrm{Set}$ is a presheaf over the *Burnside category* A_G that preserves finite products. The Burnside category A_G is the category whose objects are finite G -equivariant sets and whose morphisms are isomorphism classes of spans between these G -equivariant finite sets. Mackey functors play an important role in representation theory, equivariant homotopy theory, and they have many applications, for example in group cohomology and in the decomposition of classifying spaces with the work of Webb [Web93]. The construction of a Mackey functor from a commutative monoid with a compatible G -action provides a fully faithful functor $\mathcal{M}_{(-)} : \mathrm{CMon}^G \rightarrow \mathrm{Fun}_\times(A_G^{\mathrm{op}}, \mathrm{Set})$ that is again not an equivalence of categories in general. One might now be interested in promoting this construction to an equivalence of homotopy theories, in analogy with Elmendorf's Theorem mentioned above.

However, unlike classical algebra, in homotopy theory, commutativity is not a property, but a structure. Given a topological space X with an associative and unitary operation $\mu : X \times X \rightarrow X$, X is considered as being homotopy commutative if there exists a homotopy H as depicted in the following diagram

$$\begin{array}{ccc} X \times X & & \\ \downarrow \text{twist} & \searrow \mu & \\ X \times X & \xrightarrow{\mu} & X \end{array} \quad \begin{array}{c} H \\ \nearrow \end{array}$$

and if higher coherences are satisfied (requiring the data of higher homotopies). The data of these homotopies are part of the commutative structure of X and can be encoded with May's theory of E_∞ -operads [May72]. An E_∞ -operad is a topological operad $\mathcal{O}$ such that $\mathcal{O}_n$ is contractible and has a free Σ_n -action. This theory of “commutativity up to homotopy” is crucial in topology, as most spaces with a multiplication, such as infinite loop spaces, are not strictly commutative.

If X is a topological space with a G -action and a compatible structure of E_∞ -algebra (meaning that X is an algebra over an E_∞ -operad $\mathcal{O}$ and that all the corresponding operations and homotopies between them are G -equivariant), then one can still consider the topological presheaf $F_X : \mathcal{O}_G^{\mathrm{op}} \rightarrow \mathrm{Top}$ associated with X but the transfers do not exist anymore, as the operation μ of X is only commutative up to homotopy. To solve this issue, E_∞ -operads can be upgraded into an equivariant analogue, the N_∞ -operads introduced by Blumberg and Hill in [BH15]. In addition to the classical operations of an E_∞ -algebra, an N_∞ -operad encodes norms that can be thought of as equivariantly twisted versions

of these operations. Unlike the non-equivariant setting, not all N_∞ -operads are equivalent, and the homotopy category of N_∞ -operads is equivalent to the poset of the so-called *G-indexing systems*, by the work of Blumberg and Hill [BH15], Rubin [Rub21], and Bonventre and Pereira [BP21]. A *G-indexing system* $\mathcal{I}$ is a collection, for every subgroup H of G , of finite H -equivariant sets satisfying relations of compatibility, as described in [BH15]. If $\mathcal{I}$ is a *G-indexing system*, then every finite H -equivariant set T that belongs to $\mathcal{I}$ encodes an operation that is, roughly speaking, “*T*-equivariant”.

Suppose now that X is an algebra over an N_∞ -operad $\mathcal{O}$ associated with a *G-indexing system* $\mathcal{I}$. If $K \leq H$ are subgroups of G such that H/K belongs to $\mathcal{I}$, then we may construct a transfer map $\mathrm{Tr}_K^H : X^K \rightarrow X^H$. However, the construction of Tr_K^H requires the choice of an H/K -equivariant operation on X , that is only unique up to homotopy. The homotopy groups of X thus give rise to *$\mathcal{I}$ -incomplete Mackey functors*, defined in terms of an incomplete version of the Burnside category $\mathcal{A}_{\mathcal{I}}$, as studied by Blumberg and Hill in [BH18]. This structure cannot be naïvely promoted to the space X itself, as the transfers depend on the choice of homotopies. To take these homotopies into account, the *$\mathcal{I}$ -Burnside category* $\mathcal{A}_{\mathcal{I}}$ can be upgraded into a $(2, 1)$ -category, the *$\mathcal{I}$ -effective Burnside category* $\mathcal{A}_{\mathcal{I}}$, remembering isomorphisms between spans. The $(2, 1)$ -category $\mathcal{A}_{\mathcal{I}}$ leads also to a notion of *higher $\mathcal{I}$ -incomplete Mackey functors* (with value in the ∞ -category of spaces) defined as presheaves (of spaces) over $\mathcal{A}_{\mathcal{I}}$ that preserve finite products. In this chapter, we construct an equivalence between the ∞ -category of (higher) *$\mathcal{I}$ -incomplete Mackey functors* and the ∞ -category of algebras in *G-topological spaces* over an N_∞ -operad $\mathcal{O}$ associated with $\mathcal{I}$. More precisely, the main result of this chapter is the following theorem.

Theorem 2.A. (Theorem 2.4.37) *Let G be a finite group, let $\mathcal{I}$ be a *G-indexing system* and denote by $\mathcal{E}_{\mathcal{I}}$ Rubin’s N_∞ -operad associated with $\mathcal{I}$. The ∞ -category obtained from $\mathrm{Alg}_{\mathcal{E}_{\mathcal{I}}}(\mathrm{Top}^G)$ by inverting *G-weak homotopy equivalences* (i.e., continuous *G-equivariant maps* $f : X \rightarrow Y$ such that, for every subgroup H of G , $f^H : X^H \rightarrow Y^H$ is a weak homotopy equivalence) is equivalent to the ∞ -category of *$\mathcal{I}$ -incomplete Mackey functors* $\mathrm{Mack}_{\mathcal{I}} = \mathrm{Fun}_\times(\mathcal{A}_{\mathcal{I}}^{\mathrm{op}}, \mathcal{S})$ where $\mathcal{S}$ is the ∞ -category of spaces and $\mathcal{A}_{\mathcal{I}}$ is the incomplete $(2, 1)$ -category of spans of finite *G-equivariant sets* associated with $\mathcal{I}$.*

2.1.1 General motivations and related work

Our main motivation for Theorem 2.A is to compare the theory of N_∞ -algebras with the theory of parametrized $\mathcal{T}$ -semi-additivity as developed by Cnossen, Lenz, and Linsken in [CLL25b], which generalises the work of Nardin in [Nar16] and [Nar17]. As we already mentioned, one important feature of Elmendorf’s Theorem is that the theory of parametrized ∞ -categories gives a categorical framework for equivariant homotopy theory through the notion of *G- ∞ -categories* (here we mean $\mathcal{O}_G$ -parametrized ∞ -categories). This has been motivated by Guillou and May’s [GM24, Theorem 1.14], which describes genuine equivariant spectra in terms of spectral Mackey functors. The main

theorem of the present chapter can be thought of as an unstable version of Guillou and May's Theorem. The poset of G -indexing systems or, equivalently, the poset of N_∞ -operads is equivalent to the poset of orbital subcategories of $\mathcal{O}_G$ in the sense of [CLL25b] and it follows that every G -indexing system $\mathcal{I}$ corresponds to an associated orbital subcategory $\mathcal{T}$ of $\mathcal{O}_G$. Every orbital subcategory $\mathcal{T}$ of $\mathcal{O}_G$ comes with an associated notion of $\mathcal{T}$ -semi-additivity and it has been shown, first by Nardin in [Nar16, Theorem 6.5] and [Nar17, Theorem 2.32] in the genuine case (that is for the maximal G -indexing system), and more recently by Cnossen, Lenz, and Linskens in the general case [CLL25a, Corollary 9.14], that the G - ∞ -category of $\mathcal{I}$ -incomplete Mackey functors is the universal $\mathcal{T}$ -semi-additive G - ∞ -category over equivariant spaces. One of the consequences of Theorem 2.A should be that the G - ∞ -category of $\mathcal{O}$ -algebras for an N_∞ -operad $\mathcal{O}$ associated with a G -indexing system $\mathcal{I}$ is the universal $\mathcal{T}$ -semi-additive G - ∞ -category over equivariant spaces where $\mathcal{T}$ is the orbital subcategory of $\mathcal{O}_G$ that corresponds to $\mathcal{I}$.

The strategy of proof that we adopt to show Theorem 2.A is similar to the techniques used in [Len25]. Note that it should be possible to establish a formal comparison between $\mathcal{O}$ -algebras for an N_∞ -operad $\mathcal{O}$ and incomplete Mackey functors using the theory of parametrized operads [NS22] by showing, for every graph Σ -cofibrant simplicial G -operad $\mathcal{O}$ (in the sense of Definition 3.3.1), that the nerve functor $N(\mathrm{Alg}_{\mathcal{O}}(\mathrm{SSet}^G)) \rightarrow \mathrm{Alg}_{N^{\otimes(\mathcal{O})}}(\mathcal{S}^G)$ of [Bon19, Theorem IV] is an equivalence of ∞ -categories. However, this would require a comparison, for every graph Σ -cofibrant operad $\mathcal{O}$, between free $\mathcal{O}$ -algebras over G -equivariant spaces and the corresponding free algebras over the nerve of $\mathcal{O}$.

2.1.2 Structure of the chapter

In Section 2.2, we first recall Blumberg and Hill's notions of N_∞ -operads and G -indexing systems and the relation between them. We then define the ∞ -category $\mathbf{Alg}_{\mathcal{O}}$ of algebras over an N_∞ -operad $\mathcal{O}$ and exhibit an explicit description of it. In Section 2.3, we recall Rubin's notion of $\mathcal{I}$ -normed symmetric monoidal categories for a given G -indexing system $\mathcal{I}$ and we give an explicit construction of the operad $\mathcal{SM}_{\mathcal{I}}$ that encodes them. We then show that the nerve of $\mathcal{SM}_{\mathcal{I}}$, denoted by $\mathcal{E}_{\mathcal{I}}$, is an N_∞ -operad associated with $\mathcal{I}$. In Section 2.4, we begin by defining the $\mathcal{I}$ -effective Burnside category $\mathcal{A}_{\mathcal{I}}$ and the associated ∞ -category of $\mathcal{I}$ -Mackey functors $\mathrm{Mack}_{\mathcal{I}}$. We then construct explicitly the free $\mathcal{E}_{\mathcal{I}}$ -algebra over a finite G -set A as the nerve of an $\mathcal{I}$ -normed symmetric monoidal groupoid $\mathcal{F}(A)$ and we use a 2-categorical unfurling to construct a functor $\theta : \mathcal{A}_{\mathcal{I}} \rightarrow \mathbf{Alg}_{\mathcal{E}_{\mathcal{I}}}$ that sends A to the free $\mathcal{E}_{\mathcal{I}}$ -algebra over A . Finally, we use the results of Chapter 3 to prove Theorem 2.A (Theorem 2.4.37) by showing that the left Kan extension of $\theta : \mathcal{A}_{\mathcal{I}} \rightarrow \mathbf{Alg}_{\mathcal{E}_{\mathcal{I}}}$ along the Yoneda embedding $\mathcal{A}_{\mathcal{I}} \rightarrow \mathrm{Mack}_{\mathcal{I}}$ is an equivalence of ∞ -categories. In Section 2.5, we show a way to see the $(2, 1)$ -category of $\mathcal{I}$ -normed symmetric monoidal groupoids as a full subcategory of $\mathbf{Alg}_{\mathcal{E}_{\mathcal{I}}}$.

2.2 Preliminaries on N_∞ -operads

2.2.1 N_∞ -operads and G -indexing systems

In this subsection, we recall Blumberg and Hill's notions of N_∞ -operads in the setting of G -simplicial sets, of G -indexing systems, and the relation between them. We refer the reader to [BH15] or [Rub21] for more details on the subject.

Definition 2.2.1. Let $n \geq 0$. A *graph subgroup* of $G \times \Sigma_n$ is a subgroup $\Gamma \subseteq G \times \Sigma_n$ that intersects $\{1_G\} \times \Sigma_n$ trivially.

We recall the definition of an N_∞ -operad in G -simplicial sets.

Definition 2.2.2 ([Rub21, Definition 2.6]). An N_∞ -operad is an operad in the category $\mathbf{SSet}^G$ such that:

1. for every $n \geq 0$, the action of Σ_n on $\mathcal{O}_n$ is free;
2. for every graph subgroup $\Gamma \subseteq G \times \Sigma_n$, the simplicial set $(\mathcal{O}_n)^\Gamma$ is either contractible or empty;
3. the simplicial sets $(\mathcal{O}_2)^G$ and $(\mathcal{O}_0)^G$ are non-empty.

Recall now that graph subgroups can be characterised in terms of finite equivariant sets over subgroups of G .

Lemma 2.2.3 ([Rub21, Lemma 2.4]). *For every graph subgroup $\Gamma \subseteq G \times \Sigma_n$, there is a unique subgroup $H \subseteq G$ and a homomorphism $\sigma : H \rightarrow \Sigma_n$ such that $\Gamma = \{(h, \sigma(h)) \mid h \in H\}$.*

For every finite H -set T , we choose once and for all an ordering $T \simeq \{1, \dots, |T|\}$. The previous lemma gives a correspondence between finite H -sets and graph subgroups. If $H \subseteq G$ is a subgroup of G and T a finite H -set, we denote by $\Gamma_T \subseteq G \times \Sigma_{|T|}$ the graph subgroup associated with the homomorphism $\sigma : H \rightarrow \Sigma_{|T|}$ that corresponds to T through the chosen ordering on T . If $\mathcal{O}$ is an N_∞ -operad, then we consider, for every subgroup $H \subseteq G$, the following collection

$$\mathcal{I}_{\mathcal{O}}(H) = \left\{ T \in \mathbf{Fin}^H \mid (\mathcal{O}_{|T|})^{\Gamma_T} \neq \emptyset \right\}.$$

As proven in [BH15, Theorem 1.2], these collections of equivariant finite sets form a G -indexing system in the sense of the following definition.

Definition 2.2.4 ([Rub21, Definition 2.12]). A G -indexing system $\mathcal{I}$ consists, for every subgroup $H \subseteq G$, of a class $\mathcal{I}(H)$ of finite H -sets satisfying the following conditions:

1. for every subgroup $H \subseteq G$, the class $\mathcal{I}(H)$ contains all trivial H -sets;
2. for every subgroup $H \subseteq G$ and finite H -sets A and B such that B is isomorphic to A in $\mathbf{Fin}^H$, if $B \in \mathcal{I}(H)$, then $A \in \mathcal{I}(H)$;
3. for every sequence of subgroups $K \subseteq H \subseteq G$ and finite H -set A , if $A \in \mathcal{I}(H)$, then $\mathrm{Res}_K^H A \in \mathcal{I}(K)$;

4. for every subgroup $H \subseteq G$, $g \in G$ and finite H -set A , if $A \in \mathcal{I}(H)$, then $c_g A \in \mathcal{I}(H^g)$;
5. for every subgroup $H \subseteq G$ and finite H -sets A and B , if B is an H -equivariant subset of A and $A \in \mathcal{I}(H)$, then $B \in \mathcal{I}(H)$;
6. for every subgroup $H \subseteq G$ and finite H -sets A and B , if $A \in \mathcal{I}(H)$ and $B \in \mathcal{I}(H)$, then $A \sqcup B \in \mathcal{I}(H)$;
7. for every sequence of subgroups $K \subseteq H \subseteq G$ and finite K -set A , if $A \in \mathcal{I}(K)$ and $H/K \in \mathcal{I}(H)$, then $\text{Ind}_K^H A \in \mathcal{I}(H)$.

Given a G -indexing system $\mathcal{I}$, we say that a finite H -set T is *admissible* if it belongs to $\mathcal{I}(H)$. The G -indexing systems form a poset under the inclusion that we denote by I_G .

The N_∞ -operads for G form an ∞ -category that can be defined as follows.

Definition 2.2.5 ([BH15, Definition 3.9]). A morphism $f : \mathcal{O} \rightarrow \mathcal{P}$ of operads in SSet^G is a *graph weak equivalence of G -operads* if for every graph subgroup $\Gamma \subseteq G \times \Sigma_n$, the morphism $f^\Gamma : \mathcal{O}_n^\Gamma \rightarrow \mathcal{P}_n^\Gamma$ is a weak homotopy equivalence of simplicial sets. We denote by $\mathbf{Op}^G$ the ∞ -category obtained from $\text{Op}(\text{SSet}^G)$ by inverting graph weak equivalences and respectively by $N_\infty\text{-Op}$ and $N_\infty\text{-Op}^G$ the full subcategories of $\text{Op}(\text{SSet}^G)$ and $\mathbf{Op}^G$ spanned by N_∞ -operads.

The ∞ -category of N_∞ -operads is equivalent to the poset of G -indexing systems.

Theorem 2.2.6. *The functor $N_\infty\text{-Op} \rightarrow I_G$ that sends an N_∞ -operad $\mathcal{O}$ to its associated G -indexing system $\mathcal{I}_\mathcal{O}$ sends graph weakly equivalent N_∞ -operads to the same G -indexing system. Moreover, the induced functor $N_\infty\text{-Op}^G \rightarrow I_G$ is an equivalence of ∞ -categories.*

Proof. Combining [BH15, Theorem 1.2] and [BH15, Proposition 5.5], one deduces that the functor $N_\infty\text{-Op} \rightarrow I_G$ sends graph weakly equivalent N_∞ -operads to the same G -indexing system and that the induced functor $N_\infty\text{-Op}^G \rightarrow I_G$ is fully faithful. The fact that this functor is essentially surjective follows independently from [Rub21, Theorem 7.2] and [BP21, Corollary IV]. $\square$

2.2.2 The ∞ -categories of $\mathcal{O}$ -algebras for an N_∞ -operad $\mathcal{O}$

Let $\mathcal{O}$ be an N_∞ -operad. In this subsection, we define the ∞ -category $\mathbf{Alg}_\mathcal{O}$ of $\mathcal{O}$ -algebras. We recall first the definition of G -weak homotopy equivalences and of the ∞ -category of G -equivariant spaces.

Definition 2.2.7. A morphism $f : X \rightarrow Y$ in SSet^G is a *G -weak homotopy equivalence* if, for every subgroup $H \subseteq G$, the morphism $f^H : X^H \rightarrow Y^H$ is a weak homotopy equivalence in SSet . The *∞ -category of G -equivariant spaces* $\mathcal{S}^G$ is the ∞ -category obtained from SSet^G by inverting G -weak homotopy equivalences.

We now define the ∞ -category of $\mathcal{O}$ -algebras.

Definition 2.2.8. A morphism $f : X \rightarrow Y$ in $\text{Alg}_{\mathcal{O}}(\text{SSet}^G)$ is a *weak equivalence of $\mathcal{O}$ -algebras* if $U(f) : U(X) \rightarrow U(Y)$ is a G -weak homotopy equivalence with $U : \text{Alg}_{\mathcal{O}}(\text{SSet}^G) \rightarrow \text{SSet}^G$ the forgetful functor. The ∞ -category of $\mathcal{O}$ -algebras $\mathbf{Alg}_{\mathcal{O}}$ is the ∞ -category obtained from $\text{Alg}_{\mathcal{O}}(\text{SSet}^G)$ by inverting weak equivalences of $\mathcal{O}$ -algebras.

The ∞ -category $\mathbf{Alg}_{\mathcal{O}}$ admits the following useful description.

Proposition 2.2.9. *The category $\text{Alg}_{\mathcal{O}}(\text{SSet}^G)$ admits a combinatorial and simplicial model structure right-induced from the genuine model structure on SSet^G through the free-forgetful adjunction*

$$F : \text{SSet}^G \rightleftarrows \text{Alg}_{\mathcal{O}}(\text{SSet}^G) : U.$$

Moreover, the ∞ -category $\mathbf{Alg}_{\mathcal{O}}$ is equivalent to the homotopy coherent nerve of the simplicial subcategory of $\text{Alg}_{\mathcal{O}}(\text{SSet}^G)$ spanned by objects that are fibrant and cofibrant.

Proof. The category SSet^G is a monoidal model category with a cofibrant unit and with a symmetric monoidal fibrant replacement functor given by Ex^∞ . Moreover, the monoidal structure is given by the cartesian product and it follows that the classical interval object Δ^1 is endowed with a unique structure of cocommutative coalgebra. It follows from [BM07, Theorem 2.1] that the right-induced model structure on $\text{Alg}_{\mathcal{O}}(\text{SSet}^G)$ exists. If X is an $\mathcal{O}$ -algebra and if K is a simplicial set, then the power X^K in SSet^G is naturally endowed with the structure of an $\mathcal{O}$ -algebra and this construction induces a structure of simplicial model category on $\text{Alg}_{\mathcal{O}}(\text{SSet}^G)$. Finally, the second part of the proposition follows from [Lur17, Theorem 1.3.4.20]. $\square$

Remark 2.2.10. In this thesis, we chose to work with G -simplicial sets instead of G -topological spaces. There is a monoidal Quillen equivalence

$$|-| : \text{SSet}^G \rightleftarrows \text{Top}^G : \text{Sing}$$

between SSet^G and Top^G respectively, endowed with the genuine model structures. The functor $|-|$ preserves finite limits and it follows that the operad $|\mathcal{O}|$ is an N_∞ -operad in Top^G with the same G -indexing system as $\mathcal{O}$. Moreover, the functor $|-| : \text{SSet}^G \rightarrow \text{Top}^G$ preserves weak equivalences and it follows from [BM03, Theorem 4.7] that the ∞ -categories $\mathbf{Alg}_{\mathcal{O}}$ and $\mathbf{Alg}_{|\mathcal{O}|}$ are equivalent.

Let $\mathcal{I}$ be a G -indexing system. Different N_∞ -operads that correspond to $\mathcal{I}$ should produce equivalent ∞ -categories, and one can make the following conjecture.

Conjecture 2.2.11. *Let $\mathcal{O}$ and $\mathcal{P}$ be two N_∞ -operads associated with $\mathcal{I}$. The ∞ -categories $\mathbf{Alg}_{\mathcal{O}}$ and $\mathbf{Alg}_{\mathcal{P}}$ are equivalent.*

Note however that N_∞ -operads are not Σ -cofibrant, as pointed out in Chapter 3. In particular, many results of [BM03] do not apply to N_∞ -operads and, in particular, the proof of [BM03, Theorem 4.4] does not directly generalise. In this chapter, we will work with Rubin's combinatorial N_∞ -operad $\mathcal{E}_\mathcal{I}$ as a model, and we will recall its definition in Subsection 2.3.2. For the rest of the chapter, we will regard $\mathbf{Alg}_{\mathcal{E}_\mathcal{I}}$ as the ∞ -category of N_∞ -algebras for $\mathcal{I}$. Note however that, as long as Conjecture 2.2.11 is not proven, the results of this chapter only apply to algebras over $\mathcal{E}_\mathcal{I}$.

2.3 Normed symmetric monoidal categories

In this section, we recall Rubin's notion of $\mathcal{I}$ -normed symmetric monoidal categories, and we give an explicit description of the operad $\mathcal{SM}_\mathcal{I}$ in $\mathbf{Cat}^G$ whose algebras in $\mathbf{Cat}^G$ are $\mathcal{I}$ -normed symmetric monoidal categories. The operad $\mathcal{SM}_\mathcal{I}$ lives in the category of G -groupoids and its nerve $\mathcal{E}_\mathcal{I} := N(\mathcal{SM}_\mathcal{I})$ is an N_∞ -operad with associated G -indexing system $\mathcal{I}$. The description that we give of $\mathcal{SM}_\mathcal{I}$ is equivalent, but slightly different from the one given by Rubin in [Rub21] and [Rub20]. Our description will be useful to give an explicit construction of the free $\mathcal{E}_\mathcal{I}$ -algebras over finite G -sets in terms of $\mathcal{I}$ -normed symmetric monoidal groupoids, which is a part of the construction of the equivalence of Theorem 2.A.

2.3.1 Preliminaries on $\mathcal{I}$ -normed symmetric monoidal categories

In this subsection, we recall Rubin's notion of $\mathcal{I}$ -normed symmetric monoidal categories. We refer the reader to [Rub20] and [Rub18] for more details on the topic.

Definition 2.3.1. Let $H \subseteq G$ be a subgroup, T a finite H -set and $\mathcal{C}$ a G -category. Denote by $\mathcal{C}^{\times T}$ the H -category given by the $|T|$ -fold cartesian power $\mathcal{C}^{|T|}$ endowed with the H -action given on objects by

$$h \cdot (X_1, \dots, X_{|T|}) := (hX_{\sigma(h)^{-1}1}, \dots, hX_{\sigma(h)^{-1}|T|})$$

and by the same formula on morphisms, where $\sigma : H \rightarrow \Sigma_{|T|}$ is the homomorphism that corresponds to the action on T through the bijection $T \simeq \{1, \dots, |T|\}$ given by the chosen ordering on T . A T -external norm is an H -functor $\mathcal{C}^{\times T} \rightarrow \mathcal{C}$.

Definition 2.3.2 ([Rub20, Definition 2.3]). An $\mathcal{I}$ -normed symmetric monoidal category $\mathcal{C}$ is a G -symmetric monoidal category $(\mathcal{C}, \otimes, e, \alpha, \lambda, \rho, \beta)$ together with:

1. a T -external norm $\bigotimes_T : \mathcal{C}^{\times T} \rightarrow \mathcal{C}$ for every subgroup $H \subseteq G$ and every admissible H -set T ;
2. (untwistors) a nonequivariant natural isomorphism

$$v_T : \bigotimes_T (X_1, \dots, X_{|T|}) \rightarrow \bigotimes_{|T|} (X_1, \dots, X_{|T|})$$

for every admissible H -set T such that for every $h \in H$, the diagram

$$\begin{array}{ccc}
 h \otimes_T (X_1, \dots, X_{|T|}) & \xrightarrow{\text{id}} & \otimes_T (hX_{\sigma(h)^{-1}1}, \dots, hX_{\sigma(h)^{-1}|T|}) \\
 \downarrow hv_T & & \downarrow v_T \\
 & & \otimes_{|T|} (hX_{\sigma(h)^{-1}1}, \dots, hX_{\sigma(h)^{-1}|T|}) \\
 & & \downarrow \sigma(h)^{-1} \\
 h \otimes_{|T|} (X_1, \dots, X_{|T|}) & \xrightarrow{\text{id}} & \otimes_{|T|} (hX_1, \dots, hX_{|T|})
 \end{array}$$

commutes where $\otimes_{|T|} (X_1, \dots, X_{|T|})$ denotes the $|T|$ -fold tensor product and where the morphism $\sigma(h)^{-1}$ denotes the canonical isomorphism given by the symmetric monoidal structure of $\mathcal{C}$ that permutes the factors of $\otimes_{|T|}$ by $\sigma(h)^{-1}$.

Definition 2.3.3 ([Rub20, Definition 2.5]). Let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathcal{I}$ -normed symmetric monoidal categories. A *lax $\mathcal{I}$ -normed functor* $\phi : \mathcal{C} \rightarrow \mathcal{D}$ consists of the following data:

1. a G -functor $\phi : \mathcal{C} \rightarrow \mathcal{D}$;
2. a G -fixed morphism $\phi_e : e^{\mathcal{D}} \rightarrow \phi e^{\mathcal{C}}$;
3. a G -natural transformation $\phi_{\otimes} : \phi X \otimes^{\mathcal{D}} \phi Y \rightarrow \phi (X \otimes^{\mathcal{C}} Y)$;
4. for every admissible H -set T , an H -natural transformation

$$\phi_{\otimes_T} : \otimes_T^{\mathcal{D}} (\phi X_1, \dots, \phi X_{|T|}) \rightarrow \phi \left(\otimes_T^{\mathcal{C}} (X_1, \dots, X_{|T|}) \right)$$

such that the usual lax symmetric monoidal diagrams relating $\alpha, \lambda, \rho, \beta$ to the maps ϕ_e and $\phi_{\otimes}$ commute and such that the square

$$\begin{array}{ccc}
 \otimes_T^{\mathcal{D}} (\phi X_1, \dots, \phi X_{|T|}) & \xrightarrow{v_T^{\mathcal{D}}} & \otimes_{|T|}^{\mathcal{D}} (\phi X_1, \dots, \phi X_{|T|}) \\
 \downarrow \phi_{\otimes_T} & & \downarrow \phi_{\otimes_{|T|}} \\
 \phi \left(\otimes_T^{\mathcal{C}} (X_1, \dots, X_{|T|}) \right) & \xrightarrow{\phi(v_T^{\mathcal{C}})} & \phi \left(\otimes_{|T|}^{\mathcal{C}} (X_1, \dots, X_{|T|}) \right)
 \end{array}$$

commutes for every admissible H -set T where $\phi_{\otimes_{|T|}}$ is the map constructed with iterations of $\phi_{\otimes}$ and ϕ_e . A lax $\mathcal{I}$ -normed functor is a *strong $\mathcal{I}$ -normed functor* (respectively *strict $\mathcal{I}$ -normed functor*) if the natural transformations $\phi_e, \phi_{\otimes}$ and $\phi_{\otimes_T}$ are isomorphisms (respectively identities).

If $\phi : \mathcal{C} \rightarrow \mathcal{D}$ and $\psi : \mathcal{D} \rightarrow \mathcal{E}$ are lax $\mathcal{I}$ -normed functors, then their composition $\psi \circ \phi$ is also canonically a lax $\mathcal{I}$ -normed functor. The morphism $(\psi \circ \phi)_e$ is given by the composition

$$e_{\mathcal{E}} \xrightarrow{\psi_e} \psi e_{\mathcal{D}} \xrightarrow{\psi(\phi_e)} (\psi \circ \phi)_e e_{\mathcal{C}},$$

the morphism $(\psi \circ \phi)_{\otimes}$ by the composition

$$((\psi \circ \phi)X) \otimes^{\mathcal{E}} ((\psi \circ \phi)Y) \xrightarrow{\psi_{\otimes}} \psi(\phi X \otimes^{\mathcal{D}} \phi Y) \xrightarrow{\psi(\phi_{\otimes})} (\psi \circ \phi)(X \otimes^{\mathcal{C}} Y)$$

and, for every admissible finite H -set T , the morphism $(\psi \circ \phi)_{\otimes_T}$ is given by the composition of the morphism

$$\bigotimes_T ((\psi \circ \phi)X_1, \dots, (\psi \circ \phi)X_{|T|}) \xrightarrow{\psi_{\otimes_T}} \psi\left(\bigotimes_T (\phi X_1, \dots, \phi X_{|T|})\right)$$

with the morphism

$$\psi\left(\bigotimes_T (\phi X_1, \dots, \phi X_{|T|})\right) \xrightarrow{\psi(\phi_{\otimes_T})} (\psi \circ \phi)\left(\bigotimes_T (X_1, \dots, X_{|T|})\right).$$

If ϕ and ψ are strong $\mathcal{I}$ -normed functors (resp. strict), then $\psi \circ \phi$ is also a strong $\mathcal{I}$ -normed functor (resp. strict).

Definition 2.3.4 ([Rub20, Definition 2.6]). Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{I}$ -normed symmetric monoidal categories and $\phi, \psi : \mathcal{C} \rightrightarrows \mathcal{D}$ a pair of lax $\mathcal{I}$ -normed functors. An $\mathcal{I}$ -normed natural transformation $\omega : \phi \Rightarrow \psi$ is a G -natural transformation $\omega : \phi \Rightarrow \psi$ such that the usual monoidal transformation squares relating ϕ_e , $\phi_{\otimes}$ and $\psi_{\otimes}$ to ω commute and such that the square

$$\begin{array}{ccc} \bigotimes_T^{\mathcal{D}} (\phi X_1, \dots, \phi X_{|T|}) & \xrightarrow{\bigotimes_T (\omega, \dots, \omega)} & \bigotimes_T^{\mathcal{D}} (\psi X_1, \dots, \psi X_{|T|}) \\ \phi_{\otimes_T} \downarrow & & \downarrow \psi_{\otimes_T} \\ \phi\left(\bigotimes_T^{\mathcal{C}} (X_1, \dots, X_{|T|})\right) & \xrightarrow{\omega} & \psi\left(\bigotimes_T^{\mathcal{C}} (X_1, \dots, X_{|T|})\right) \end{array}$$

commutes for every admissible H -set T .

Notation 2.3.5. The structure of a 2-category on Cat^G lifts to $\mathcal{I}$ -normed symmetric monoidal categories and we denote by $\mathcal{ISMLax}$ (resp. $\mathcal{ISMStg}$ and $\mathcal{ISMSt}$) the 2-category of $\mathcal{I}$ -normed symmetric monoidal categories, lax $\mathcal{I}$ -normed functors (resp. strong and strict) and $\mathcal{I}$ -normed natural transformations.

The category of strong $\mathcal{I}$ -normed functors $\mathcal{ISMStg}(\mathcal{C}, \mathcal{D})$ between two $\mathcal{I}$ -normed symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is naturally endowed with a structure of a symmetric monoidal category.

Proposition 2.3.6. *Let $\phi, \psi : \mathcal{C} \rightrightarrows \mathcal{D}$ be two strong $\mathcal{I}$ -normed functors. The composition*

$$\phi \otimes \psi : \mathcal{C} \xrightarrow{\Delta} \mathcal{C} \times \mathcal{C} \xrightarrow{\phi \times \psi} \mathcal{D} \times \mathcal{D} \xrightarrow{\otimes_{\mathcal{D}}} \mathcal{D}$$

is naturally endowed with the structure of a strong $\mathcal{I}$ -normed functor. Moreover, if $\tau_1 : \phi_1 \Rightarrow \phi_2$ and $\tau_2 : \psi_1 \Rightarrow \psi_2$ are two $\mathcal{I}$ -normed natural transformations between strong $\mathcal{I}$ -normed functors, the natural transformation $\tau_1 \otimes \tau_2$ defined as $\bigotimes_{\mathcal{D}} \circ (\tau_1 \times \tau_2) \circ \Delta$ is an $\mathcal{I}$ -normed natural transformation between $\phi_1 \otimes \psi_1$ and $\phi_2 \otimes \psi_2$. Given two $\mathcal{I}$ -normed symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$, these constructions endow $\mathcal{ISMStg}(\mathcal{C}, \mathcal{D})$ with the structure of a symmetric monoidal category.

Proof. The G -fixed isomorphism $(\phi \otimes \psi)_e$ is the composition

$$e_{\mathcal{D}} \simeq e_{\mathcal{D}} \otimes e_{\mathcal{D}} \xrightarrow{\phi_e \otimes \psi_e} \phi(e_{\mathcal{C}}) \otimes \psi(e_{\mathcal{C}}),$$

the G -natural transformation $(\phi \otimes \psi)_{\otimes}$ is given by the composition of the morphism

$$(\phi(X) \otimes \psi(Y)) \otimes (\phi(Y) \otimes \psi(Y)) \simeq (\phi(X) \otimes \phi(Y)) \otimes (\psi(X) \otimes \psi(Y))$$

with the morphism

$$(\phi(X) \otimes \phi(Y)) \otimes (\psi(X) \otimes \psi(Y)) \xrightarrow{(\phi_{\otimes}) \otimes (\psi_{\otimes})} \phi(X \otimes Y) \otimes \psi(X \otimes Y)$$

and, for every admissible H -set T , the H -natural transformation $(\phi \otimes \psi)_{\otimes_T}$ is given by the composition

$$\begin{aligned} & \bigotimes_T (\phi(X_1) \otimes \psi(X_1), \dots, \phi(X_{|T|}) \otimes \psi(X_{|T|})) \\ & \xrightarrow{\alpha} \left(\bigotimes_T (\phi(X_1), \dots, \phi(X_{|T|})) \right) \otimes \left(\bigotimes_T (\psi(X_1), \dots, \psi(X_{|T|})) \right) \\ & \xrightarrow{(\phi_{\otimes_T}) \otimes (\psi_{\otimes_T})} \phi \left(\bigotimes_T (X_1, \dots, X_{|T|}) \right) \otimes \psi \left(\bigotimes_T (X_1, \dots, X_{|T|}) \right) \end{aligned}$$

where α is the H -natural transformation given by the composition

$$\begin{aligned} & \bigotimes_T (\phi(X_1) \otimes \psi(X_1), \dots, \phi(X_{|T|}) \otimes \psi(X_{|T|})) \\ & \xrightarrow{v_T} \bigotimes_{|T|} (\phi(X_1) \otimes \psi(X_1), \dots, \phi(X_{|T|}) \otimes \psi(X_{|T|})) \\ & \simeq \left(\bigotimes_{|T|} (\phi(X_1), \dots, \phi(X_{|T|})) \right) \otimes \left(\bigotimes_{|T|} (\psi(X_1), \dots, \psi(X_{|T|})) \right) \\ & \xrightarrow{v_T^{-1} \otimes v_T^{-1}} \left(\bigotimes_T (\phi(X_1), \dots, \phi(X_{|T|})) \right) \otimes \left(\bigotimes_T (\psi(X_1), \dots, \psi(X_{|T|})) \right). \end{aligned}$$

A direct inspection shows that $\phi \otimes \psi$ is a strong $\mathcal{I}$ -normed functor and that the result also stands on $\mathcal{I}$ -normed natural transformations. The structural morphisms of the symmetric monoidal category $\mathcal{ISMStg}(\mathcal{C}, \mathcal{D})$ are given componentwise by the ones of $\mathcal{D}$. $\square$

Remark 2.3.7. Note that if $\phi, \psi : \mathcal{C} \rightrightarrows \mathcal{D}$ is a pair of strict $\mathcal{I}$ -normed functors, then the strong $\mathcal{I}$ -normed functor $\phi \otimes \psi$ is not strict in general. In particular, the previous proposition fails for strict $\mathcal{I}$ -normed functors.

It follows from Proposition 2.3.6 that the 2-category $\mathcal{ISMStg}$ is a **SMC**-category in the sense of [Gui10, Definition 2.1]. Moreover, $\mathcal{ISMStg}$ admits finite products and is, in particular, a semi-additive 2-category.

Proposition 2.3.8. *The 2-category $\mathcal{ISMStg}$ is semi-additive, ie, $\mathcal{ISMStg}$ admits direct sums.*

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathcal{I}$ -normed symmetric monoidal categories. The product $\mathcal{C} \times \mathcal{D}$ is canonically endowed with a structure of $\mathcal{I}$ -normed symmetric monoidal category which is the cartesian product of $\mathcal{C}$ and $\mathcal{D}$ in the 2-category $\mathcal{ISMStg}$. The final G -category $*$ is also canonically an $\mathcal{I}$ -normed symmetric monoidal category and is a zero object in the 2-category $\mathcal{ISMStg}$. If we denote by $0_{\mathcal{C}, \mathcal{D}} : \mathcal{C} \rightarrow \mathcal{D}$ the strong $\mathcal{I}$ -normed functor constant on $e_{\mathcal{D}}$, then we have to show that $\mathcal{C} \times \mathcal{D}$ endowed with the strong $\mathcal{I}$ -normed functors $i_{\mathcal{C}} = (\text{id}_{\mathcal{C}}, 0_{\mathcal{C}, \mathcal{D}}) : \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{D}$ and $i_{\mathcal{D}} = (0_{\mathcal{D}, \mathcal{C}}, \text{id}_{\mathcal{D}}) : \mathcal{D} \rightarrow \mathcal{C} \times \mathcal{D}$ is the coproduct of $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{ISMStg}$. More precisely, if we denote by $p_{\mathcal{C}} : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C}$ and $p_{\mathcal{D}} : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$ the projections, then we prove that the functor

$$((-) \circ p_{\mathcal{C}}) \otimes ((-) \circ p_{\mathcal{D}}) : \mathcal{ISMStg}(\mathcal{C}, \mathcal{E}) \times \mathcal{ISMStg}(\mathcal{D}, \mathcal{E}) \rightarrow \mathcal{ISMStg}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

that sends a pair of strong $\mathcal{I}$ -normed functors $\phi : \mathcal{C} \rightarrow \mathcal{E}$ and $\psi : \mathcal{D} \rightarrow \mathcal{E}$ to $(\phi \circ p_{\mathcal{C}}) \otimes (\psi \circ p_{\mathcal{D}})$ is an inverse of the functor

$$((-) \circ i_{\mathcal{C}}, (-) \circ i_{\mathcal{D}}) : \mathcal{ISMStg}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \rightarrow \mathcal{ISMStg}(\mathcal{C}, \mathcal{E}) \times \mathcal{ISMStg}(\mathcal{D}, \mathcal{E}).$$

Given a pair of strong $\mathcal{I}$ -normed functors $\phi : \mathcal{C} \rightarrow \mathcal{E}$ and $\psi : \mathcal{D} \rightarrow \mathcal{E}$, we have $((\phi \circ p_{\mathcal{C}}) \otimes (\psi \circ p_{\mathcal{D}})) \circ i_{\mathcal{C}} = \phi \otimes 0_{\mathcal{C}, \mathcal{E}} \simeq \phi$ and $((\phi \circ p_{\mathcal{C}}) \otimes (\psi \circ p_{\mathcal{D}})) \circ i_{\mathcal{D}} = 0_{\mathcal{D}, \mathcal{E}} \otimes \psi \simeq \psi$ and if $\zeta : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ is a strong $\mathcal{I}$ -normed functor, then $(\zeta \circ i_{\mathcal{C}} \circ p_{\mathcal{C}}) \otimes (\zeta \circ i_{\mathcal{D}} \circ p_{\mathcal{D}})$ is the strong $\mathcal{I}$ -normed functor that sends an object (X, Y) of $\mathcal{C} \times \mathcal{D}$ to $\zeta(X, e_{\mathcal{D}}) \otimes \zeta(e_{\mathcal{C}}, Y) \stackrel{\zeta_{\otimes}}{\simeq} \zeta(X \otimes e_{\mathcal{C}}, Y \otimes e_{\mathcal{D}}) \simeq \zeta(X, Y)$ and the result follows. $\square$

2.3.2 Rubin's combinatorial models of N_∞ -operads

In this subsection, we give an explicit description of the operad $\mathcal{SM}_{\mathcal{I}}$ in Cat^G defined in [Rub21] and [Rub20] whose algebras in Cat^G are $\mathcal{I}$ -normed symmetric monoidal categories. We then use this description to give a short proof of the fact that the operad $\mathcal{E}_{\mathcal{I}} = N(\mathcal{SM}_{\mathcal{I}})$ is an N_∞ -operad associated with our chosen G -indexing system $\mathcal{I}$.

Construction 2.3.9. If X is a G -set, then the *translation G -groupoid* $\widetilde{X}$ of X is the G -category defined by the following diagram

$$(X \times X) \times_X (X \times X) \simeq X \times X \times X \xrightarrow{(p_1, p_3)} X \times X \begin{array}{c} \xrightarrow{p_1} \\ \xleftarrow{\Delta} \\ \xrightarrow{p_2} \end{array} X.$$

The associated functor $\widetilde{(-)} : \text{Set}^G \rightarrow \text{Cat}^G$ is a right adjoint of the object functor $\text{Ob} : \text{Cat}^G \rightarrow \text{Set}^G$ and in particular, preserves limits. It follows that if $\mathcal{O}$ is an operad in Set^G , then $\widetilde{\mathcal{O}}$ is canonically an operad in Cat^G .

Definition 2.3.10. The operad $\mathcal{SM}_{\mathcal{I}}$ is the operad $\widetilde{F(S_{\mathcal{I}})}$ with $F(S_{\mathcal{I}})$ the free operad in the category Set^G over the symmetric sequence $S_{\mathcal{I}}$ defined by

$$(S_{\mathcal{I}})_n = \bigsqcup_{T \in \mathcal{I}, |T|=n} \frac{G \times \Sigma_n}{\Gamma_T}.$$

We will now give an explicit description of $F(S_{\mathcal{I}})$. The construction of the free operad over a symmetric sequence in Set^G can be made explicit in general, as in [Rub21, Section 5] for instance, though its general description can be complicated to handle. However, if we denote by $V_{\mathcal{I}}$ the non-symmetric sequence of G -sets defined by

$$(V_{\mathcal{I}})_n = \bigsqcup_{\substack{H \subseteq G, |T|=n, \\ T \in \mathcal{I}(H)}} G/H,$$

the choice for every subgroup $H \subseteq G$ of a set of coset representatives of G/H induces a nonequivariant isomorphism of symmetric sequences $S_{\mathcal{I}} \simeq V_{\mathcal{I}} \times \Sigma_*$ where Σ_* denotes the symmetric sequence of sets given by the symmetric groups. If we forget the action of G , then it follows that the operad $F(S_{\mathcal{I}})$ can be described as the product $F(V_{\mathcal{I}}) \times \Sigma_*$ with $F(V_{\mathcal{I}})$ the free nonsymmetric operad over the sequence $V_{\mathcal{I}}$. We recall now the construction of the nonsymmetric operad $F(V_{\mathcal{I}})$ in Set in terms of planar rooted trees and we then give an explicit description of the required G -action on $F(S_{\mathcal{I}})$.

Remark 2.3.11. The freeness of the symmetric sequence of sets $S_{\mathcal{I}}$ was already used in [Rub21, Theorem 5.21] to give an explicit description of $F(S_{\mathcal{I}})$. However, [Rub21, Theorem 5.21] does not construct $F(S_{\mathcal{I}})$ as the product $F(V_{\mathcal{I}}) \times \Sigma_*$. We use this presentation to construct explicitly the free N_{∞} -algebras over finite G -sets in Subsection 2.4.3.

Construction 2.3.12. For every subgroup $H \subseteq G$, choose a set of coset representatives $\{1_G = g_1, \dots, g_{[G:H]}\}$ of G/H . Consider the formal symbols $r \otimes_T$ and id with r a G/H coset representative and T an admissible H -set. A *formal external norm* is a formal composite of the previous symbols defined by recursion as follows:

1. the symbol id is a formal external norm;
2. if $\theta_1, \dots, \theta_{|T|}$ are formal external norms, then $r \otimes_T (\theta_1, \dots, \theta_{|T|})$ is a formal external norm.

If θ is a formal external norm, then its *length* $|\theta|$ is defined by the following recursion:

1. if $\theta = \text{id}$, then $|\theta| = 1$;

2. if $\theta = r \otimes_T (\theta_1, \dots, \theta_{|T|})$, then $|\theta| = |\theta_1| + \dots + |\theta_{|T|}|$.

The nonsymmetric operad $F(V_{\mathcal{I}})$ in Set can now be described as the operad whose set of n -ary operations $F(V_{\mathcal{I}})_n$ is the one of formal external norms of length n . The identity of $F(V_{\mathcal{I}})$ is id and if θ and $\tau_1, \dots, \tau_n$ are formal external norms with $|\theta| = n$, their composition $\gamma(\theta, \tau_1, \dots, \tau_n)$ is defined by recursion as follows:

1. if $\theta = \text{id}$, then $\gamma(\theta, \tau_1) = \tau_1$;
2. if $\theta = r \otimes_T (\theta_1, \dots, \theta_k)$, then

$$\gamma(\theta, \tau_1, \dots, \tau_n) = r \otimes_T (\gamma(\theta_1, \tau_1, \dots, \tau_{|\theta_1|}), \dots, \gamma(\theta_k, \tau_{|\theta_1|+|\theta_2|+ \dots + |\theta_{k-1}|}, \dots, \tau_n)).$$

There is an action of G on formal external norms defined by recursion as follows:

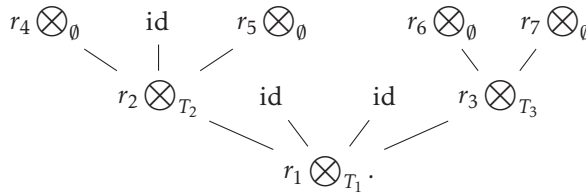
1. if $\theta = \text{id}$, then $g \cdot \theta = \theta$;
2. if $\theta = r \otimes_T (\theta_1, \dots, \theta_{|T|})$, then $g \cdot \theta = r' \otimes_T (h \cdot \theta_{\sigma(h)^{-1}1}, \dots, h \cdot \theta_{\sigma(h)^{-1}|T|})$ with r' the chosen coset representative of gr and h the unique element of H such that $gr = r'h$ and where $\sigma : H \rightarrow \Sigma_{|T|}$ is the homomorphism that corresponds to T through the chosen ordering $T \simeq \{1, \dots, |T|\}$.

The G -action on formal external norms is compatible with the operadic structure of $F(V_{\mathcal{I}})$ and it follows that $F(V_{\mathcal{I}})$ is canonically an operad in Set^G .

Remark 2.3.13. As we already mentioned, $F(V_{\mathcal{I}})$ is the free nonsymmetric operad in Set over the sequence

$$(V_{\mathcal{I}})_n = \bigsqcup_{\substack{H \subseteq G, |T|=n, \\ T \in \mathcal{I}(H)}} G/H,$$

and it follows that a formal external norm can be seen as a planar rooted tree with n -ary nodes decorated with the symbols $r \otimes_T$ with T an admissible H -set of cardinality n , r a chosen coset representative of G/H , where leaves represent occurrences of the identity id . The figure below shows an example



Note that for the construction of $F(V_{\mathcal{I}})$, the choices for every subgroup H of G , of orderings on admissible H -sets, and of coset representatives of G/H , play only a role in the definition of the G -action. The choices of coset representatives of G/H correspond to the necessary choices to construct the nonequivariant isomorphism $S_{\mathcal{I}} \simeq V_{\mathcal{I}} \times \Sigma_*$ with Σ_* the symmetric sequence associated with the symmetric groups. Moreover, note that $F(V_{\mathcal{I}})$ is the free operad over $V_{\mathcal{I}}$ only in the category Set and not as an operad in Set^G .

Notation 2.3.14. If $\delta_1 \in \Sigma_{k_1}, \dots, \delta_n \in \Sigma_{k_n}$ and $\delta \in \Sigma_n$ are permutations, then we denote by $\delta\langle\delta_1, \dots, \delta_n\rangle$ the permutation $\delta(k_1, \dots, k_n) \circ \bigsqcup_{i=1}^n \delta_i \in \Sigma_{k_1 + \dots + k_n}$ with $\delta(k_1, \dots, k_n)$ the block permutation.

We can now give the promised explicit description of the G -action on the operad $F(S_{\mathcal{I}})$.

Construction 2.3.15. Let θ be a formal external norm and g an element of G . We define a permutation $\omega_\theta(g)$ in $\Sigma_{|\theta|}$ by recursion as follows:

1. if $\theta = \text{id}$, then $\omega_\theta(g) = \text{id}_{\Sigma_1}$;
2. if $\theta = r \otimes_T (\theta_1, \dots, \theta_{|T|})$, then $\omega_\theta(g) = \sigma(h)\langle\omega_{\theta_1}(h), \dots, \omega_{\theta_{|T|}}(h)\rangle$ with h and $\sigma : H \rightarrow \Sigma_{|T|}$ given as in the definition of the G -action on formal external norms.

The G -action on the operad $F(S_{\mathcal{I}})$ is now defined on an element (θ, δ) of $F(S_{\mathcal{I}})_n = F(V_{\mathcal{I}})_n \times \Sigma_n$ by

$$g \cdot (\theta, \delta) = (g \cdot \theta, \omega_\theta(g) \circ \delta).$$

Remark 2.3.16. We can verify by induction that the action on formal external norms is well-defined. If θ is a formal external norm and g and g' are elements of G , we can check by induction that $g'(g\theta) = g'g\theta$. First note that the result is trivial for $\theta = \text{id}$. Assume now that $\theta = r \otimes_T (\theta_1, \dots, \theta_{|T|})$ and that the result is true for $\theta_1, \dots, \theta_{|T|}$. If we write $gr = r'h$ and $g'r' = r''h'$, we have

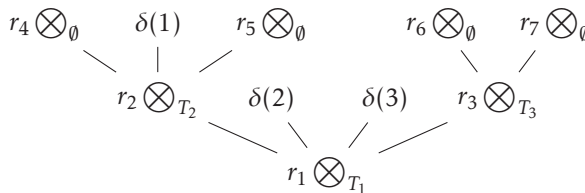
$$\begin{aligned} g'(g\theta) &= g'r' \otimes_T (h\theta_{\sigma(h)^{-1}1}, \dots, h\theta_{\sigma(h)^{-1}|T|}) \\ &= r'' \otimes_T (h'h\theta_{\sigma(h'h)^{-1}1}, \dots, h'h\theta_{\sigma(h'h)^{-1}|T|}) \\ &= g'g\theta. \end{aligned}$$

A similar argument shows that $\omega_{g\theta}(g') \circ \omega_\theta(g) = \omega_\theta(g'g)$ and thus the action of G on $F(S_{\mathcal{I}})$ is also well defined. As a consequence, if θ is an H -fixed formal external norm, the restriction of $\omega_\theta : G \rightarrow \Sigma_{|\theta|}$ to H is a homomorphism.

Remark 2.3.17. If θ is an H -fixed formal external norm, the homomorphism $\omega_\theta : H \rightarrow \Sigma_{|\theta|}$ depends on the chosen orderings on the admissible H -sets appearing in θ and on the choices of coset representatives of G/H . However, up to isomorphism, the corresponding H -set does not depend on these choices.

Notation 2.3.18. If θ is an H -fixed formal external norm, we denote by T_θ the finite H -set that corresponds to the homomorphism $\omega_\theta : H \rightarrow \Sigma_{|\theta|}$.

Remark 2.3.19. An operation (θ, δ) of $F(S_{\mathcal{I}})$ can be seen as the planar rooted tree that corresponds to θ with an ordering on the leaves given by the permutation δ . The picture below illustrates an example.



If θ is a formal external norm and g is an element of G , the permutation $\omega_\theta(g)$ corresponds to the permutation of the leaves obtained by acting on θ with g .

Notation 2.3.20. In what follows, if T is an admissible finite H -set, then we denote by $\bigotimes_T$ the formal external norm $1_G \bigotimes_T (\text{id}, \dots, \text{id})$. If θ is any formal external norm, then we also denote by θ the operation $(\theta, \text{id}_{\Sigma_{| \theta |}})$ of $F(S_{\mathcal{I}})$.

In order to show that the description of the G -action of $F(S_{\mathcal{I}})$ that we gave is the right one, we show now, under our description, that $F(S_{\mathcal{I}})$ is the free operad in Set^G over the symmetric sequence $S_{\mathcal{I}}$.

Proposition 2.3.21. *The G -operad $F(S_{\mathcal{I}})$ is the free operad in Set^G over the symmetric sequence $S_{\mathcal{I}}$.*

Proof. Note first that, given a symmetric sequence of G -sets S , a morphism $S_{\mathcal{I}} \rightarrow S$ is the data of, for every admissible H -set T , a Γ_T -fixed element of $S_{|T|}$. The operation $\bigotimes_T$ of $F(S_{\mathcal{I}})$ is Γ_T -fixed and we thus get a morphism $S_{\mathcal{I}} \rightarrow F(S_{\mathcal{I}})$ of symmetric sequences of G -sets. Consider now any morphism of symmetric sequences $\phi : S_{\mathcal{I}} \rightarrow \mathcal{O}$ with $\mathcal{O}$ any G -operad. For every admissible H -set T , denote by $\bigotimes_T^{\mathcal{O}}$ the Γ_T -fixed operation of $\mathcal{O}$ that corresponds to ϕ . If θ is a formal external norm, consider the operation $\Phi(\theta)$ of $\mathcal{O}$ defined by the following recursion:

1. if $\theta = \text{id}$, then $\Phi(\theta) = \text{id}_{\mathcal{O}}$;
2. if $\theta = r \bigotimes_T (\theta_1, \dots, \theta_{|T|})$, then $\Phi(\theta) = r \cdot \gamma \left(\bigotimes_T^{\mathcal{O}}, \Phi(\theta_1), \dots, \Phi(\theta_{|T|}) \right)$.

We can now define a morphism of G -operads $\Phi : F(S_{\mathcal{I}}) \rightarrow \mathcal{O}$ by setting $\Phi(\theta, \delta) = \Phi(\theta) \cdot \delta$. It is clear that Φ is a morphism of operads in Set and that it is the unique morphism that extends ϕ . We must now check that Φ is G -equivariant. Consider any operation (θ, δ) in $F(S_{\mathcal{I}})$. We have to show by induction that, for every g in G ,

$$\Phi(g\theta, \omega_\theta(g) \cdot \delta) = g\Phi(\theta, \delta).$$

If θ is the identity id , then the result is trivially verified. Let us suppose now by induction that $\theta = r \bigotimes_T (\theta_1, \dots, \theta_{|T|})$ and that the result is known for $\theta_1, \dots, \theta_{|T|}$. If we follow the notation from the definition of the G -action on formal external norms again, then we have

$$\begin{aligned} & \Phi(g\theta, \omega_\theta(g) \circ \delta) \\ &= r' \gamma \left(\bigotimes_T^{\mathcal{O}}, \Phi(h\theta_{\sigma(h)^{-1}1}), \dots, \Phi(h\theta_{\sigma(h)^{-1}|T|}) \right) \cdot (\omega_\theta(g) \circ \delta) \\ &= r' \gamma \left(\bigotimes_T^{\mathcal{O}}, \Phi(h\theta_{\sigma(h)^{-1}1}), \dots, \Phi(h\theta_{\sigma(h)^{-1}|T|}) \right) \cdot (\sigma(h) \langle \omega_1(h), \dots, \omega_{|T|(h)} \rangle \circ \delta) \\ &= r' \gamma \left(\left(\bigotimes_T^{\mathcal{O}}, \sigma(h) \right), \Phi(h\theta_1, \omega_{\theta_1}(h)), \dots, \Phi(h\theta_{|T|}, \omega_{\theta_{|T|}}(h)) \right) \cdot \delta \\ &= r' \gamma \left(h \bigotimes_T^{\mathcal{O}}, h\Phi(\theta_1), \dots, h\Phi(\theta_{|T|}) \right) \cdot \delta \\ &= g\Phi(\theta, \delta). \end{aligned}$$

□

We deduce from Proposition 2.3.21 that Rubin's coherence theorem applies in our situation.

Theorem 2.3.22 ([Rub20, Theorem 5.6],[Rub18, Theorem 2.10]). *The 2-categories $\mathcal{ISMLax}$, $\mathcal{ISMStg}$ and $\mathcal{ISMSt}$ are respectively equivalent to the 2-categories $\mathbf{AlgLax}_{\mathcal{SM}_{\mathcal{I}}}$, $\mathbf{AlgStg}_{\mathcal{SM}_{\mathcal{I}}}$ and $\mathbf{AlgSt}_{\mathcal{SM}_{\mathcal{I}}}$ in the sense of Notation 2.5.4.*

Notation 2.3.23. In what follows, we denote by $\mathcal{E}_{\mathcal{I}}$ the operad $N(\mathcal{SM}_{\mathcal{I}})$ in $\mathbf{SSet}^G$.

The following corollary is a direct consequence of Proposition 2.5.8.

Corollary 2.3.24. *There is a fully faithful simplicial functor*

$$N: \mathcal{ISMSt} \rightarrow \mathbf{Alg}_{\mathcal{E}_{\mathcal{I}}}(\mathbf{SSet}^G)$$

that sends an $\mathcal{I}$ -normed symmetric monoidal category $\mathcal{C}$ to its nerve $N(\mathcal{C})$ where the strict 2-category $\mathcal{ISMSt}$ is seen as a simplicial category.

We will now show that the operad $\mathcal{E}_{\mathcal{I}}$ is an N_{∞} -operad with associated G -indexing system $\mathcal{I}$. We start with the following proposition, which is a reformulation in terms of our description of $\mathcal{SM}_{\mathcal{I}}$ of [Rub21, Theorem 4.6].

Proposition 2.3.25. *Let $H \subseteq G$ be a subgroup and θ an H -fixed formal external norm. The finite H -set T_{θ} (see Notation 2.3.18) is admissible.*

Proof. We show the result by induction on θ . Note first that if θ is the identity id , then T_{id} is the final G -set $*$ which is admissible by Axiom 1 of Definition 2.2.4. Assume now that $\theta = r \otimes_T (\tau_1, \dots, \tau_{|T|})$ with T an admissible K -set and r a chosen coset representative of G/K with $K \subseteq G$ any subgroup. It follows from the fact that θ is H -fixed that the coset of G/K represented by r is H -fixed and thus $H \subseteq {}^rK$. If we denote by T' the H -set $\text{Res}_H^{{}^rK} c_{r^{-1}} T$ and by θ' the formal external norm $\otimes_{T'} (\tau_1, \dots, \tau_{|T|})$, then there is an isomorphism $T_{\theta} \simeq T_{\theta'}$ of finite H -sets and the result holds for θ if it holds for θ' . We can now assume without loss of generality that $\theta = \otimes_T (\tau_1, \dots, \tau_{|T|})$ with T an admissible H -set. Choose a decomposition $T \simeq \bigsqcup_{i=1}^n H/K_i$ and denote by $\alpha_i: \{1, \dots, [H:K_i]\} \rightarrow \{1, \dots, |T|\}$ the map that corresponds to the inclusion $H/K_i \hookrightarrow T$ through the ordering of T . If we denote by θ_i the formal external norm $\otimes_{H/K_i} (\tau_{\alpha_i(1)}, \dots, \tau_{\alpha_i([H:K_i])})$, then we have $T_{\theta} \simeq \bigsqcup_{i=1}^n T_{\theta_i}$ and using Axioms 6 and 5 of Definition 2.2.4, the result holds for θ if and only if it holds for every θ_i . We can now assume without loss of generality that $T = H/K$. Since the formal external norm $\theta = \otimes_{H/K} (\tau_1, \dots, \tau_{[H:K]})$ is H -fixed, we have for every h in H that $h\tau_{\sigma(h)^{-1}j} = \tau_j$ and we deduce that τ_1 is K -fixed and that $h_j\tau_1 = \tau_j$ with h_j a representative of the j -th coset of H/K through the chosen ordering of H/K . By induction, we can assume that the K -set T_{τ_1} is admissible and we obtain that $T_{\theta} \simeq \text{Ind}_K^H T_{\tau_1}$. The result now follows from Axiom 7 of Definition 2.2.4. $\square$

We can now show that $\mathcal{E}_{\mathcal{I}}$ is an N_{∞} -operad with associated G -indexing system $\mathcal{I}$.

Proposition 2.3.26. *The operad $\mathcal{E}_\mathcal{I}$ in $\mathbf{SSet}^G$ is an N_∞ -operad with $\mathcal{I}$ as its G -indexing system.*

Proof. Let $\Gamma \subseteq G \times \Sigma_n$ be any subgroup. The functors $\widetilde{(-)}$ and N preserve limits and it follows that

$$(\mathcal{E}_\mathcal{I})_n^\Gamma \simeq N\left(\widetilde{F(S_\mathcal{I})}_n^\Gamma\right).$$

For every G -set X , the simplicial set $N(\widetilde{X})$ is respectively contractible or empty if and only if X is non-empty or empty and we deduce that $(\mathcal{E}_\mathcal{I})_n^\Gamma$ is respectively contractible or empty if and only if $F(S_\mathcal{I})_n^\Gamma$ is non-empty or empty. The freeness of the symmetric actions on $\mathcal{E}_\mathcal{I}$ now follows from the freeness of the symmetric actions on $F(S_\mathcal{I})$ by considering subgroups of $G \times \Sigma_n$ of the form $\{1_G\} \times \Lambda$ with $\Lambda \subseteq \Sigma_n$. To conclude that $\mathcal{E}_\mathcal{I}$ is an N_∞ -operad with $\mathcal{I}$ as G -indexing system, it is now enough to show that, for every finite H -set T , $F(S_\mathcal{I})_n^{\Gamma_T}$ is not empty if and only if T is admissible. Assume first that T is admissible. The n -ary operation $\bigotimes_T$ of $F(S_\mathcal{I})$ is clearly fixed by Γ_T and $F(S_\mathcal{I})_n^{\Gamma_T}$ is not empty. Suppose now that (θ, δ) is a n -ary operation of $F(S_\mathcal{I})$ fixed by Γ_T . If we denote by $\sigma : H \rightarrow \Sigma_n$ the homomorphism corresponding to T through the chosen ordering $T \simeq \{1, \dots, |T|\}$, we deduce that θ is H -fixed and that $\omega_\theta(h) \circ \delta = \delta \circ \sigma(h)$. The permutation δ can now be seen as an H -equivariant isomorphism between T_θ and T and, by Proposition 2.3.25 and Axiom 2 of Definition 2.2.4, T is admissible. The result follows. $\square$

2.4 Mackey functors vs $\mathcal{E}_\mathcal{I}$ -algebras

In this section, we define the $\mathcal{I}$ -effective Burnside category $\mathcal{A}_\mathcal{I}$ and we use the 2-categorical unfurling of [Len25, Proposition 2.24] to construct a fully faithful functor $\mathfrak{G} : \mathcal{A}_\mathcal{I} \rightarrow N_\Delta(\mathcal{I}\mathbf{SMSt}_{(2,1)})$ with N_Δ the homotopy coherent nerve and $\mathcal{I}\mathbf{SMSt}_{(2,1)}$ the underlying $(2,1)$ -category of $\mathbf{ISMS}\mathbf{t}$. We conclude the section by showing that $\mathcal{G}$ induces a functor $\theta : \mathcal{A}_\mathcal{I} \rightarrow \mathbf{Alg}_{\mathcal{E}_\mathcal{I}}$ whose left Kan extension $\Theta : \mathbf{Fun}_\times(\mathcal{A}_\mathcal{I}^{\text{op}}, \mathcal{S}) \rightarrow \mathbf{Alg}_{\mathcal{E}_\mathcal{I}}$ along the Yoneda embedding $\mathcal{A}_\mathcal{I} \rightarrow \mathbf{Fun}_\times(\mathcal{A}_\mathcal{I}^{\text{op}}, \mathcal{S})$ is an equivalence of ∞ -categories. This will prove Theorem 2.A in Subsection 2.4.4.

2.4.1 Orbital subcategories and the $\mathcal{I}$ -effective Burnside category

In this subsection, we explain how the G -indexing system $\mathcal{I}$ leads to the definition of the $\mathcal{I}$ -effective Burnside category $\mathcal{A}_\mathcal{I}$.

Definition 2.4.1 ([Nar16, Definition 2.5]). Let $\mathcal{O}$ be a small ∞ -category. The *finite cocompletion* of $\mathcal{O}$ denoted by $\mathbf{FO}$ is the full subcategory of $\mathbf{Psh}(\mathcal{O})$ spanned by finite coproducts of representables. The ∞ -category $\mathbf{FO}$ admits finite coproducts and satisfies the following universal property: for every ∞ -category

$\mathcal{C}$ that admits finite coproducts, the natural functor $\mathcal{O} \rightarrow \mathbf{F}\mathcal{O}$ induces an equivalence of ∞ -categories

$$\mathrm{Fun}_{\sqcup}(\mathbf{F}\mathcal{O}, \mathcal{C}) \rightarrow \mathrm{Fun}(\mathcal{O}, \mathcal{C})$$

where $\mathrm{Fun}_{\sqcup}(\mathbf{F}\mathcal{O}, \mathcal{C})$ is the full subcategory of $\mathrm{Fun}(\mathbf{F}\mathcal{O}, \mathcal{C})$ of those functors $\phi : \mathbf{F}\mathcal{O} \rightarrow \mathcal{C}$ that preserve finite coproducts.

Example 2.4.2. The finite cocompletion of $\mathcal{O}_G$ is equivalent to the category Fin^G .

We recall now from [CLL25b] the notion of orbital subcategories.

Definition 2.4.3 ([CLL25b, Definition 4.2.2]). Let $\mathcal{O}$ be a small ∞ -category. A wide subcategory $\mathcal{T} \subseteq \mathcal{O}$ is *orbital* if the base change of a morphism in $\mathbf{F}\mathcal{T}$ along any morphism in $\mathbf{F}\mathcal{O}$ exists and belongs to $\mathbf{F}\mathcal{T}$. Equivalently, for every pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ v \downarrow & & \downarrow w \\ C & \xrightarrow{z} & D \end{array}$$

in $\mathrm{Psh}(\mathcal{O})$ with B, C and D in $\mathcal{O}$, we can decompose A as a disjoint union $\bigsqcup_{i=1}^n A_i$ of objects of $\mathcal{O}$ and, if we write v through such decomposition as $(v_i)_{i=1}^n : \bigsqcup_{i=1}^n A_i \rightarrow C$, then v_i belongs to $\mathcal{T}$. A small ∞ -category $\mathcal{C}$ is *orbital* if it is orbital when regarded as a subcategory of itself.

Example 2.4.4. For every G -indexing system $\mathcal{J}$, consider the subcategory $\mathcal{O}_{\mathcal{J}}$ of the orbit category $\mathcal{O}_G$ spanned by the morphisms $u : G/K \rightarrow G/H$ such that H/K^g belongs to $\mathcal{J}(H)$, where g is a representative of the K -fixed coset of G/H that corresponds to u . The category $\mathcal{O}_{\mathcal{J}}$ is an orbital subcategory of $\mathcal{O}_G$.

Definition 2.4.5. If $\mathcal{O}$ is an ∞ -category and $\mathcal{T}$ an orbital subcategory of $\mathcal{O}$, then we denote by $\mathcal{A}(\mathcal{O}, \mathcal{T})$ the effective Burnside category $A^{\mathrm{eff}}(\mathbf{F}\mathcal{O}, \mathbf{F}\mathcal{O}, \mathbf{F}\mathcal{T})$ in the sense of [Bar17, Definition 5.10].

The following proposition is a direct consequence of [BH21, Lemma C.3].

Proposition 2.4.6. *Let $\mathcal{O}$ be an ∞ -category and $\mathcal{T}$ an orbital subcategory of $\mathcal{O}$. The ∞ -category $\mathcal{A}(\mathcal{O}, \mathcal{T})$ is semi-additive.*

We now prove that the poset of G -indexing systems, and equivalently the poset of N_{∞} -operads, is equivalent to the poset of orbital subcategories of the orbit category $\mathcal{O}_G$.

Proposition 2.4.7. *Denote by $\mathrm{SubOrb}_{\mathcal{O}_G}$ the poset of orbital subcategories of $\mathcal{O}_G$ under inclusion. The morphism of posets $\mathrm{I}_G \rightarrow \mathrm{SubOrb}_{\mathcal{O}_G}$ that sends $\mathcal{J}$ to the orbital subcategory $\mathcal{O}_{\mathcal{J}}$ of Example 2.4.4 is an isomorphism.*

Proof. The category Fin^G is equivalent to the finite cocompletion of $\mathcal{O}_G$ and there is an isomorphism of posets between $\mathrm{SubOrb}_{\mathcal{O}_G}$ and the poset of wide, pullback stable, finite coproduct complete subcategories of Fin^G that sends an orbital subcategory $\mathcal{T} \subseteq \mathcal{O}_G$ to its finite cocompletion $\mathbf{F}\mathcal{T}$ seen as a wide subcategory of Fin^G . The result now follows from [BH18, Theorem 1.4]. $\square$

Definition 2.4.8. The $\mathcal{I}$ -effective Burnside category $\mathcal{A}_{\mathcal{I}}$ is the ∞ -category $\mathcal{A}(\mathcal{O}_G, \mathcal{O}_{\mathcal{I}})$ in the sense of Definition 2.4.5. A G -equivariant map $u : A \rightarrow B$ between finite G -sets is *admissible* if it belongs to $\mathcal{O}_{\mathcal{I}}$.

The following proposition gives a simple way to characterise admissible maps.

Proposition 2.4.9. *Let $u : A \rightarrow B$ be a G -equivariant map between finite G -sets. If we choose a decomposition $B \simeq \bigsqcup_{i=1}^n G/H_i$, then the map u is admissible if and only if for every $i = 1, \dots, n$, the finite H_i -set $u^{-1}(1_G H_i)$ is admissible.*

Proof. Choose a decomposition $B \simeq \bigsqcup_{i=1}^n G/H_i$ and write u as $\bigsqcup_{i=1}^n u_i$ with $u_i : A_i \rightarrow G/H_i$ where $A_i = u^{-1}(G/H_i)$. By definition, the result stands for u if and only if it holds for every u_i and we can assume without loss of generality that $B = G/H$. Choose a decomposition $A \simeq \bigsqcup_{j=1}^m G/K_j$ and write u through this decomposition as $(u_j)_{j=1}^m$ with $u_j : G/K_j \rightarrow G/H$. By definition, u is admissible if and only if every u_j is admissible. Moreover, $u^{-1}(1_G H) \simeq \bigsqcup_{j=1}^m u_j^{-1}(1_G H)$, and it follows from Axioms 5 and 6 of Definition 2.2.4 that the result stands for u if and only if it holds for every u_j so we can assume without loss of generality that $A = G/K$. If $u : G/K \rightarrow G/H$ is now given by a K -fixed coset representative g of G/H , the result follows from the fact that $u^{-1}(1_G H)$ is exactly H/K^g , which is by definition an admissible H -set if and only if u is admissible. $\square$

The $\mathcal{I}$ -effective Burnside category $\mathcal{A}_{\mathcal{I}}$ induces a notion of $\mathcal{I}$ -Mackey functors.

Definition 2.4.10. An $\mathcal{I}$ -Mackey functor is a functor $\mathcal{M} : \mathcal{A}_{\mathcal{I}}^{\text{op}} \rightarrow \mathcal{S}$ that preserves finite products. We denote by $\text{Mack}_{\mathcal{I}}$ the ∞ -category of $\mathcal{I}$ -Mackey functors $\text{Fun}_{\times}(\mathcal{A}_{\mathcal{I}}^{\text{op}}, \mathcal{S})$.

Remark 2.4.11. By Proposition 2.4.6, the ∞ -category $\mathcal{A}_{\mathcal{I}}$ is semi-additive and it follows from [GGN15, Corollary 2.5] that the ∞ -category $\text{Mack}_{\mathcal{I}}$ is equivalent to the ∞ -category $\text{Fun}_{\oplus}(\mathcal{A}_{\mathcal{I}}, \text{Mon}_{E_\infty}(\mathcal{S}))$ with $\text{Mon}_{E_\infty}(\mathcal{S})$ the ∞ -category of E_∞ -monoids in $\mathcal{S}$. The ∞ -category $\text{Mack}_{\mathcal{I}}$ is thus a natural ∞ -categorical generalisation of the category of classical $\mathcal{I}$ -Mackey functors with values in commutative monoids.

2.4.2 Unfurling for 2-categories and orbital subcategories

In this subsection, we recall the 2-categorical unfurling of [Len25, Proposition 2.24] in the case of $\mathcal{A}(\mathcal{O}, \mathcal{T})$ with $\mathcal{O}$ a category and $\mathcal{T} \subseteq \mathcal{O}$ an orbital subcategory of $\mathcal{O}$.

Proposition 2.4.12 ([Len25, Proposition 2.22]). *Let $\mathcal{C}$ be a strict 2-category and let $\phi : \mathbf{F}\mathcal{O} \rightarrow \mathcal{C}$ be a strict 2-functor such that for every morphism $u : A \rightarrow B$ in $\mathbf{F}\mathcal{T}$, the 1-cell $u_! := \phi(u)$ admits a right adjoint u^* satisfying the Beck-Chevalley*

condition: for every pullback square in $\mathbf{FO}$ as on the left in

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ v \downarrow & & \downarrow w \\ C & \xrightarrow{z} & D \end{array} \qquad \begin{array}{ccc} \phi(A) & \xrightarrow{u_!} & \phi(B) \\ v^* \uparrow & \cong & \uparrow w^* \\ \phi(C) & \xrightarrow{z_!} & \phi(D) \end{array}$$

with vertical arrows in $\mathbf{FJ}$, the canonical mate depicted on the right is an isomorphism. Then there is a unique functor $\Phi : \mathcal{A}(\mathcal{O}, \mathcal{T}) \rightarrow N_{\Delta}(\mathcal{C}_{(2,1)})$ that sends a 2-simplex of the form

$$\begin{array}{ccccc} & & A & & \\ & i \swarrow & & \searrow j & \\ B & & & & C \\ k \swarrow & & & \searrow m & \\ D & & E & & F \end{array}$$

to the 2-simplex

$$\begin{array}{ccc} & \phi(E) & \\ l_! k^* \nearrow & \Uparrow & \searrow n_! m^* \\ \phi(D) & \xrightarrow{(nj)_!(ki)^*} & \phi(F) \end{array}$$

given by the pasting

$$\begin{array}{ccccc} & & A & & \\ & i^* \nearrow & & \searrow j_! & \\ B & & & & C \\ k^* \nearrow & & \Downarrow & \searrow m^* & \\ D & & E & & F \end{array}$$

(A curved arrow labeled $(ki)^*$ goes from D to B , and another curved arrow labeled $(nj)_!$ goes from C to F .)

Here $\mathcal{C}_{(2,1)}$ is the $(2,1)$ -category obtained from $\mathcal{C}$ by throwing away the non-invertible 2-cells.

Definition 2.4.13. Let $\mathcal{C}$ be a strict 2-category. A strict 2-functor $\phi : \mathbf{FO} \rightarrow \mathcal{C}$ satisfies the $\mathcal{T}$ -Beck Chevalley condition if it satisfies the conditions of Proposition 2.4.12. In the case where $\mathcal{O}$ is $\mathcal{O}_G$ and where $\mathcal{T}$ is the orbital subcategory $\mathcal{O}_{\mathcal{I}}$ of $\mathcal{O}_G$ that corresponds to $\mathcal{I}$, we say that ϕ satisfies the $\mathcal{I}$ -Beck-Chevalley condition.

Notation 2.4.14. Let $\mathcal{D}$ be a category with finite coproducts. If $A_1, \dots, A_n$ and $B_1, \dots, B_m$ are objects of $\mathcal{D}$, for every $i = 1, \dots, n$, $f_i : A \rightarrow B_{\alpha(i)}$ is a morphism in $\mathcal{D}$ and $\alpha : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ is a map, then we denote by $((f_i)_{i=1}^n, \alpha) : \bigsqcup_{i=1}^n A_i \rightarrow \bigsqcup_{j=1}^m B_j$ the morphism $(\iota_{\alpha(i)} \circ f_i)_{i=1}^n$ with $\iota_j : B_j \rightarrow \bigsqcup_{j=1}^m B_j$ the canonical inclusion. Note that if $\mathcal{E}$ is a small category, then every morphism in $\mathbf{FE}$

can be written canonically as $((f_i)_{i=1}^n, \alpha) : \sqcup_{i=1}^n A_i \rightarrow \sqcup_{j=1}^m B_j$ with A_i and B_j in $\mathcal{E}$.

In the case where $\mathcal{C}$ is a semi-additive strict 2-category, we now give a simple criterion to check whether a strict 2-functor $\phi : \mathbf{FO} \rightarrow \mathcal{C}$ that preserves finite coproducts satisfies the $\mathcal{T}$ -Beck-Chevalley condition.

Proposition 2.4.15. *Let $\mathcal{C}$ be a semi-additive strict 2-category and $\phi : \mathbf{FO} \rightarrow \mathcal{C}$ a strict 2-functor that preserves finite coproducts. The strict 2-functor $\phi : \mathbf{FO} \rightarrow \mathcal{C}$ satisfies the $\mathcal{T}$ -Beck-Chevalley condition if and only if for every morphism $u : A \rightarrow B$ in $\mathbf{FT}$ with B in $\mathcal{O}$, the 1-cell $\phi(u) := u_!$ admits a right adjoint u^* such that for every pullback square in $\mathbf{FO}$ as on the left in*

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ v \downarrow & & \downarrow w \\ C & \xrightarrow{z} & D \end{array} \qquad \begin{array}{ccc} \phi(A) & \xrightarrow{u_!} & \phi(B) \\ v^* \uparrow & \cong & \uparrow w^* \\ \phi(C) & \xrightarrow{z_!} & \phi(D) \end{array}$$

with w in $\mathbf{FT}$, z in $\mathcal{O}$, and C and D in $\mathcal{O}$, the canonical mate depicted on the right is an isomorphism. Moreover, if a strict 2-functor $\phi : \mathbf{FO} \rightarrow \mathcal{C}$ that preserves finite coproducts satisfies the $\mathcal{T}$ -Beck-Chevalley condition, the induced functor $\Phi : \mathcal{A}(\mathcal{O}, \mathcal{T}) \rightarrow N_\Delta(\mathcal{C}_{(2,1)})$ is semi-additive.

Proof. It is clear that the $\mathcal{T}$ -Beck-Chevalley condition implies the second condition of the proposition. Assume now that ϕ satisfies the second condition. Let $u : A \rightarrow B$ be a morphism in $\mathbf{FT}$. Choose a decomposition $B \simeq \sqcup_{k=1}^p B_k$ with B_k in $\mathcal{O}$ through which u can be written as $\sqcup_{k=1}^p u_k$ with $u_k : A_k \rightarrow B_k$ in $\mathbf{FT}$. By assumptions, $(u_k)_!$ admits a right adjoint $(u_k)^*$ and it follows that $u_!$ also admits a right adjoint given by $u^* = \bigoplus_{k=1}^p (u_k)^*$. Consider now any pullback square

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ v \downarrow & & \downarrow w \\ C & \xrightarrow{z} & D \end{array} \tag{S}$$

in $\mathbf{FO}$ with v and w in $\mathbf{FT}$. Choose decompositions $C \simeq \sqcup_{i=1}^n C_i$ and $D \simeq \sqcup_{j=1}^m D_j$ with C_i and D_j in $\mathcal{O}$ through which we have $z = ((z_i)_{i=1}^n, \alpha)$ and $w = \sqcup_{j=1}^m w_j$ with $w_j : B_j \rightarrow D_j$. For every $i = 1, \dots, n$, consider the following pullback square

$$\begin{array}{ccc} A_i & \xrightarrow{u_i} & B_{\alpha(i)} \\ v_i \downarrow & & \downarrow w_{\alpha(i)} \\ C_i & \xrightarrow{z_i} & D_{\alpha(i)}. \end{array} \tag{S_i}$$

By disjointness and universality of finite sums in $\mathbf{FO}$, we have an isomorphism $A \simeq \sqcup_{i=1}^n A_i$ through which we can write u and v respectively as $((u_i)_{i=1}^n, \alpha) : \sqcup_{i=1}^n A_i \rightarrow \sqcup_{j=1}^m B_j$ and $\sqcup_{i=1}^n v_i$. The morphisms v_i and w_j both belong in $\mathbf{FT}$ and $(v_i)_!$ and $(w_j)_!$ respectively admit right adjoints $(v_i)^*$ and

$(w_j)^*$ by assumption. Moreover, the mate ω_i depicted in the following diagram associated with the square (S_i) is an isomorphism by assumption

$$\begin{array}{ccc} \phi(A_i) & \xrightarrow{(u_i)!} & \phi(B_{\alpha(i)}) \\ (v_i)^* \uparrow & \Downarrow^{\omega_i} & \uparrow (w_{\alpha(i)})^* \\ \phi(C_i) & \xrightarrow{(z_i)!} & \phi(D_{\alpha(i)}). \end{array}$$

The morphisms $w_i = \bigoplus_{j=1}^m (w_j)_!$ and $v_i = \bigoplus_{i=1}^n (v_i)_!$ respectively admit right adjoints $w^* = \bigoplus_{j=1}^m (w_j)^*$ and $v^* = \bigoplus_{i=1}^n (v_i)^*$ and by direct inspection, the mate ω depicted in the following diagram associated with the square (S)

$$\begin{array}{ccc} \varphi(A) & \xrightarrow{u!} & \phi(B) \\ v^* \uparrow & \Downarrow^{\omega} & \uparrow w^* \\ \phi(C) & \xrightarrow{z!} & \phi(D) \end{array}$$

corresponds to $(\omega_{i,j})_{i,j}$ through the equivalence

$$\mathcal{C}(\phi(C), \phi(B)) \simeq \prod_{i=1}^n \prod_{j=1}^m \mathcal{C}(\phi(C_i), \phi(B_j)),$$

where we define $\omega_{i,j}$ by $\text{id}_{0_{\phi(C_j), \phi(B_i)}}$ whenever $\alpha(i) \neq j$. Every $\omega_{i,j}$ is an isomorphism and it follows that ω is an isomorphism and that ϕ satisfies the $\mathcal{I}$ -Beck-Chevalley condition. If $\phi : \mathbf{FO} \rightarrow \mathcal{C}$ is a strict 2-functor that preserves finite coproducts, then it follows directly from [Bar17, Proposition 4.3] that the induced functor $\Phi : \mathcal{A}(\mathcal{O}, \mathcal{I}) \rightarrow N_{\Delta}(\mathcal{C}_{(2,1)})$ is semi-additive. $\square$

2.4.3 From $\mathcal{A}_{\mathcal{I}}$ to $\mathcal{ISMS}_{\mathcal{I}}$

In this subsection, we construct a strict 2-functor $\mathfrak{G} : \mathbf{Fin}^G \rightarrow \mathcal{ISMS}_{\mathcal{I}}$ that satisfies the $\mathcal{I}$ -Beck Chevalley condition in the sense of Definition 2.4.13.

Notation 2.4.16. If $n \geq 0$, we denote by $\langle n \rangle$ the finite set $\{1, \dots, n\}$.

Definition 2.4.17. Let A be a finite G -set. The G -category $\mathfrak{G}(A)$ has as objects pairs $(\theta, \mathcal{L})$ with θ a formal external norm and $\mathcal{L} = (a_1, \dots, a_{|\theta|})$ a sequence of elements of A . A morphism $\alpha : (\theta, (a_1, \dots, a_{|\theta|})) \rightarrow (\tau, (b_1, \dots, b_{|\tau|}))$ in $\mathfrak{G}(A)$ is a map $\alpha : \langle |\theta| \rangle \rightarrow \langle |\tau| \rangle$ such that $a_i = b_{\alpha(i)}$ for every i in $\langle |\theta| \rangle$. The action of G on $\mathfrak{G}(A)$ is defined on objects by

$$g \cdot (\theta, (a_1, \dots, a_{|\theta|})) = (g\theta, (g a_{\omega_{\theta}(g)^{-1}1}, \dots, g a_{\omega_{\theta}(g)^{-1}|\theta|}))$$

and on a morphism $\alpha : (\theta, (a_1, \dots, a_{|\theta|})) \rightarrow (\tau, (b_1, \dots, b_{|\tau|}))$ by $g \cdot \alpha = \omega_{\tau}(g) \circ \alpha \circ \omega_{\theta}(g)^{-1}$. It follows from Remark 2.3.16 that these G -actions are well-defined. The G -category $\mathfrak{G}(A)$ is an $\mathcal{I}$ -normed symmetric monoidal category with the following structure:

1. the unit e is the object $(\bigotimes_{\emptyset}, ())$ with $()$ the empty list;
2. the tensor product $\otimes : \mathfrak{G}(A) \times \mathfrak{G}(A) \rightarrow \mathfrak{G}(A)$ is defined on objects by

$$(\theta, \mathcal{L}_1) \otimes (\tau, \mathcal{L}_2) = (\theta \otimes \tau, \mathcal{L}_1 \sqcup \mathcal{L}_2)$$

and on morphisms by $\alpha \otimes \beta = \alpha \sqcup \beta$ where $\theta \otimes \tau$ is the formal external norm given by $\gamma(\bigotimes_{\langle 2 \rangle}, \theta, \tau)$ with $\langle 2 \rangle$ seen as a trivial G -set;

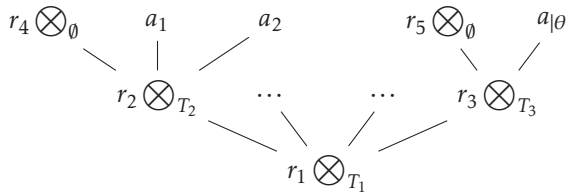
3. given an admissible H -set T , we define the external T -norm $\bigotimes_T : \mathfrak{G}(A)^{\times T} \rightarrow \mathfrak{G}(A)$ on objects by

$$\bigotimes_T ((\theta_1, \mathcal{L}_1), \dots, (\theta_{|T|}, \mathcal{L}_{|T|})) = (\bigotimes_T (\theta_1, \dots, \theta_{|T|}), \bigsqcup_{i=1}^{|T|} \mathcal{L}_i).$$

and on morphisms by $\bigotimes_T (\alpha_1, \dots, \alpha_{|T|}) = \bigsqcup_{i=1}^{|T|} \alpha_i$.

If $X = (\theta, (a_1, \dots, a_{|\theta|}))$ is an object of $\mathfrak{G}(A)$, then the *length* of X is the length of θ and we denote it by $|X|$. If X, Y and Z are objects of $\mathfrak{G}(A)$ with respective lengths n, m and k , the natural equivariant isomorphisms $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$, $\lambda_X : e \otimes X \rightarrow X$, $\rho_X : X \otimes e \rightarrow X$ and $\beta_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ are respectively given by the permutations $1_{\Sigma_{n+m+k}}$, 1_{Σ_n} , 1_{Σ_n} and $\tau(n, m)$ with $\tau(n, m)$ the block permutation associated with the unique non trivial permutation τ of Σ_2 . If $X_1, \dots, X_{|T|}$ are objects of $\mathfrak{G}(A)$ with respective lengths $n_1, \dots, n_{|T|}$, the natural nonequivariant isomorphism $v_T : \bigotimes_T (X_1, \dots, X_{|T|}) \rightarrow \bigotimes_{|T|} (X_1, \dots, X_{|T|})$ is given by the permutation $1_{\Sigma_{n_1 + \dots + n_{|T|}}}$.

Remark 2.4.18. If A is a finite G -set, an object $(\theta, (a_1, \dots, a_{|\theta|}))$ of $\mathfrak{G}(A)$ can be seen as the planar rooted tree associated with θ where the i -th leaf represents a_i .



Notation 2.4.19. In what follows, given an element a of A , we will denote by a the object $(\text{id}, (a))$ of $\mathfrak{G}(A)$. This gives a canonical inclusion $A \hookrightarrow \mathfrak{G}(A)$.

The construction of Definition 2.4.17 is functorial. For every G -equivariant map $u : A \rightarrow B$, the definition of the strict $\mathcal{I}$ -normed functor $\mathfrak{G}(u) : \mathfrak{G}(A) \rightarrow \mathfrak{G}(B)$ is part of the following construction.

Construction 2.4.20. Let A and B be two finite G -sets and $\phi : A \rightarrow \mathfrak{G}(B)$ a G -functor with A seen as a discrete G -category. We define a strict $\mathcal{I}$ -normed functor $\mathcal{G}(\phi) : \mathfrak{G}(A) \rightarrow \mathfrak{G}(B)$ as follows: on an object $X = (\theta, (a_1, \dots, a_{|\theta|}))$ of $\mathfrak{G}(A)$, the strict $\mathcal{I}$ -normed functor $\mathcal{G}(\phi)$ is defined by the following recursion:

1. if $\theta = \text{id}$, then $\mathcal{G}(\phi)(X) = \phi(a)$;
2. if $\theta = r \bigotimes_T (\theta_1, \dots, \theta_{|T|})$, then $\mathcal{G}(\phi)(X)$ is

$$r \bigotimes_T \left(\mathcal{G}(\phi) \left(\theta_1, (a_1, \dots, a_{|\theta_1|}) \right), \dots, \mathcal{G}(\phi) \left(\theta_{|T|}, (a_{|\theta_1|+|\theta_2|}, \dots, a_{|\theta_1|+|\theta_2|+|\theta_3|}) \right) \right)$$

and on a morphism $\alpha : (\theta, (a_1, \dots, a_{|\theta|})) \rightarrow (\tau, (b_1, \dots, b_{|\tau|}))$, the map $\mathcal{G}(\phi)(\alpha)$ is defined by the block map

$$\alpha(k_1, \dots, k_{|\theta|})(l_1, \dots, l_{|\tau|}) : \langle k_1 + \dots + k_{|\theta|} \rangle \rightarrow \langle l_1 + \dots + l_{|\tau|} \rangle$$

that sends the copy of $\langle k_i \rangle$ included in $\langle k_1 \rangle \sqcup \dots \sqcup \langle k_{|\theta|} \rangle \simeq \langle k_1 + \dots + k_{|\theta|} \rangle$ to the copy of $\langle k_i \rangle = \langle l_{\alpha(i)} \rangle$ included in $\langle l_1 \rangle \sqcup \dots \sqcup \langle l_{|\tau|} \rangle \simeq \langle l_1 + \dots + l_{|\tau|} \rangle$ where k_i and l_j are respectively the length of the image of a_i and b_j by ϕ (we use here that $a_i = b_{\alpha(i)}$). This construction is functorial: if $\phi, \psi : A \rightrightarrows \mathfrak{G}(B)$ are two G -functors and $\omega : \phi \Rightarrow \psi$ a G -natural transformation between them, then we obtain an $\mathcal{I}$ -normed natural transformation $\mathcal{G}(\omega) : \mathcal{G}(\phi) \Rightarrow \mathcal{G}(\psi)$ defined by

$$\mathcal{G}(\omega)_{(\theta, (a_1, \dots, a_{|\theta|}))} = \bigsqcup_{i=1}^{|\theta|} \omega_{a_i}.$$

We denote by $\mathcal{G} : \text{Fun}_G(A, \mathfrak{G}(B)) \rightarrow \mathcal{I}\text{SMSt}(\mathfrak{G}(A), \mathfrak{G}(B))$ the functor provided by this construction.

Lemma 2.4.21. *For finite G -sets A and B , the functor*

$$\mathcal{G} : \text{Fun}_G(A, \mathfrak{G}(B)) \rightarrow \mathcal{I}\text{SMSt}(\mathfrak{G}(A), \mathfrak{G}(B))$$

is fully faithful.

Proof. By definition, if $\phi, \psi : A \rightrightarrows \mathfrak{G}(B)$ are two G -functors, then, because $\mathcal{G}(\phi)$ and $\mathcal{G}(\psi)$ are strict $\mathcal{I}$ -normed functors, an $\mathcal{I}$ -normed natural transformation $\Omega : \mathcal{G}(\phi) \Rightarrow \mathcal{G}(\psi)$ is determined by its values on elements of A and Ω must be equal to $\mathcal{G}(\omega)$ where ω is the natural transformation $\omega : \phi \Rightarrow \psi$ obtained by restricting Ω to A . The result follows. $\square$

Proposition 2.4.22. *The following diagram of functors is strictly commutative.*

$$\begin{array}{ccc} \text{Fun}_G(A, \mathfrak{G}(B)) \times \text{Fun}_G(B, \mathfrak{G}(C)) & \xrightarrow{\text{id}_{\text{Fun}_G(A, \mathfrak{G}(B))} \times \mathcal{G}} & \text{Fun}_G(A, \mathfrak{G}(B)) \times \mathcal{I}\text{SMSt}(\mathfrak{G}(B), \mathfrak{G}(C)) \\ \downarrow \mathcal{G} \times \mathcal{G} & & \downarrow \circ \\ & & \text{Fun}_G(A, \mathfrak{G}(C)) \\ & & \downarrow \mathcal{G} \\ \mathcal{I}\text{SMSt}(\mathfrak{G}(A), \mathfrak{G}(B)) \times \mathcal{I}\text{SMSt}(\mathfrak{G}(B), \mathfrak{G}(C)) & \xrightarrow{\circ} & \mathcal{I}\text{SMSt}(\mathfrak{G}(A), \mathfrak{G}(B)) \end{array}$$

Proof. Let $\phi : A \rightarrow \mathfrak{G}(B)$ and $\psi : B \rightarrow \mathfrak{G}(C)$ be two G -functors. It follows from a direct induction on formal external norms that the strict $\mathcal{I}$ -normed functors $\mathcal{G}(\psi) \circ \mathcal{G}(\phi)$ and $\mathcal{G}(\mathcal{G}(\psi) \circ \phi)$ agree on objects. Consider now a morphism $\alpha : X \rightarrow Y$ in $\mathfrak{G}(A)$ between two objects $X = (\theta, (a_1, \dots, a_{|\theta|}))$ and $Y = (\tau, (b_1, \dots, b_{|\tau|}))$ of $\mathfrak{G}(A)$. If we denote by $X_i = (\lambda_i, (c_1^i, \dots, c_{|\lambda_i|}^i))$ the image of a_i by ϕ , then by definition, the functor $\mathcal{G}(\psi) \circ \mathcal{G}(\phi)$ sends α on the block map $\alpha(|\lambda_1|, \dots, |\lambda_\theta|)(k_1^1, \dots, k_{|\lambda_1|}^1, \dots, k_{|\lambda_\theta|}^{|\theta|})$ where k_j^i is the length of $\psi(c_j^i)$. It follows that the image of α under $\mathcal{G}(\psi) \circ \mathcal{G}(\phi)$ is equal to the block map $\alpha(k_1^1 + \dots + k_{|\lambda_1|}^1, \dots, k_1^{|\theta|} + \dots + k_{|\lambda_\theta|}^{|\theta|})$ which is also the image of α under $\mathcal{G}(\mathcal{G}(\psi) \circ \phi)$. A similar argument shows that the diagram of the proposition commutes on morphisms. $\square$

If $u : A \rightarrow B$ is a G -equivariant map, then we denote by $\mathfrak{G}(u) : \mathfrak{G}(A) \rightarrow \mathfrak{G}(B)$ the image under $\mathcal{G}$ of the composition $A \xrightarrow{u} B \rightarrow \mathfrak{G}(B)$ and it follows from Proposition 2.4.22 that we get a strict 2-functor $\mathfrak{G} : \text{Fin}^G \rightarrow \mathcal{I}\text{SMSt}$. The 2-category $\mathcal{I}\text{SMStg}$ is semi-additive by Proposition 2.3.8 and we can thus prove the following proposition.

Proposition 2.4.23. *The functor $\mathfrak{G}$, seen as a strict 2-functor $\mathfrak{G} : \text{Fin}^G \rightarrow \mathcal{I}\text{SMStg}$ through the inclusion $\mathcal{I}\text{SMSt} \hookrightarrow \mathcal{I}\text{SMStg}$, preserves finite coproducts.*

Proof. By Proposition 2.3.8, the 2-category $\mathcal{I}\text{SMStg}$ is semi-additive and, given two finite G -sets A and B , the coproduct of $\mathfrak{G}(A)$ and $\mathfrak{G}(B)$ in $\mathcal{I}\text{SMStg}$ is given by their product $\mathfrak{G}(A) \times \mathfrak{G}(B)$. We have to show now that the canonical strong $\mathcal{I}$ -normed functor

$$\Phi = (\mathfrak{G}(i_A) \circ p_A) \otimes (\mathfrak{G}(i_B) \circ p_B) : \mathfrak{G}(A) \times \mathfrak{G}(B) \rightarrow \mathfrak{G}(A \sqcup B)$$

is an equivalence, where $i_A : A \rightarrow A \sqcup B$, $i_B : B \rightarrow A \sqcup B$, $p_A : \mathfrak{G}(A) \times \mathfrak{G}(B) \rightarrow \mathfrak{G}(A)$ and $p_B : \mathfrak{G}(A) \times \mathfrak{G}(B) \rightarrow \mathfrak{G}(B)$ are respectively the canonical inclusions and projections. The functor Φ sends an object $((\theta, (a_1, \dots, a_{|\theta|})), (\tau, (b_1, \dots, b_{|\tau|})))$ of $\mathfrak{G}(A) \times \mathfrak{G}(B)$ to $(\theta \otimes \tau, (a_1, \dots, a_\theta, b_1, \dots, b_{|\tau|}))$ and a morphism (α, β) of $\mathfrak{G}(A) \times \mathfrak{G}(B)$ to $\alpha \otimes \beta$. It admits an inverse in $\mathcal{I}\text{SMStg}$ that can be constructed as follows: consider the G -functor $\psi_A : A \sqcup B \rightarrow \mathfrak{G}(A)$ given respectively by the canonical inclusion $A \rightarrow \mathfrak{G}(A)$ and by the G -functor $B \rightarrow \mathfrak{G}(A)$ constant on the unit e . Denote by $\Psi_A : \mathfrak{G}(A \sqcup B) \rightarrow \mathfrak{G}(A)$ the strict $\mathcal{I}$ -normed functor $\mathcal{G}(\psi_A)$. Symmetrically, we obtain a strict $\mathcal{I}$ -normed functor $\Psi_B : \mathfrak{G}(A \sqcup B) \rightarrow \mathfrak{G}(B)$ and we get a strict $\mathcal{I}$ -normed functor $\Psi = (\Psi_A, \Psi_B) : \mathfrak{G}(A \sqcup B) \rightarrow \mathfrak{G}(A) \times \mathfrak{G}(B)$. Let us show that Ψ is an inverse of Φ in $\mathcal{I}\text{SMStg}$. We first construct by induction an $\mathcal{I}$ -normed natural transformation $\eta : \Phi \circ \Psi \simeq \text{id}_{\mathfrak{G}(A \sqcup B)}$. Consider any object $X = (\theta, (c_1, \dots, c_\theta))$ of $\mathfrak{G}(A \sqcup B)$. If $\theta = \text{id}$, then $(c_1, \dots, c_\theta) = (c)$ and c belongs either to A or B and we can define η_X respectively by the isomorphisms $\rho_c : c \otimes e \simeq c$ or $\lambda_c : e \otimes c \simeq c$. Assume now that $\theta = r \otimes_T (\theta_1, \dots, \theta_{|T|})$, denote by X_i the object $(\theta_i, (c_{|\theta_1|+\dots+\theta_{i-1}|}, \dots, c_{|\theta_1|+\dots+\theta_i|}))$ and assume that η_{X_i} is already

constructed. By definition $\Phi \circ \Psi(X)$ is

$$\begin{aligned} & \Psi_A(X) \otimes \Psi_B(X) \\ &= r \otimes_T (\Psi_A(X_1), \dots, \Psi_A(X_{|\theta|})) \otimes r \otimes_T (\Psi_B(X_1), \dots, \Psi_B(X_{|\theta|})) \\ &\stackrel{\alpha}{\simeq} r \otimes_T (\Psi_A(X_1) \otimes \Psi_B(X_1), \dots, \Psi_A(X_{|\theta|}) \otimes \Psi_B(X_{|\theta|})) \end{aligned}$$

and η_X can be defined by $\eta_X = \bigsqcup_{i=1}^{|\tau|} \eta_{X_i}$ through the G -equivariant isomorphism α . It follows from a direct inductive argument that η is an $\mathcal{I}$ -normed natural transformation. We can also construct an $\mathcal{I}$ -normed natural transformation $\epsilon : \Psi \circ \Phi \simeq \text{id}_{\mathfrak{G}(A) \times \mathfrak{G}(B)}$. Given an object $X = ((\theta, (a_1, \dots, a_{|\theta|})), (\tau, (b_1, \dots, b_{|\tau|})))$ of $\mathfrak{G}(A) \times \mathfrak{G}(B)$, ϵ_X is the G -equivariant isomorphism ϵ_X between $((\theta \otimes \tau)_A, (a_1, \dots, a_{|\theta|}))$, $((\theta \otimes \tau)_B, (b_1, \dots, b_{|\tau|}))$ and $((\theta, (a_1, \dots, a_{|\theta|})), (\tau, (b_1, \dots, b_{|\tau|})))$ given by $(\text{id}_{\langle |\theta| \rangle}, \text{id}_{\langle |\tau| \rangle})$ where $(\theta \otimes \tau)_A$ is the formal external norm obtained by replacing each occurrence of the identity that corresponds to an element of B by $\bigotimes_{\emptyset}$. $\square$

Our goal is now to use Proposition 2.4.15 to show that $\mathfrak{G} : \text{Fin}^G \rightarrow \mathcal{I}\text{SMStg}$ satisfies the $\mathcal{I}$ -Beck Chevalley condition in the sense of Definition 2.4.13. For every subgroup $H \subseteq G$ and admissible map $u : A \rightarrow G/H$, we have to construct a right adjoint in $\mathcal{I}\text{SMStg}$ to the strict $\mathcal{I}$ -normed functor $\mathfrak{G}(u)$. Note that in the category $\text{Fin}^G_{/G/H}$, the map $u : A \rightarrow G/H$ is isomorphic to the projection $\mathcal{P}_T : \text{Ind}_H^G T \rightarrow G/H$ where T is the admissible H -set given by $u^{-1}(1_{G/H})$. We must then construct a right adjoint to $\mathfrak{G}(\mathcal{P}_T)$ in $\mathcal{I}\text{SMStg}$.

Construction 2.4.24. If T is an admissible H -set, then we denote by $\mathfrak{G}(\mathcal{P}_T)^* : \mathfrak{G}(G/H) \rightarrow \mathfrak{G}(\text{Ind}_H^G T)$ the image under $\mathcal{G}$ of the G -functor $G/H \rightarrow \mathfrak{G}(\text{Ind}_H^G T)$ that corresponds through the equivalence

$$\text{Fun}(G/H, \mathfrak{G}(\text{Ind}_H^G T)) \simeq \mathfrak{G}(\text{Ind}_H^G T)^H$$

to the H -fixed object $(\bigotimes_T, (t_1, \dots, t_{|T|}))$ of $\mathfrak{G}(\text{Ind}_H^G T)$ where $T = \{t_1, \dots, t_{|T|}\}$ is the chosen ordering on T .

Proposition 2.4.25. *Let T be an admissible H -set. The strict $\mathcal{I}$ -normed functor $\mathfrak{G}(\mathcal{P}_T)^*$ is a right adjoint of the strict $\mathcal{I}$ -normed functor $\mathfrak{G}(\mathcal{P}_T)$ in the 2-category $\mathcal{I}\text{SMStg}$.*

Proof. It follows from Proposition 2.4.22 that the strict $\mathcal{I}$ -normed functor $\mathfrak{G}(\mathcal{P}_T)^* \mathfrak{G}(\mathcal{P}_T)$ is the image under $\mathcal{G}$ of the G -functor $\text{Ind}_H^G T \rightarrow \mathfrak{G}(\text{Ind}_H^G T)$ that corresponds by adjunction to the H -functor $T \rightarrow \mathfrak{G}(\text{Ind}_H^G T)$ that sends t to $(\bigotimes_T, (t_1, \dots, t_{|T|}))$. The unit $\eta : \text{id}_{\mathfrak{G}(\text{Ind}_H^G T)} \rightarrow \mathfrak{G}(\mathcal{P}_T)^* \mathfrak{G}(\mathcal{P}_T)$ can be defined as the image under $\mathcal{G}$ of the G -natural transformation that corresponds by adjunction to the H -natural transformation defined on t_i by the morphism $t_i \rightarrow (\bigotimes_T, (t_1, \dots, t_{|T|}))$ given by the map $\langle 1 \rangle \rightarrow \langle |T| \rangle$ that sends 1 to i . Using

again Proposition 2.4.22, the strict $\mathcal{I}$ -normed functor $\mathfrak{G}(\mathcal{P}_T)\mathfrak{G}(\mathcal{P}_T)^*$ is the image under $\mathcal{G}$ of the G -functor $G/H \rightarrow \mathfrak{G}(G/H)$ that corresponds through the equivalence $\text{Fun}_G(G/H, \mathfrak{G}(\text{Ind}_H^G T)) \simeq \mathfrak{G}(\text{Ind}_H^G T)^H$ to the H -fixed object $(\bigotimes_T, (1_G, \dots, 1_G))$ of $\mathfrak{G}(G/H)$ and the counit can be defined as the image under $\mathcal{G}$ of the G -natural transformation that corresponds to the H -fixed morphism $(\bigotimes_T, (1_G, \dots, 1_G)) \rightarrow 1_G$ given by the unique map $\langle |T| \rangle \rightarrow \langle 1 \rangle$. The triangle identities can be verified using Proposition 2.4.22. $\square$

We can now show, using Proposition 2.4.15, the following result.

Proposition 2.4.26. *The functor $\mathfrak{G} : \text{Fin}^G \rightarrow \mathcal{I}\text{SMStg}$ satisfies the $\mathcal{I}$ -Beck-Chevalley condition in the sense of Definition 2.4.13*

Proof. Let $u : A \rightarrow G/H$ be any admissible map. The map $u : A \rightarrow G/H$ is isomorphic in the category $\text{Fin}_{/(G/H)}^G$ to the projection $\mathcal{P}_T : \text{Ind}_H^G T \rightarrow G/H$ where T is the finite H -set $u^{-1}(1_G H)$, which is admissible by Proposition 2.4.9. It follows from Proposition 2.4.25 that $\mathfrak{G}(u)$ admits a right adjoint $\mathfrak{G}(u)^*$. Let $K \subseteq G$ be a subgroup of G and let $w : G/K \rightarrow G/H$ be a G -equivariant map. Without loss of generality, we can assume that K is included in H and that w corresponds to this inclusion. If we denote by $v : \text{Ind}_K^G \text{Res}_K^H T \rightarrow \text{Ind}_H^G T$ the G -equivariant map that corresponds by adjunction to the K -equivariant map $\text{Res}_K^H T \rightarrow \text{Ind}_H^G T$ that sends t to $(1_G H, t)$, the square

$$\begin{array}{ccc} \text{Ind}_K^G \text{Res}_K^H T & \xrightarrow{v} & \text{Ind}_H^G T \\ \mathcal{P}_{\text{Res}_K^H T} \downarrow & & \downarrow \mathcal{P}_T \\ G/K & \xrightarrow{w} & G/H \end{array}$$

is Cartesian. A direct inspection shows that the mate depicted in the following diagram

$$\begin{array}{ccc} \mathfrak{G}(\text{Ind}_K^G \text{Res}_K^H T) & \xrightarrow{\mathfrak{G}(v)} & \mathfrak{G}(\text{Ind}_H^G T) \\ \mathfrak{G}(\mathcal{P}_{\text{Res}_K^H T})^* \uparrow & \Rightarrow & \uparrow \mathfrak{G}(\mathcal{P}_T)^* \\ \mathfrak{G}(G/K) & \xrightarrow{\mathfrak{G}(w)} & \mathfrak{G}(G/H) \end{array}$$

is the $\mathcal{I}$ -normed natural transformation $\mathfrak{G}(v)\mathfrak{G}(\mathcal{P}_{\text{Res}_K^H T})^* \rightarrow \mathfrak{G}(\mathcal{P}_T)^*\mathfrak{G}(w)$ given by the image under $\mathcal{G}$ of the G -natural transformation that corresponds through the equivalence $\text{Fun}_G(G/H, \mathfrak{G}(\text{Ind}_H^G T)) \simeq \mathfrak{G}(\text{Ind}_H^G T)^H$ to the K -fixed morphism $(\bigotimes_{\text{Res}_K^H T}, (t_1, \dots, t_{|T|})) \rightarrow (\bigotimes_T, (t_1, \dots, t_{|T|}))$ given by the identity $\text{id}_{\langle |T| \rangle} : \langle |T| \rangle \rightarrow \langle |T| \rangle$, which is clearly an isomorphism. The result now follows from Proposition 2.4.23, Proposition 2.3.8, and Proposition 2.4.15. $\square$

It follows from Proposition 2.4.12 that $\mathfrak{G} : \text{Fin}^G \rightarrow \mathcal{I}\text{SMStg}$ induces a semi-additive functor $\mathcal{G} : \mathcal{A}_{\mathcal{I}} \rightarrow N_\Delta(\mathcal{I}\text{SMStg}_{(2,1)})$. By the following proposition,

$\mathfrak{G} : \text{Fin}^G \rightarrow \mathcal{I}\text{SMSt}$ also satisfies the $\mathcal{I}$ -Beck-Chevalley condition and $\mathcal{G}$ factors as a functor $\mathcal{A}_{\mathcal{I}} \rightarrow N_{\Delta}(\mathcal{I}\text{SMSt}_{(2,1)})$.

Proposition 2.4.27. *For every admissible morphism $u : A \rightarrow B$, the strict $\mathcal{I}$ -normed functor $\mathfrak{G}(u) : \mathfrak{G}(A) \rightarrow \mathfrak{G}(B)$ admits a right adjoint in the 2-category $\mathcal{I}\text{SMSt}$.*

Proof. Choose a decomposition $B \simeq \bigsqcup_{i=1}^n G/H_i$ and write u through this decomposition as $\bigsqcup_{i=1}^n \mathcal{P}_{T_i}$ with $\mathcal{P}_{T_i} : \text{Ind}_{H_i}^G T_i \rightarrow G/H_i$ the canonical projection associated with an admissible finite H_i -set T_i . In the 2-category $\mathcal{I}\text{SMStg}$, the strict $\mathcal{I}$ -normed functor $\mathfrak{G}(u)$ admits a right adjoint $\mathfrak{G}(u)^*$ given by the strong $\mathcal{I}$ -normed functor $\mathfrak{G}(\mathcal{P}_{T_1})^* \otimes \cdots \otimes \mathfrak{G}(\mathcal{P}_{T_n})^*$. Let us now show that the strong $\mathcal{I}$ -normed functor $\mathfrak{G}(u)^*$ is isomorphic to a strict $\mathcal{I}$ -normed functor. Every $\mathfrak{G}(\mathcal{P}_{T_i})^*$ is a strict $\mathcal{I}$ -normed functor provided as the image of some G -functor under $\mathcal{G}$ and it is enough to show, given two G -functors $\phi, \psi : A \rightrightarrows \mathfrak{G}(B)$, that the strong $\mathcal{I}$ -normed functor $\mathcal{G}(\phi) \otimes \mathcal{G}(\psi)$ is isomorphic to a strict $\mathcal{I}$ -normed functor. More precisely, a direct induction shows that the strong $\mathcal{I}$ -normed functor $\mathcal{G}(\phi) \otimes \mathcal{G}(\psi)$ is isomorphic to the image under $\mathcal{G}$ of the G -functor given by the composition $A \xrightarrow{(\phi, \psi)} \mathfrak{G}(B) \times \mathfrak{G}(B) \xrightarrow{\otimes} \mathfrak{G}(B)$. The result follows. $\square$

2.4.4 The proof of the main Theorem

In this section, we prove Theorem 2.A. We have morphisms of simplicial sets

$$\mathcal{A}_{\mathcal{I}} \xrightarrow{\mathcal{G}} N_{\Delta}(\mathcal{I}\text{SMSt}_{(2,1)}) \xrightarrow{N_{\Delta}(\iota)} N_{\Delta}(\mathcal{I}\text{SMSt}) \hookrightarrow N_{\Delta}(\text{Alg}_{\mathcal{E}_{\mathcal{I}}}(\text{SSet}^G))$$

where $\iota : \mathcal{I}\text{SMSt}_{(2,1)} \rightarrow \mathcal{I}\text{SMSt}$ denotes the strict 2-functor that sends an $\mathcal{I}$ -normed symmetric monoidal category to its core and where the third morphism is given by Corollary 2.3.24. We show now that the composition of these morphisms factors as a functor $\theta : \mathcal{A}_{\mathcal{I}} \rightarrow \text{Alg}_{\mathcal{E}_{\mathcal{I}}}$. This composition sends a finite G -set A to the nerve of the core of the $\mathcal{I}$ -normed symmetric monoidal category $\mathfrak{G}(A)$ and we have to show that this object is fibrant and cofibrant in $\text{Alg}_{\mathcal{E}_{\mathcal{I}}}(\text{SSet}^G)$. If we denote by $\mathcal{F}(A)$ the core of $\mathfrak{G}(A)$, the fibrancy of $N(\mathcal{F}(A))$ follows from the fact that its underlying G -simplicial set is the nerve of a G -groupoid. To show that $N(\mathcal{F}(A))$ is cofibrant, we show that it is the free $\mathcal{E}_{\mathcal{I}}$ -algebra over A . We show first that $\mathcal{F}(A)$ is the free $\mathcal{I}$ -normed symmetric monoidal category over A seen as a discrete G -category. We start with the following elementary lemma.

Lemma 2.4.28. *Let Γ be a discrete group and let X and Y be two $G \times \Gamma$ -sets such that the Γ -action on X is free. The canonical G -equivariant map*

$$f : (X^{n+1} \times Y)_{\Gamma} \rightarrow \underbrace{(X \times X \times Y)_{\Gamma} \times_{(X \times Y)_{\Gamma}} \cdots \times_{(X \times Y)_{\Gamma}} (X \times X \times Y)_{\Gamma}}_{n \text{ times}}$$

is a bijection.

Proof. We prove first that f is injective. Consider two tuples $A = (x_1, \dots, x_{n+1}, y)$ and $B = (x'_1, \dots, x'_{n+1}, y')$ in $(X^{n+1} \times Y)$ with the same image under f . This means there are elements $\gamma_1, \dots, \gamma_n$ in Γ such that for all $i = 1, \dots, n$, we have

$$\gamma_i(x_i, x_{i+1}, y) = (x'_i, x'_{i+1}, y').$$

In particular, it follows that $\gamma_i x_{i+1} = x'_{i+1}$ and $\gamma_{i+1} x_{i+1} = x'_{i+1}$ and we deduce that $\gamma_i = \gamma_{i+1}$ using that the Γ -action on X is free. This implies that A and B represent the same element of $(X^{n+1} \times Y)_\Gamma$. We prove now that f is surjective. Consider an element A of

$$\underbrace{(X \times X \times Y)_\Gamma \times_{(X \times Y)_\Gamma} \cdots \times_{(X \times Y)_\Gamma} (X \times X \times Y)_\Gamma}_{n \text{ times}}$$

represented by tuples (x_i, z_i, y_i) for all $i = 1, \dots, n$. By assumptions, there exists γ_i in Γ such that $z_i = \gamma_i x_{i+1}$ and $\gamma_i y_{i+1} = y_i$ for all $i = 1, \dots, n-1$. For all $i = 1, \dots, n+1$, consider the element

$$w_i = \begin{cases} x_1 & \text{if } i = 1; \\ z_1 & \text{if } i = 2; \\ (\gamma_1 \gamma_2 \cdots \gamma_{i-2}) \cdot z_{i-1} & \text{if } i = 3, \dots, n+1. \end{cases}$$

Note that we have $(w_i, w_{i+1}, y_1) = (\gamma_1 \gamma_2 \cdots \gamma_{i-1}) \cdot (x_i, z_i, y_i)$ and, in particular, the image of the orbit of $(w_1, \dots, w_{n+1}, y_1)$ under f is A . □

One important direct consequence of Lemma 2.4.28 is the following result.

Proposition 2.4.29. *Let Γ , X and Y be as in Lemma 2.4.28. The G -category $(\tilde{X} \times Y)_\Gamma$ is the G -category whose G -set of objects is $(X \times Y)_\Gamma$, whose G -set of morphisms is $(X \times X \times Y)_\Gamma$ and whose composition is given by the G -equivariant map*

$$(X \times X \times Y)_\Gamma \times_{(X \times Y)_\Gamma} (X \times X \times Y)_\Gamma \simeq (X \times X \times X \times Y)_\Gamma \xrightarrow{((p_1, p_3) \times \text{id}_Y)_\Gamma} (X \times X \times Y)_\Gamma.$$

Using this proposition, one can prove that $\mathcal{F}(A)$ is the free $\mathcal{I}$ -normed symmetric monoidal category on A .

Proposition 2.4.30. *The $\mathcal{I}$ -normed symmetric monoidal category $\mathcal{F}(A)$ is free over A for strict $\mathcal{I}$ -normed functors. Moreover, for every $\mathcal{I}$ -normed symmetric monoidal category $\mathcal{C}$, the G -functor $A \rightarrow \mathcal{F}(A)$ induces an equivalence of categories*

$$\mathcal{ISMSt}(\mathcal{F}(A), \mathcal{C}) \simeq \text{Fun}_G(A, \mathcal{C}).$$

Proof. By Theorem 2.3.22, it is enough to show that $\mathcal{F}(A)$ is the $\mathcal{SM}_{\mathcal{I}}$ -algebra given by the image of A under the monad $\mathcal{SM}_{\mathcal{I}} : \text{Cat}^G \rightarrow \text{Cat}^G$. Given a

G -category $\mathcal{C}$, if we denote respectively by $\text{Ob}(\mathcal{C})$ and $\text{Mor}(\mathcal{C})$ the associated G -sets of objects and morphisms, then, using Proposition 2.4.29, we have

$$\text{Ob}\left(\bigsqcup_{n \geq 0} (\mathcal{SM}_{\mathcal{I}})_n \times_{\Sigma_n} A^n\right) \simeq \bigsqcup_{n \geq 0} F(S_{\mathcal{I}})_n \times_{\Sigma_n} A^n \simeq \bigsqcup_{n \geq 0} F(V_{\mathcal{I}})_n \times A^n \simeq \text{Ob}(\mathcal{F}(A))$$

and

$$\begin{aligned} \text{Mor}\left(\bigsqcup_{n \geq 0} (\mathcal{SM}_{\mathcal{I}})_n \times_{\Sigma_n} A^n\right) &\simeq \bigsqcup_{n \geq 0} F(S_{\mathcal{I}})_n^2 \times_{\Sigma_n} A^n \simeq \bigsqcup_{n \geq 0} F(V_{\mathcal{I}})_n^2 \times A^n \times \Sigma_n \\ &\simeq \text{Mor}(\mathcal{F}(A)), \end{aligned}$$

with $F(V_{\mathcal{I}})$ defined in Subsection 2.3.2. It is clear that these isomorphisms induce an isomorphism $\mathcal{F}(A) \simeq \mathcal{SM}_{\mathcal{I}}(A)$. The equivalence of categories

$$\mathcal{ISMSt}(\mathbf{F}(A), \mathcal{C}) \simeq \text{Fun}_G(A, \mathcal{C})$$

is a consequence of the freeness of $\mathcal{F}(A)$ and of the description of $\mathcal{ISMSt}(\mathcal{F}(A), \mathcal{C})$ in terms of the arrow category of $\mathcal{C}$ given by Proposition 2.5.7. $\square$

Remark 2.4.31. Given a G -functor $\phi : A \rightarrow \mathfrak{G}(B)$, the core of the strict $\mathcal{I}$ -normed functor $\mathcal{G}(\phi)$ is the strict $\mathcal{I}$ -normed functor $\mathcal{F}(A) \rightarrow \mathcal{F}(B)$ induced by $\phi : A \rightarrow \mathcal{F}(B)$ through the equivalence $\mathcal{ISMSt}(\mathcal{F}(A), \mathcal{F}(B)) \simeq \text{Fun}_G(A, \mathcal{F}(B))$ of Proposition 2.4.30.

The following lemma is the key point to transfer the freeness of $\mathcal{F}(A)$ as an $\mathcal{I}$ -normed symmetric monoidal category to the freeness of $N(\mathcal{F}(A))$ as an $\mathcal{E}_{\mathcal{I}}$ -algebra.

Lemma 2.4.32. *Let Γ , X and Y be as in Lemma 2.4.28. The natural G -equivariant map*

$$N(\widetilde{X} \times Y)/\Gamma \rightarrow N((\widetilde{X} \times Y)/\Gamma)$$

is an isomorphism of G -simplicial sets where $\widetilde{X}$ is the translation G -groupoid of X , as defined in Construction 2.3.9.

Proof. This is a direct consequence of Lemma 2.4.28. $\square$

Corollary 2.4.33. *Let A be a finite G -set. The nerve of $\mathcal{F}(A)$ is the free N_{∞} -algebra over A .*

Proof. We have to show that the nerve of $\mathcal{F}(A)$ is the image of A by the monad $\mathcal{E}_{\mathcal{I}} : \text{SSet}^G \rightarrow \text{SSet}^G$. We proved in Proposition 2.4.30 that $\mathcal{F}(A)$ is the free $\mathcal{I}$ -normed symmetric monoidal category over A and by definition $(\mathcal{E}_{\mathcal{I}})_n = N((\mathcal{SM}_{\mathcal{I}})_n)$. It is then enough to show that the canonical morphism

$$\bigsqcup_{n \geq 0} A^n \times_{\Sigma_n} N((\mathcal{SM}_{\mathcal{I}})_n) \rightarrow N\left(\bigsqcup_{n \geq 0} A^n \times_{\Sigma_n} (\mathcal{SM}_{\mathcal{I}})_n\right)$$

is an isomorphism. This is a consequence of the fact that the nerve preserves finite coproducts and of the previous lemma using that $(\mathcal{SM}_{\mathcal{I}})_n = F(\widetilde{S}_{\mathcal{I}})_n$. $\square$

If A is a finite G -set, then the $\mathcal{E}_T$ -algebra $N(\mathcal{F}(A))$ is fibrant and cofibrant and we get a functor $\theta : \mathcal{A}_T \rightarrow \mathbf{Alg}_{\mathcal{E}_T}$ that sends A to $N(\mathcal{F}(A))$. We will show now that this functor is fully faithful. For every subgroup $H \subseteq G$ and finite G -set A , there are equivalences

$$\mathcal{ISMS}\mathbf{t}(\mathcal{F}(G/H), \mathcal{F}(A)) \simeq \mathbf{Fun}_G(G/H, \mathcal{F}(A)) \simeq \mathcal{F}(A)^H$$

and the latter groupoid can be described by the following proposition.

Proposition 2.4.34. *For every finite G -set A , there is an equivalence of categories*

$$\mathcal{I}(H)_{/\mathrm{Res}_H^G A} \simeq \mathfrak{G}(A)^H.$$

Proof. Let $u : T \rightarrow \mathrm{Res}_H^G A$ be an H -equivariant map with T an admissible H -set. Consider the H -fixed object $(\bigotimes_T, (u(t_1), \dots, u(t_{|T|})))$ of $\mathfrak{G}(A)$ where $T = \{t_1, \dots, t_{|T|}\}$ is given by the chosen ordering on T . If $w : u \rightarrow v$ is a morphism in $\mathcal{I}(H)_{/\mathrm{Res}_H^G A}$ between two H -equivariant maps $u : T \rightarrow \mathrm{Res}_H^G A$ and $v : S \rightarrow \mathrm{Res}_H^G A$, we get an H -fixed morphism $(\bigotimes_T, (u(t_1), \dots, u(t_{|T|}))) \rightarrow (\bigotimes_S, (v(s_1), \dots, v(s_{|S|})))$ given by the map $w : \langle |T| \rangle \rightarrow \langle |S| \rangle$ that corresponds to $w : T \rightarrow S$ through the chosen orderings $T = \{t_1, \dots, t_{|T|}\}$ and $S = \{s_1, \dots, s_{|S|}\}$. We thus get a functor $\mathcal{I}(H)_{/\mathrm{Res}_H^G A} \rightarrow \mathfrak{G}(A)^H$ which is fully faithful. Given now an H -fixed object $(\theta, (a_1, \dots, a_{|\theta|}))$ of $\mathfrak{G}(A)^H$, the identity $\mathrm{id}_{\langle |\theta| \rangle} : \langle |\theta| \rangle \rightarrow \langle |\theta| \rangle$ induces an H -fixed isomorphism $(\bigotimes_{T_\theta}, (u(t_1), \dots, u(t_{|\theta|}))) \rightarrow (\theta, (a_1, \dots, a_{|\theta|}))$ where $u : T_\theta \rightarrow A$ is the H -equivariant map that sends t_i to a_i . We conclude that the functor $\mathcal{I}(H)_{/\mathrm{Res}_H^G A} \rightarrow \mathfrak{G}(A)^H$ is essentially surjective and the result follows. $\square$

The $\mathcal{I}$ -normed symmetric monoidal category $\mathcal{F}(A)$ is by definition the core of $\mathfrak{G}(A)$ and we deduce the following corollary.

Corollary 2.4.35. *For every finite G -set A , there is an equivalence of groupoids*

$$\iota(\mathcal{I}(H)_{/\mathrm{Res}_H^G A}) \simeq \mathcal{F}(A)^H.$$

We can now use these descriptions to show the following result.

Proposition 2.4.36. *The functor $\theta : \mathcal{A}_T \rightarrow \mathbf{Alg}_{\mathcal{E}_T}$ is fully faithful.*

Proof. By definition, the functor $\theta : \mathcal{A}_T \rightarrow \mathbf{Alg}_{\mathcal{E}_T}$ is the composition of morphisms of simplicial sets

$$\mathcal{A}_T \xrightarrow{\mathcal{G}} N_\Delta(\mathcal{ISMS}\mathbf{t}_{(2,1)}) \xrightarrow{N_\Delta(\iota)} N_\Delta(\mathcal{ISMS}\mathbf{t}) \hookrightarrow N_\Delta(\mathbf{Alg}_{\mathcal{E}_T}(\mathbf{SSet}^G))$$

which factors through the inclusion $\mathbf{Alg}_{\mathcal{E}_T} \hookrightarrow N_\Delta(\mathbf{Alg}_{\mathcal{E}_T}(\mathbf{SSet}^G))$. By Proposition 2.5.8, the simplicial functor $\mathcal{ISMS}\mathbf{t} \hookrightarrow \mathbf{Alg}_{\mathcal{E}_T}(\mathbf{SSet}^G)$ is fully faithful and it is enough to show that the functor $N_\Delta(\iota) \circ \mathcal{G} : \mathcal{A}_T \rightarrow N_\Delta(\mathcal{ISMS}\mathbf{t})$ is fully

faithful. The ∞ -category $\mathcal{A}_I$ is a $(2,1)$ -category and, if A and B are finite G -sets, the groupoid $\text{Hom}_{\mathcal{A}_I}(A, B)$ can be described as follow: the objects in $\text{Hom}_{\mathcal{A}_I}(A, B)$ are spans $A \xleftarrow{f} C \xrightarrow{g} B$ and morphisms are isomorphisms of spans

$$\begin{array}{ccccc} & & C & & \\ & \swarrow f & \downarrow \simeq & \searrow g & \\ A & & & & B \\ & \nwarrow h & \downarrow & \nearrow i & \\ & & D & & \end{array}$$

Seen as a functor $\mathcal{A}_I \rightarrow N_\Delta(\mathcal{I}\mathbf{SMStg})$, the functor $N_\Delta(\iota) \circ \mathcal{G}$ is semi-additive by Proposition 2.4.15 and it is enough to show, given a subgroup $H \subseteq G$ and a finite G -set A , that $(N_\Delta(\iota) \circ \mathcal{G})_{G/H, A} : \text{Hom}_{\mathcal{A}_I}(G/H, A) \rightarrow \text{Hom}_{N_\Delta(\mathcal{I}\mathbf{SMSt})}(\mathcal{F}(G/H), \mathcal{F}(A))$ is an equivalence. Every span in $\text{Hom}_{\mathcal{A}_I}(G/H, A)$ is isomorphic to a span of the form $G/H \xleftarrow{\mathcal{P}_T} \text{Ind}_H^G T \xrightarrow{u} A$ with T an admissible H -set and there is an equivalence of groupoids $\text{Hom}_{\mathcal{A}_I}(G/H, A) \simeq \iota(\mathcal{I}(H)_{/\text{Res}_H^G A})$ that sends an H -equivariant map $u : T \rightarrow A$ to the span $G/H \xleftarrow{\mathcal{P}_T} \text{Ind}_H^G T \xrightarrow{u} A$ with $v : \text{Ind}_H^G T \rightarrow A$ the G -equivariant map that corresponds to u by adjunction. By definition, the functor $(N_\Delta(\iota) \circ \mathcal{G})_{G/H, A} : \text{Hom}_{\mathcal{A}_I}(G/H, A) \rightarrow \text{Hom}_{N_\Delta(\mathcal{I}\mathbf{SMSt})}(\mathcal{F}(G/H), \mathcal{F}(A))$ sends the span $G/H \xleftarrow{\mathcal{P}_T} \text{Ind}_H^G T \xrightarrow{u} A$ to the composition $\mathfrak{G}(v)\mathfrak{G}(\mathcal{P}_T)^*$ which is the image under $\mathcal{G}$ of the G -functor $G/H \rightarrow \mathcal{F}(A)$ that corresponds to the H -fixed object $(\bigotimes_T, (u(t_1), \dots, u(t_{|T|})))$. Moreover, the functor $(N_\Delta(\iota) \circ \mathcal{G})_{G/H, A} : \text{Hom}_{\mathcal{A}_I}(G/H, A) \rightarrow \text{Hom}_{N_\Delta(\mathcal{I}\mathbf{SMSt})}(\mathcal{F}(G/H), \mathcal{F}(A))$ sends an isomorphism of spans

$$\begin{array}{ccccc} & & \text{Ind}_H^G T & & \\ & \swarrow \mathcal{P}_T & \downarrow \text{Ind}_H^G w & \searrow u & \\ G/H & & & & A \\ & \nwarrow \mathcal{P}_S & \downarrow & \nearrow v & \\ & & \text{Ind}_H^G S & & \end{array}$$

to the I -normed natural transformation

$$\mathfrak{G}(u)\mathfrak{G}(\mathcal{P}_T)^* \simeq \mathfrak{G}(v)\mathfrak{G}(\text{Ind}_H^G w)\mathfrak{G}(\text{Ind}_H^G w^{-1})\mathfrak{G}(\mathcal{P}_S)^* = \mathfrak{G}(v)\mathfrak{G}(\mathcal{P}_S)^*$$

which corresponds through the equivalence $\mathcal{I}\mathbf{SMSt}(\mathcal{F}(G/H), \mathcal{F}(A)) \simeq \mathcal{F}(A)^H$ to the isomorphism

$$\bigotimes_T (u(t_1), \dots, u(t_{|T|})) \rightarrow \bigotimes_S (v(s_1), \dots, v(s_{|S|}))$$

given by $w : T \rightarrow S$ seen as a map $w : \langle |T| \rangle \rightarrow \langle |S| \rangle$. Finally, it follows that the functor $(N_\Delta(\iota) \circ \mathcal{G})_{G/H, A} : \text{Hom}_{\mathcal{A}_I}(G/H, A) \rightarrow \text{Hom}_{N_\Delta(\mathcal{I}\mathbf{SMSt})}(\mathcal{F}(G/H), \mathcal{F}(A))$ corresponds to the equivalences

$$\text{Hom}_{\mathcal{A}_I}(G/H, A) \simeq \iota(\mathcal{I}(H)_{/\text{Res}_H^G A}) \simeq \mathcal{F}(A)^H \simeq \mathcal{I}\mathbf{SMSt}(\mathcal{F}(G/H), \mathcal{F}(A)).$$

□

We can finally prove our main theorem.

Theorem 2.4.37. *The left Kan extension $\Theta : \text{Mack}_{\mathcal{I}} \rightarrow \mathbf{Alg}_{\mathcal{E}_{\mathcal{I}}}$ of $\theta : \mathcal{A}_{\mathcal{I}} \rightarrow \mathbf{Alg}_{\mathcal{E}_{\mathcal{I}}}$ along the Yoneda embedding $\mathcal{A}_{\mathcal{I}} \rightarrow \text{Mack}_{\mathcal{I}}$ is an equivalence of ∞ -categories.*

Proof. Combining Corollary 3.3.22 with Elmendorf's Theorem, we obtain that the free-forgetful adjunction

$$F : \mathcal{S}^G \rightleftarrows \mathbf{Alg}_{\mathcal{E}_{\mathcal{I}}} : U$$

satisfies the conditions of [Lur17, Corollary 4.7.3.18]. It follows that $\mathbf{Alg}_{\mathcal{E}_{\mathcal{I}}}$ is projectively generated and that the $\mathcal{E}_{\mathcal{I}}$ -algebras $N(\mathcal{F}(A))$ for all finite G -sets A form a set of compact projective generators. Finally, it follows from Proposition 2.4.36 and [Lur09, Proposition 5.5.8.22] that the left Kan extension $\Theta : \text{Mack}_{\mathcal{I}} \rightarrow \mathbf{Alg}_{\mathcal{E}_{\mathcal{I}}}$ of $\theta : \mathcal{A}_{\mathcal{I}} \rightarrow \mathbf{Alg}_{\mathcal{E}_{\mathcal{I}}}$ is an equivalence of ∞ -categories. $\square$

Remark 2.4.38. By definition, the inverse of the functor $\Theta : \text{Mack}_{\mathcal{I}} \rightarrow \mathbf{Alg}_{\mathcal{E}_{\mathcal{I}}}$ sends a (fibrant) N_∞ -algebra X to the $\mathcal{I}$ -Mackey functor $\mathcal{M}_X : \mathcal{A}_{\mathcal{I}}^{\text{op}} \rightarrow \mathcal{S}$ that sends G/H to the space

$$\text{Hom}_{\mathbf{Alg}_{\mathcal{E}_{\mathcal{I}}}}(N(\mathcal{F}(G/H)), X) \simeq X^H.$$

If $K \subseteq H$ are subgroups of G , then the functor $\text{Res}_K^H : X^H \rightarrow X^K$ obtained as the image of the span $G/K \xleftarrow{\text{id}_{G/K}} G/K \xrightarrow{u} G/H$ by $\mathcal{M}_X$ corresponds to the canonical inclusion $X^H \hookrightarrow X^K$. If H/K is admissible, then the functor $\text{Tr}_K^H : X^K \rightarrow X^H$ obtained as the image of the span $G/H \xleftarrow{u} G/K \xrightarrow{\text{id}_{G/K}} G/K$ by $\mathcal{M}_X$ sends a n -simplex x of X^K to the n -simplex $\bigotimes_{H/K}^X (h_1 x, \dots, h_{[H:K]} x)$ of X^H where $h_1, \dots, h_{[H:K]}$ are representatives of H/K and with $\bigotimes_{H/K}^X : X^{\times[H:K]} \rightarrow X$ the image of $\bigotimes_{H/K}$ by the morphism of operads $|-|_X : \mathcal{E}_{\mathcal{I}} \rightarrow \text{End}_X$ that corresponds to the $\mathcal{E}_{\mathcal{I}}$ -algebra structure of X .

2.5 Appendix: Operads in G -categories

In this appendix, we recall the definitions of the 2-categories of $\mathcal{O}$ -algebras over an operad $\mathcal{O}$ in Cat^G . The goal is to prove Proposition 2.5.8, which is used to prove Proposition 2.4.36. We follow [Rub20, Section 4] for the definitions and refer the reader to it for more details.

Notation 2.5.1. If $\mathcal{O}$ is an operad in Cat^G and $\mathcal{C}$ an $\mathcal{O}$ -algebra in Cat^G , then we will denote by $|-|_{\mathcal{C}} : \mathcal{O} \rightarrow \text{End}_{\mathcal{C}}$ the morphism of operads in Cat^G that corresponds to the structure of $\mathcal{O}$ -algebra of $\mathcal{C}$.

Definition 2.5.2 ([Rub20, Definition 4.2]). Let $\mathcal{O}$ be an operad in Cat^G and $\mathcal{C}$ and $\mathcal{D}$ two $\mathcal{O}$ -algebras in Cat^G . A *lax $\mathcal{O}$ -algebra morphism* $\phi : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

1. a G -functor $\phi : \mathcal{C} \rightarrow \mathcal{D}$;

2. for every $n \geq 0$ and $x \in \mathcal{O}_n$, a natural transformation $\phi_x : |x|_{\mathcal{D}} \circ \phi^{\times n} \Rightarrow \phi \circ |x|_{\mathcal{C}}$ of functors $\mathcal{C}^{\times n} \rightarrow \mathcal{D}$;

such that the following conditions are satisfied:

1. for every $n \geq 0$, the maps ϕ_x vary naturally in $x \in \mathcal{O}_n$;
2. for every $n \geq 0$, $x \in \mathcal{O}_n$ and $(g, \sigma) \in G \times \Sigma_n$, the equation $(g, \sigma) \cdot \phi_x = \phi_{(g, \sigma) \cdot x}$ holds, ie, ϕ_x is $(G \times \Sigma_n)$ -equivariant;
3. $\phi_{\text{id}_{\mathcal{O}}} = \text{id}_{\phi}$;
4. for every $y \in \mathcal{O}_n$ and $x_i \in \mathcal{O}_{k_i}$, the transformation $\phi_{\gamma(y, x_1, \dots, x_n)}$ is equal to the composite

$$\begin{aligned}
 & |y|_{\mathcal{D}} \circ \left((|x_1|_{\mathcal{D}} \circ \phi^{\times k_1}) \times \dots \times (|x_n|_{\mathcal{D}} \circ \phi^{\times k_n}) \right) \\
 & \quad \Downarrow \text{id}_{|y|_{\mathcal{D}}} \circ (\phi_{x_1} \times \dots \times \phi_{x_n}) \\
 & (|y|_{\mathcal{D}} \circ \phi^{\times n}) \circ (|x_1|_{\mathcal{C}} \times \dots \times |x_n|_{\mathcal{C}}) \\
 & \quad \Downarrow \phi_y \circ (\text{id}_{|x_1|_{\mathcal{C}}} \times \dots \times \text{id}_{|x_n|_{\mathcal{C}}}) \\
 & \phi \circ |y|_{\mathcal{C}} \circ (|x_1|_{\mathcal{C}} \times \dots \times |x_n|_{\mathcal{C}}).
 \end{aligned}$$

A *strong $\mathcal{O}$ -algebra morphism* (resp. *strict $\mathcal{O}$ -algebra morphism*) is a lax $\mathcal{O}$ -algebra morphism $\phi : \mathcal{C} \rightarrow \mathcal{D}$ such that ϕ_x is an isomorphism (resp. the identity) for every $n \geq 0$ and $x \in \mathcal{O}_n$.

If $\phi : \mathcal{C} \rightarrow \mathcal{D}$ and $\psi : \mathcal{D} \rightarrow \mathcal{E}$ are two lax $\mathcal{O}$ -algebra morphisms, the composition $\psi \circ \phi$ has a natural structure of lax $\mathcal{O}$ -algebra morphism where for every $n \geq 0$ and $x \in \mathcal{O}_n$, the natural transformation $(\psi \circ \phi)_x$ is given by the composition

$$|x|_{\mathcal{E}} \circ (\psi \circ \phi)^{\times n} = |x|_{\mathcal{E}} \circ \psi^{\times n} \circ \phi^{\times n} \xrightarrow{\psi_x \circ (\text{id}_{\phi^{\times n}})} \psi \circ |x|_{\mathcal{D}} \circ \phi^{\times n} \xrightarrow{\text{id}_{\psi} \circ \phi_x} \psi \circ \phi \circ |x|_{\mathcal{C}}.$$

If ϕ and ψ are strong $\mathcal{O}$ -algebra morphisms (resp. strict), their composition $\psi \circ \phi$ is also a strong $\mathcal{O}$ -algebra morphism (resp. strict).

Definition 2.5.3 ([Rub20, Definition 4.3]). Let $\mathcal{O}$ be an operad in Cat^G and $\phi, \psi : \mathcal{C} \rightrightarrows \mathcal{D}$ a pair of lax $\mathcal{O}$ -algebra morphisms. An *$\mathcal{O}$ -natural transformation* $\omega : \phi \Rightarrow \psi$ is a G -natural transformation $\omega : \phi \Rightarrow \psi$ such that for every $n \geq 0$ and $x \in \mathcal{O}_n$, the following diagram of natural transformations commutes

$$\begin{array}{ccc}
 |x|_{\mathcal{D}} \circ \phi^{\times n} & \xrightarrow{\text{id}_{|x|_{\mathcal{D}}} \circ \omega^{\times n}} & |x|_{\mathcal{D}} \circ \psi^{\times n} \\
 \phi_x \Downarrow & & \Downarrow \psi_x \\
 \phi \circ |x|_{\mathcal{C}} & \xrightarrow{\omega \circ \text{id}_{|x|_{\mathcal{C}}}} & \psi \circ |x|_{\mathcal{C}}.
 \end{array}$$

Notation 2.5.4. We denote by $\mathbf{AlgLax}_\mathcal{O}$ (resp. $\mathbf{AlgStg}_\mathcal{O}$ and $\mathbf{AlgSt}_\mathcal{O}$) the 2-category of $\mathcal{O}$ -algebras in $\mathbf{Cat}^G$, lax $\mathcal{O}$ -algebra morphisms (resp. strong and strict) and $\mathcal{O}$ -natural transformations.

Let $\mathcal{O}$ be an operad in $\mathbf{Cat}^G$, $\mathcal{C}$ an $\mathcal{O}$ -algebra in $\mathbf{Cat}^G$ and $\mathcal{D}$ a G -category. The functor $\mathrm{Fun}(\mathcal{D}, (-)) : \mathbf{Cat}^G \rightarrow \mathbf{Cat}^G$ preserves products and $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ is naturally a $\mathrm{Fun}(\mathcal{D}, \mathcal{O})$ -algebra. It follows that the G -category of nonequivariant functors $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ is naturally endowed with the structure of an $\mathcal{O}$ -algebra where the morphism of G -operads $|-|_{\mathrm{Fun}(\mathcal{D}, \mathcal{C})} : \mathcal{O} \rightarrow \mathrm{End}_{\mathrm{Fun}(\mathcal{D}, \mathcal{C})}$ is given by the composition

$$\mathcal{O} \rightarrow \mathrm{Fun}(\mathcal{D}, \mathcal{O}) \xrightarrow{|-|_{\mathrm{Fun}(\mathcal{C}, \mathcal{D})}} \mathrm{End}_{\mathrm{Fun}(\mathcal{D}, \mathcal{C})}.$$

Here the first morphism $\mathcal{O} \simeq \mathrm{Fun}(*, \mathcal{O}) \rightarrow \mathrm{Fun}(\mathcal{D}, \mathcal{O})$ is induced by the unique functor $\mathcal{D} \rightarrow *$. In particular, the arrow category $\mathrm{Arr}(\mathcal{C}) = \mathrm{Fun}(\Delta^1, \mathcal{C})$ of $\mathcal{C}$ has a natural structure of an $\mathcal{O}$ -algebra.

Remark 2.5.5. If $\mathcal{O}$ is an operad in $\mathbf{Cat}^G$, $\mathcal{C}$ an $\mathcal{O}$ -algebra in $\mathbf{Cat}^G$ and $\mathcal{D}$ a G -category, then, for every $n \geq 0$ and $x \in \mathcal{O}_n$, the operation $|x|_{\mathrm{Fun}(\mathcal{D}, \mathcal{C})}$ is given by the composition $\mathrm{Fun}(\mathcal{D}, \mathcal{C})^{\times n} \xrightarrow{x} \mathrm{Fun}(\mathcal{D}^{\times n}, \mathcal{C}^{\times n}) \xrightarrow{(-) \circ |x|_\mathcal{C}} \mathrm{Fun}(\mathcal{D}, \mathcal{C})$.

If $\mathcal{C}$ and $\mathcal{D}$ are two G -categories, the data of a G -natural transformation $\omega : \phi \Rightarrow \psi$ between two G -functors $\phi, \psi : \mathcal{C} \rightrightarrows \mathcal{D}$ is equivalent to the data of a G -functor $\Omega : \mathcal{C} \rightarrow \mathrm{Arr}(\mathcal{D})$. This characterisation of G -natural transformations lifts to the case of $\mathcal{O}$ -natural transformations.

Proposition 2.5.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathcal{O}$ -algebras in $\mathbf{Cat}^G$. The data of an $\mathcal{O}$ -natural transformation $\omega : \phi \Rightarrow \psi$ between two lax $\mathcal{O}$ -algebra morphisms (resp. strong or strict) $\phi, \psi : \mathcal{C} \rightrightarrows \mathcal{D}$ is equivalent to the data of a lax $\mathcal{O}$ -algebra morphism (resp. strong or strict) $\Omega : \mathcal{C} \rightarrow \mathrm{Arr}(\mathcal{D})$.*

Proof. Let $\omega : \phi \Rightarrow \psi$ be an $\mathcal{O}$ -natural transformation between two lax $\mathcal{O}$ -algebra morphisms (resp. strong or strict) $\phi, \psi : \mathcal{C} \rightrightarrows \mathcal{D}$. The G -functor $\Omega : \mathcal{C} \rightarrow \mathrm{Arr}(\mathcal{D})$ associated with the G -natural transformation ω admits a structure of lax $\mathcal{O}$ -algebra morphism (resp. strong or strict) where for $n \geq 0$ and $x \in \mathcal{O}_n$, the natural transformation $\Omega_x : |x|_{\mathrm{Arr}(\mathcal{D})} \circ \Omega^{\times n} \Rightarrow \Omega \circ |x|_\mathcal{C}$ is given by the diagram

$$\begin{array}{ccc} |x|_\mathcal{D} \circ \phi^{\times n} & \xrightarrow{\mathrm{id}_{|x|_\mathcal{D}} \circ \omega^{\times n}} & |x|_\mathcal{D} \circ \psi^{\times n} \\ \phi_x \downarrow & & \downarrow \psi_x \\ \phi \circ |x|_\mathcal{C} & \xrightarrow{\omega \circ \mathrm{id}_{|x|_\mathcal{C}}} & \psi \circ |x|_\mathcal{C} \end{array}$$

seen as a morphism in the category $\mathrm{Fun}(\mathcal{C}^{\times n}, \mathrm{Arr}(\mathcal{D})) \simeq \mathrm{Arr}(\mathrm{Fun}(\mathcal{C}^{\times n}, \mathcal{D}))$, which is commutative by assumption using that ω is an $\mathcal{O}$ -natural transformation. It follows directly from the fact that ϕ and ψ are lax $\mathcal{O}$ -algebra morphisms that Ω is a lax $\mathcal{O}$ -algebra morphism (resp. strong or strict). Assume now that $\Omega : \mathcal{C} \rightarrow \mathrm{Arr}(\mathcal{D})$ is a lax $\mathcal{O}$ -algebra morphism (resp. strong or strict) and consider the G -natural transformation $\omega : \phi \Rightarrow \psi$ associated with Ω seen as a G -functor. The G -functors ϕ and ψ are by definition respectively defined by

$d^0 \circ \Omega$ and $d^1 \circ \Omega$ where $d^0 : \text{Arr}(\mathcal{D}) \rightarrow \mathcal{D}$ and $d^1 : \text{Arr}(\mathcal{D}) \rightarrow \mathcal{D}$ are the strict $\mathcal{O}$ -algebra morphisms induced by the inclusions $\partial^0 : \Delta^0 \rightarrow \Delta^1$ and $\partial^1 : \Delta^0 \rightarrow \Delta^1$. It follows that ϕ and ψ are naturally lax $\mathcal{O}$ -algebra morphisms (resp. strong and strict). Finally, the natural transformation $\Omega_x : |x|_{\text{Arr}(\mathcal{D})} \circ \Omega^{\times n} \Rightarrow \Omega \circ |x|_{\mathcal{C}}$ seen as a morphism in $\text{Arr}(\text{Fun}(\mathcal{C}^{\times n}, \mathcal{D})) \simeq \text{Fun}(\mathcal{C}^{\times n}, \text{Arr}(\mathcal{D}))$ directly ensures that the diagram

$$\begin{array}{ccc} |x|_{\mathcal{D}} \circ \phi^{\times n} & \xrightarrow{\text{id}_{|x|_{\mathcal{D}}} \circ \omega^{\times n}} & |x|_{\mathcal{D}} \circ \psi^{\times n} \\ \phi_x \downarrow & & \downarrow \psi_x \\ \phi \circ |x|_{\mathcal{C}} & \xrightarrow{\omega \circ \text{id}_{|x|_{\mathcal{C}}}} & \psi \circ |x|_{\mathcal{C}} \end{array}$$

is commutative and it follows that ω is an $\mathcal{O}$ -natural transformation. The result follows. $\square$

If $\mathcal{O}$ is an operad in Cat^G and $\mathcal{C}$ and $\mathcal{D}$ two $\mathcal{O}$ -algebras in $\mathbf{Cat}^G$, then it follows directly from the previous proposition that the categories $\mathbf{AlgLax}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$, $\mathbf{AlgStg}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$ and $\mathbf{AlgSt}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$ admit the following description.

Proposition 2.5.7. *Let $\mathcal{O}$ be an operad in Cat^G and $\mathcal{C}$ and $\mathcal{D}$ two $\mathcal{O}$ -algebras in $\mathbf{Cat}^G$. If $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$ denotes either $\mathbf{AlgLax}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$, $\mathbf{AlgStg}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$ or $\mathbf{AlgSt}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$ and if $\text{Alg}_{\mathcal{O}}(\mathcal{C}, \mathcal{D}) = \text{Ob}(\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}, \mathcal{D}))$ denotes the corresponding class of G -functors, then the category given by the following diagram*

$$\text{Alg}_{\mathcal{O}}(\mathcal{C}, \text{Arr}(\mathcal{D})) \times_{\text{Alg}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})} \text{Alg}_{\mathcal{O}}(\mathcal{C}, \text{Arr}(\mathcal{D})) \rightarrow \text{Alg}_{\mathcal{O}}(\mathcal{C}, \text{Arr}(\mathcal{D})) \xrightleftharpoons{\quad} \text{Alg}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$$

induced by the canonical strict $\mathcal{O}$ -algebra morphisms $\text{Arr}(\mathcal{D}) \times_{\mathcal{D}} \text{Arr}(\mathcal{D}) \rightarrow \text{Arr}(\mathcal{D})$, $\text{Arr}(\mathcal{D}) \rightrightarrows \mathcal{D}$ and $\mathcal{D} \rightarrow \text{Arr}(\mathcal{D})$ is equivalent to $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$.

Finally, we prove the important result of this appendix.

Proposition 2.5.8. *There is a fully faithful simplicial functor $N : \mathbf{AlgSt}_{\mathcal{O}} \rightarrow \text{Alg}_{N(\mathcal{O})}(\mathbf{SSet}^G)$ that sends an $\mathcal{O}$ -algebra $\mathcal{C}$ in $\mathbf{Cat}^G$ to its nerve $N(\mathcal{C})$ where the strict 2-category $\mathbf{AlgSt}_{\mathcal{O}}$ is seen as a simplicial category.*

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathcal{O}$ -algebras in $\mathbf{Cat}^G$. It follows from Proposition 2.5.7 that we have bijections

$$\begin{aligned} & N(\mathbf{AlgSt}_{\mathcal{O}}(\mathcal{C}, \mathcal{D}))_n \\ &= \text{Alg}_{\mathcal{O}}(\mathcal{C}, \text{Arr}(\mathcal{D})) \times_{\text{Alg}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})} \cdots \times_{\text{Alg}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})} \text{Alg}_{\mathcal{O}}(\mathcal{C}, \text{Arr}(\mathcal{D})) \\ &\simeq \text{Alg}_{\mathcal{O}}(\mathcal{C}, \text{Arr}(\mathcal{D}) \times_{\mathcal{D}} \cdots \times_{\mathcal{D}} \text{Arr}(\mathcal{D})) \\ &\simeq \text{Alg}_{N(\mathcal{O})}(\mathbf{SSet}^G)(N(\mathcal{C}), N(\mathcal{D})^{\Delta^n}) \\ &= \text{Hom}_{\text{Alg}_{N(\mathcal{O})}(\mathbf{SSet}^G)}(N(\mathcal{C}), N(\mathcal{D}))_n \end{aligned}$$

which give an isomorphism of simplicial sets

$$N(\mathbf{AlgSt}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})) \simeq \text{Hom}_{\text{Alg}_{N(\mathcal{O})}(\mathbf{SSet}^G)}(N(\mathcal{C}), N(\mathcal{D})).$$

It is clear that these isomorphisms induce a fully faithful simplicial functor $\mathbf{AlgSt}_{\mathcal{O}} \rightarrow \text{Alg}_{N(\mathcal{O})}(\mathbf{SSet}^G)$. $\square$

Chapter 3

On sifted homotopy colimits of algebras over an N_∞ -operad

3.1 Introduction

In this chapter, we generalise different results of [BM03] and [PS18a] to N_∞ -operads. These results use the fact that the considered operad is Σ -cofibrant, in the sense of the following definition (we recall the definition specified to the case of G -simplicial sets).

Definition 3.1.1. An operad $\mathcal{O}$ in $\mathbf{sSet}^G$ is Σ -cofibrant if the morphism of collections $I \rightarrow \mathcal{O}$ corresponding to the identity of $\mathcal{O}$ is a cofibration of collections for the projective model structure.

Note however that N_∞ -operads are not Σ -cofibrant in $\mathbf{sSet}^G$.

Proposition 3.1.2. *Let $\mathcal{O}$ be an N_∞ -operad in $\mathbf{sSet}^G$. If the G -indexing system associated with $\mathcal{O}$ is not trivial, then $\mathcal{O}$ is not Σ -cofibrant.*

Proof. The projective model structure on $(\mathbf{sSet}^G)^{\Sigma_n}$ corresponds to the model structure on $\mathbf{sSet}^{G \times \Sigma_n}$ given by the family $\mathcal{F}$ of $G \times \Sigma_n$ spanned by the subgroups $H \times \{e\}$, with H a subgroup of G . If $\mathcal{O}$ is not a trivial N_∞ -operad, there exists a graph subgroup $\Gamma \subseteq G \times \Sigma_n$ for some n such that $\Gamma \notin \mathcal{F}$ and such that $\mathcal{O}_n^\Gamma \neq \emptyset$. Using [Ste16, Proposition 2.16], we deduce that $\mathcal{O}_n$ is not cofibrant in $\mathbf{sSet}^{G \times \Sigma_n}$ for the model structure corresponding to $\mathcal{F}$. In particular, this implies that $\mathcal{O}$ is not Σ -cofibrant. $\square$

It follows from Proposition 3.1.2 that the results of [BM03] and [PS18a] do not apply to N_∞ -operads. In fact, we prove in Section 3.2 that [BM03, Corollary 5.5] is false for algebras over an N_∞ -operad in a category of equivariant simplicial presheaves. This turns out to be problematic for the study of sifted homotopy colimits in N_∞ -algebras. In Section 3.3, we resolve this issue by replacing the property of Σ -cofibrancy by a weaker property of *graph Σ -cofibrancy* which is satisfied by N_∞ -operads. For graph Σ -cofibrant operads,

the results that we obtain are a bit weaker than the classical result, but are sufficient to prove that sifted homotopy colimits are computed at the level of the underlying equivariant spaces.

3.2 Counter example

Let I be a small category and consider the equivariant model structure on $\text{Fun}(I, \text{sSet}^G)$ obtained from the projective model structure on $\text{Fun}(I, \text{sSet})$. Note that this model structure exists using [Ste16, Lemma 2.12]. In this section, we prove that [BM03, Corollary 5.5] is false for N_∞ -operads.

Proposition 3.2.1. *For every cellular map f in $\text{Fun}(I, \text{sSet}^G)$, the map $f(n) \in \text{Fun}(I, \text{Set}^G)$ is of the form*

$$X \rightarrow X \sqcup \left(\bigsqcup_{j \in J} G/H_j \times \text{hom}(A_j, (-)) \right). \quad (3.1)$$

In particular, cellular objects are of the form

$$\bigsqcup_{j \in J} G/H_j \times \text{hom}(A_j, (-)).$$

Proof. Sequential colimits of maps of the form (3.1) also have this form and it is enough to show that for every pushout square

$$\begin{array}{ccc} G/H \times \text{hom}(i, (-)) \times \partial \Delta^n & \longrightarrow & X \\ \downarrow & & \downarrow f \\ G/H \times \text{hom}(i, (-)) \times \Delta^n & \longrightarrow & Y, \end{array} \quad (3.2)$$

the map $f(n)$ is of the form (3.1). If we specify the square (3.2) at some simplicial degree k , we obtain a pushout square

$$\begin{array}{ccc} \bigsqcup_{\partial \Delta^n(k)} G/H \times \text{hom}(i, (-)) & \longrightarrow & X(k) \\ \downarrow & & \downarrow f \\ \bigsqcup_{\Delta^n(k)} G/H \times \text{hom}(i, (-)) & \longrightarrow & Y(k) \end{array}$$

and because the map of sets $\partial \Delta^n(k) \hookrightarrow \Delta^n(k)$ is an inclusion, we deduce that $Y(k)$ is the coproduct

$$X(k) \sqcup \left(\bigsqcup_{\Delta^n(k) \setminus \partial \Delta^n(k)} G/H \times \text{hom}(i, (-)) \right)$$

and that $f(k)$ corresponds to the associated summand inclusion. $\square$

Proposition 3.2.2. *For every cofibrant X in $\text{Fun}(I, \text{sSet}^G)$, for every $f: i \rightarrow j$ in I and every n , $X(f)(n): X(i)(n) \rightarrow Y(i)(n)$ preserves the stabilisers of elements.*

Proof. Note first that every retract of an object X in $\text{Fun}(I, \text{sSet}^G)$ that satisfies the condition from the statement also satisfies this condition. The result now follows from the fact that cellular objects clearly satisfy the condition from the statement using Proposition 3.2.1. $\square$

Proposition 3.2.3. *Let $\mathcal{O}$ be an N_∞ -operad for C_2 with a complete indexing system. The forgetful functor $U: \text{Alg}_{\mathcal{O}}(\text{Fun}(\text{Fin}, \text{sSet})^{C_2}) \rightarrow \text{Fun}(\text{Fin}, \text{sSet})^{C_2}$ does not preserve cofibrant objects.*

Proof. Consider the graph subgroup $\Gamma \subseteq C_2 \times \Sigma_2$ given by the diagonal. The simplicial set $\mathcal{O}_2^\Gamma$ is contractible and, in particular, there is a Γ -fixed point in the set $\mathcal{O}_2(0)$. Because the Σ_2 -action on $\mathcal{O}_2$ is free, we obtain that the corresponding $C_2 \times \Sigma_2$ -equivariant map $(C_2 \times \Sigma_2)/\Gamma \rightarrow \mathcal{O}_2(0)$ is injective. Define X as the free $\mathcal{O}$ -algebra on $\text{hom}(*, (-))$ seen as a discrete simplicial presheaf. We obtain a C_2 -monomorphism

$$\text{hom}(\{1, 2\}, (-)) \simeq (C_2 \times \Sigma_2)/\Gamma \times_{\Sigma_2} \text{hom}(*, (-))^2 \hookrightarrow \mathcal{O}_2(0) \times_{\Sigma_2} \text{hom}(*, (-))^2$$

where the last term is a C_2 -subset of $U(X)_0$ and where the C_2 -action on $\text{hom}(\{1, 2\}, (-))$ is induced by the free C_2 -action on $\{1, 2\}$. The identity map $\text{id}_{\{1, 2\}}$ seen as an element of $\text{hom}(\{1, 2\}, \{1, 2\})$ has the trivial group as stabiliser and the unique map $\{1, 2\} \rightarrow *$ seen as an element of $\text{hom}(\{1, 2\}, *)$ has C_2 as stabiliser. This contradicts Proposition 3.2.2 and it follows, as $\text{hom}(\{1, 2\}, (-))$ is a C_2 -subobject of $U(X)_0$ that $U(X)$ is not cofibrant. $\square$

3.3 Cellular extensions of strong fixed cofibrant operads

Until the end of this chapter, I is a small sifted category with finite coproducts. The category $\text{Fun}(I, \text{sSet})$ will always be endowed with the projective model structure.

Definition 3.3.1. For every $n \geq 0$, the *graph model structure* on $\text{Fun}(I, \text{sSet})^{G \times \Sigma_n}$ is the model structure induced by the projective model structure on $\text{Fun}(I, \text{sSet})$ and associated with the family of graph subgroups of $G \times \Sigma_n$, which exists using [Ste16, Lemma 2.12]. We also call the corresponding model structure on collections in $\text{Fun}(I, \text{sSet})^G$ the *graph model structure*. An operad $\mathcal{O}$ in $\text{Fun}(I, \text{sSet}^G)$ is *graph Σ -cofibrant* if its identity map $\eta_{\mathcal{O}}: 1 \rightarrow \mathcal{O}$ is a graph cofibration of collections.

Remark 3.3.2. Note that because I admits coproducts, $\text{Fun}(I, \text{sSet})$ is a monoidal model category.

Definition 3.3.3. A morphism $f: A \rightarrow B$ in $\text{Fun}(I, \text{sSet})^G$ is a *fixed cofibration* if, for every subgroup H of G , the following conditions are satisfied:

1. the morphism $f^H: A^H \rightarrow B^H$ is a cofibration in $\text{Fun}(I, \text{sSet})$;
2. the colimit of f , seen as a functor $I \rightarrow \text{Arr}(\text{sSet})^G$, commutes with H -fixed points.

Proposition 3.3.4. *The class of fixed cofibrations is closed under pushouts and transfinite compositions.*

Proof. We prove first the result for transfinite compositions. The point (1) is a direct consequence of the fact that fixed points commute with sequential colimits and of the fact that transfinite compositions of cofibrations are cofibrations. For the point (2), this follows from the fact that colimits commute with colimits and from the fact that fixed points commute with sequential colimits. For pushouts, the proof is similar and uses the fact that the underlying map (if we forget the G -action) of a fixed cofibration is a monomorphism and the fact that fixed points commute with pushouts along monomorphisms in any presheaf category. $\square$

Proposition 3.3.5. *Every cellular map in $\text{Fun}(I, \text{sSet})^G$ (for the model structure of Definition 3.3.1) is a fixed cofibration.*

Proof. Using Proposition 3.3.4, it is enough to prove the result for generating cofibrations. Consider $f \times G/H$ with f a cofibration in $\text{Fun}(I, \text{sSet})$. We have $(G/H \times f)^K \simeq f \times (G/H)^K$ which is a cofibration as f is a cofibration. Moreover, $\text{colim}(G/H \times f)^K \simeq (G/H)^K \times \text{colim} f \simeq \text{colim}((G/H)^K \times f)$. $\square$

Notation 3.3.6. For every partition $\mathcal{P} = (U_1, \dots, U_k)$ of $\{1, \dots, n\}$, we denote by $\Sigma_{\mathcal{P}}$ the subgroup of Σ_n spanned by the permutations that preserve $\mathcal{P}$.

Definition 3.3.7. A *marked partitioned tree* is a marked tree T (in the sense of [PS18a, Section 5]) with a partition $\mathcal{P}$ of the set of its leaves.

For a given marked partitioned tree T with partition $\mathcal{P}$, we denote by $\text{Aut}_{\mathcal{P}}(T)$ the subgroup of $\text{Aut}(T)$ spanned by the automorphisms of T that preserve $\mathcal{P}$.

Remark 3.3.8. The subgroup $\text{Aut}_{\mathcal{P}}(T)$ of $\text{Aut}(T)$ is the inverse image of $\Sigma_{\mathcal{P}}$ by the homomorphism $\alpha: \text{Aut}(T) \rightarrow \Sigma_n$ that sends an automorphism of T to the associated permutation of its leaves.

Construction 3.3.9. Consider a partitioned marked tree T with partition $\mathcal{P}$ and suppose we have the grafting operation $t_m(T_1, \dots, T_m)$ representing T . We can endow every T_i with a partition $\mathcal{P}_i$ induced by $\mathcal{P}$. As in [BM03, Lemma 5.9], we can partition the set $\{T_1, \dots, T_m\}$ into subsets of pairwise isomorphic partitioned marked trees $\{T_1^1, \dots, T_{m_1}^1\} \cup \dots \cup \{T_1^k, \dots, T_{m_k}^k\}$ with $m_1 + \dots + m_k = m$. We finally obtain that

$$\text{Aut}_{\mathcal{P}}(T) \simeq \left(\text{Aut}_{\mathcal{P}_1}(T_1)^{m_1} \times \dots \times \text{Aut}_{\mathcal{P}_k}(T_k)^{m_k} \right) \rtimes \left(\Sigma_{m_1} \times \dots \times \Sigma_{m_k} \right).$$

Notation 3.3.10. Given a partition $\mathcal{P} = (U_1, \dots, U_k)$ of $\{1, \dots, n\}$ with $|U_i| = n_i$ and a tuple $g = (g_1, \dots, g_k)$ with $g_i: A_i \rightarrow B_i$ a map in $\text{Fun}(I, \text{sSet})^G$, we denote by $\square_{\mathcal{P}} f$ the $\Sigma_{\mathcal{P}}$ -object

$$\square_{\mathcal{P}} g := \square_{i=1}^k g_i^{\square^{n_i}},$$

where $\square$ denotes the pushout-product.

Definition 3.3.11. A morphism $f: A \rightarrow B$ in $\text{Fun}(I, \text{sSet})^{G \times \Sigma_n}$ is a *strong fixed cofibration* if for every partition $\mathcal{P} = (U_1, \dots, U_k)$ of $\{1, \dots, n\}$ with $|U_i| = n_i$ and for every tuple $g = (g_1, \dots, g_k)$ of fixed cofibrations, the morphism

$$\left(f \square \square_{\mathcal{P}} g \right)_{\Sigma_{\mathcal{P}}}$$

is a fixed cofibration, where $\Sigma_{\mathcal{P}}$ acts on f by restriction of the Σ_n -action. A morphism $f: A \rightarrow B$ of collections in $\text{Fun}(I, \text{sSet})^G$ is a *strong fixed cofibration* if $f(n): A(n) \rightarrow B(n)$ is a strong fixed cofibration for every n in $\mathbb{N}$. An operad $\mathcal{O}$ in $\text{Fun}(I, \text{sSet})^G$ is a *strong fixed cofibrant operad* if the morphism of collections $\eta_{\mathcal{O}}: 1 \rightarrow \mathcal{O}$ provided by the identity of $\mathcal{O}$ is a strong fixed cofibration.

Proposition 3.3.12. *The class of strong fixed cofibrations is closed under pushouts and transfinite compositions.*

Proof. We prove first the result for pushouts. Consider a strong fixed cofibration $f: A \rightarrow B$ and a pushout $g: C \rightarrow D$ of f . Using [PS18b, Proposition 3.1.5], we obtain that $g \square \square_{\mathcal{P}} h$ is a pushout of $f \square \square_{\mathcal{P}} h$ and, in particular, $(g \square \square_{\mathcal{P}} h)_{\Sigma_{\mathcal{P}}}$ is a pushout of $(f \square \square_{\mathcal{P}} h)_{\Sigma_{\mathcal{P}}}$. The result then follows from Proposition 3.3.4. We prove now the result for transfinite compositions. Note first that, using [PS18b, Proposition 3.1.6], strong fixed cofibrations are closed under finite compositions. Suppose that $f: A_0 \rightarrow A$ is the transfinite composition of

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots$$

where each f_i is a strong fixed cofibration. We denote by $F_i: A_0 \rightarrow A_{i+1}$ the composition of

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \dots \xrightarrow{f_i} A_{i+1}.$$

As every f_i is a strong fixed cofibration, every F_i is also a strong fixed cofibration. Finally, the pushout-product $f \square \square_{\mathcal{P}} h$ is the sequential colimit

$$\text{colim}_{i \in \mathbb{N}} \left(F_i \square \square_{\mathcal{P}} h \right)$$

and in particular $(f \square \square_{\mathcal{P}} h)_{\Sigma_{\mathcal{P}}}$ is the colimit

$$\text{colim}_{i \in \mathbb{N}} \left(\left(F_i \square \square_{\mathcal{P}} h \right)_{\Sigma_{\mathcal{P}}} \right).$$

As the colimit of every $(F_i \square \square_{\mathcal{P}} h)_{\Sigma_{\mathcal{P}}}$ commutes with fixed points, we deduce that the colimit of $(f \square \square_{\mathcal{P}} h)_{\Sigma_{\mathcal{P}}}$ also commutes with fixed points using that sequential colimits commute with fixed points. We now prove that the sequential diagram $i \mapsto (F_i \square \square_{\mathcal{P}} h)_{\Sigma_{\mathcal{P}}}^H$ is a cofibration in $\text{Fun}(\mathbb{N}, \text{Fun}(I, \text{sSet}))$, where $\mathbb{N}$ is seen as a category via its poset structure. In particular, this will imply that $(f \square \square_{\mathcal{P}} h)_{\Sigma_{\mathcal{P}}}^H$ is a cofibration. We denote respectively by B and C the source and target of $\square_{\mathcal{P}} h$. Using the classical description of cofibrations in $\text{Fun}(\mathbb{N}, \text{Fun}(I, \text{sSet}))$ in terms of the Reedy model structure, we have to prove that the morphism

$$\left((A_n \times C) \cup_{(A_n \times B \cup_{A_0 \times B} A_0 \times C)} (A_{n+1} \times B \cup_{A_0 \times B} A_0 \times C) \right)_{\Sigma_{\mathcal{P}}}^H \rightarrow (A_{n+1} \times C)_{\Sigma_{\mathcal{P}}}^H \quad (3.3)$$

induced by the square

$$\begin{array}{ccc} (A_n \times B \cup_{A_0 \times B} A_0 \times C)_{\Sigma_{\mathcal{P}}}^H & \longrightarrow & (A_{n+1} \times B \cup_{A_0 \times B} A_0 \times C)_{\Sigma_{\mathcal{P}}}^H \\ (F_n \square \square_{\mathcal{P}} h)_{\Sigma_{\mathcal{P}}}^H \downarrow & & \downarrow (F_{n+1} \square \square_{\mathcal{P}} h)_{\Sigma_{\mathcal{P}}}^H \\ (A_n \times C)_{\Sigma_{\mathcal{P}}}^H & \longrightarrow & (A_{n+1} \times C)_{\Sigma_{\mathcal{P}}}^H \end{array}$$

is a cofibration. Using that pushouts commute we pushouts, the morphism (3.3) can be identified with $(f_n \square \square_{\mathcal{P}} h)_{\Sigma_{\mathcal{P}}}^H$ and the result follows from the fact that f_n is a strong fixed cofibration. $\square$

Proposition 3.3.13. *For every partition $\mathcal{P} = (U_1, \dots, U_k)$, for every tuple $g = (f_1, \dots, f_k)$ of fixed cofibrations and for every group homomorphism $\alpha: H \rightarrow \Sigma_{\mathcal{P}}$, the map*

$$\alpha^* \left(\square_{\mathcal{P}} f \right)$$

is a fixed cofibration.

Proof. The homomorphism $\alpha: H \rightarrow \Sigma_{\mathcal{P}}$ gives an H -set T , which we can decompose T into orbits as $T \simeq H/K_1 \sqcup \dots \sqcup H/K_l$. In particular, the homomorphism α factors as a homomorphism $\alpha: H \rightarrow \Sigma_{[H:K_1]} \times \dots \times \Sigma_{[H:K_l]}$ and we denote by $\alpha_i: H \rightarrow \Sigma_{[H:K_i]}$ the associated homomorphism. Using the fact that fixed points commute with products and that pushouts along a monomorphism commute with fixed points, we deduce that

$$\alpha^* \left(\square_{\mathcal{P}} f \right)^H \simeq \square_{i=1}^l \alpha_i^* \left(f_i^{\square_{[H:K_i]}} \right)^H$$

with f_i the component of f for which $H/K_i \subset U_i$. If we assume that for every $i = 1, \dots, l$, $\alpha_i^* \left(\square_{\mathcal{P}_i} f_i \right)$ is a fixed cofibration, we obtain that $\alpha^* \left(\square_{\mathcal{P}} f \right)$ is also a fixed cofibration using that pushout-products of cofibrations are cofibrations and the fact that the functor

$$\square: \text{Arr} \left(\text{Fun}(I, \text{sSet})^G \right)^2 \rightarrow \text{Arr} \left(\text{Fun}(I, \text{sSet})^G \right)$$

preserves sifted colimits. We can now assume that $T = H/K$ and that α is a homomorphism $\alpha: H \rightarrow \Sigma_n$ with $n = [H : K]$. We now prove that

$$\alpha^*(f^{\square^n})^H \simeq f^K.$$

If we write $f: A \rightarrow B$ as an inclusion, the iterated pushout-product $f^{\square^n}$ can be described as the inclusion map

$$f^{\square^n}: \bigcup_{i=1}^n B^{i-1} \times A \times B^{n-i} \hookrightarrow B^n.$$

We prove first that $\alpha^*(B^n)^H \simeq B^K$. Consider the map $p: B^K \rightarrow B^n$ that sends a K -fixed point x to the element $p(x) = (h_1x, \dots, h_nx)$ with $e_H = h_1, \dots, h_n$ representatives in H of the cosets in H/K with the ordering given by the chosen isomorphism $G/H \simeq \{1, \dots, n\}$. The element $p(x)$ is an H -fixed point, moreover, p induces an isomorphism $\alpha^*(B^n)^H \simeq B^K$ where the inverse of p is given by the projection $\alpha^*(B^n)^H \rightarrow B^K$ to the first component. Finally, using a similar argument, we have that

$$\alpha^*\left(\bigcup_{i=1}^n B^{i-1} \times A \times B^{n-i}\right)^H \simeq A^K$$

and that the map $\alpha^*(f^{\square^n})^H$ corresponds to the inclusion $f^K: A^K \hookrightarrow B^K$. In particular, we obtain that $\alpha^*(f^{\square^n})^H$ is a cofibration as f is a fixed cofibration. Moreover, we have that

$$\begin{aligned} & \operatorname{colim} \left(\alpha^*(f^{\square^n})^H \right) \\ & \simeq \operatorname{colim} (f^K) \simeq (\operatorname{colim} f)^K \simeq \left(\alpha^*((\operatorname{colim} f)^{\square^n}) \right)^H \simeq (\operatorname{colim} \alpha^*(f^{\square^n}))^H \end{aligned}$$

and it follows that $\alpha^*(f^{\square^n})$ is a fixed cofibration. $\square$

Construction 3.3.14. Let Θ be a finite group, let X be a $(G \times \Theta)$ -set with a free Θ -action and let Y be in $\operatorname{Fun}(I, \mathbf{sSet})^{G \times \Theta}$. We will now describe the quotient $(X \times Y)_{\Theta}$ as a G -object in $\operatorname{Fun}(I, \mathbf{sSet})$. Choose first a set R of representatives in X of the classes in X_{Θ} . For every representative x and every g in G , there is a unique representative y and a unique $\theta \in \Theta$ such that $gx = \theta y$. The uniqueness of θ is given by the fact that the Θ -action on X is free. We now construct a G -action on $X_{\Theta} \times Y$. For every g in G , the associated map $g \cdot (-): X_{\Theta} \times Y \rightarrow X_{\Theta} \times Y$ sends the a -copy of Y to the $(g \cdot a)$ -copy of Y and is given on Y by the map $(\theta, g) \cdot (-)$ where θ is the unique element of Θ such that $g \cdot x = \theta \cdot y$ with x and y the representatives of a and $g \cdot a$ respectively. There is a G -equivariant map $X \times Y \rightarrow X_{\Theta} \times Y$ that sends (a, y) to $(x, \theta^{-1}y)$ with x the representative of a and θ the unique element of Θ such that $\theta x = a$. This map exhibits $X_{\Theta} \times Y$ with the G -action that we described above as the quotient $(X \times Y)_{\Theta}$. Let H be a subgroup of G . We now describe the H -fixed points of $(X \times Y)_{\Theta}$. For every

H -fixed point a in X_Θ represented by x , the map $\alpha_x: H \rightarrow \Theta$ that sends an element h to the unique θ in Θ such that $hx = \theta x$ is a homomorphism. We now obtain that the H -fixed points of $(X \times Y)_\Theta$ is given by

$$\bigsqcup_{x \in X_\Theta} \alpha_x^*(Y)^H.$$

Note that this construction also applies to maps in $\text{Fun}(I, \text{sSet})^{G \times \Theta}$.

Proposition 3.3.15. *Every cellular cofibration in $\text{Fun}(I, \text{sSet})^{G \times \Sigma_n}$ for the graph model structure is a strong fixed cofibration.*

Proof. Using Proposition 3.3.12, it is enough to prove the result for generating cofibrations. Consider the cofibration $f = T \times f'$ with T a finite $G \times \Sigma_n$ -set with a free Σ_n -action and $f': A \rightarrow B$ a cofibration in $\text{Fun}(I, \text{sSet})$. Using Construction 3.3.14, we have that $\left((f \square \square_{\mathcal{P}} g)_{\Sigma_{\mathcal{P}}}\right)^H$ is the map

$$\bigsqcup_{x \in T_{\Sigma_{\mathcal{P}}}^H} f' \square \alpha_x^* \left(\square_{\mathcal{P}} g \right)^H$$

which is a cofibration as a coproduct of pushout-products of cofibrations using Proposition 3.3.13. We now have to prove that the colimit of $(f \square \square_{\mathcal{P}} g)_{\Sigma_{\mathcal{P}}}$ commutes with H -fixed points. We have that

$$\begin{aligned} & \text{colim} \left(\bigsqcup_{x \in T_{\Sigma_{\mathcal{P}}}^H} f' \square \alpha_x^* \left(\square_{\mathcal{P}} g \right)^H \right) \\ & \simeq \bigsqcup_{x \in T_{\Sigma_{\mathcal{P}}}^H} \text{colim} \left(f' \square \alpha_x^* \left(\square_{\mathcal{P}} \text{colim} g \right) \right)^H \simeq \left(\text{colim} \left(f \square \square_{\mathcal{P}} g \right)_{\Sigma_{\mathcal{P}}} \right)^H \end{aligned}$$

using Proposition 3.3.13, the fact that $\square$ preserves colimits in both variables, and the fact that I is sifted. $\square$

Proposition 3.3.16. *Let $\mathcal{C}$ be a bicomplete category, let $\alpha: G \rightarrow H$ be a group homomorphism, and let K be a subgroup of H . The functor $(\alpha_1(-))_K$ is the functor that sends a object X in $\mathcal{C}^G$ to the following object in $\mathcal{C}$*

$$\bigsqcup_{\alpha(G)hK \in \alpha(G) \backslash H/K} X_{\alpha^{-1}(hKh^{-1})}.$$

Proof. We compute the right adjoint of this functor. It sends an object Y of $\mathcal{C}$ to the image under α^* of $\text{hom}(H/K, Y)$. The latter object is the same as $\text{hom}(\alpha^*(H/K), Y)$. Moreover, we have that

$$\alpha^*(H/K) \simeq \bigsqcup_{\alpha(G)hK \in \alpha(G) \backslash H/K} G/\alpha^{-1}(hKh^{-1}).$$

We deduce the right adjoint of the functor we care about sends an object Y of $\mathcal{C}$ to

$$\prod_{\alpha(G)hK \in \alpha(G) \backslash H/K} \text{hom}(G/\alpha^{-1}(hKh^{-1}), Y)$$

which implies that the image of a G -object X in $\mathcal{C}$ under $(\alpha_!(X))_K$ is

$$\bigsqcup_{\alpha(G)hK \in \alpha(G) \backslash H/K} X_{\alpha^{-1}(hKh^{-1})}.$$

□

Proposition 3.3.17. *Let $\mathcal{C}$ be a cocomplete and closed monoidal category. For every group morphism $\alpha: G \rightarrow H$, for every X in $\mathcal{C}^G$, and for every Y in $\mathcal{C}^H$, there is an isomorphism*

$$\alpha_!(X \otimes \alpha^* Y) \simeq \alpha_! X \otimes Y. \quad (3.4)$$

Proof. For every Y in $\mathcal{C}^H$, both sides of (3.4) give functors $\mathcal{C}^G \rightarrow \mathcal{C}^H$ and we compute their right adjoints. For the right-hand side, the right adjoint is given by the functor that sends an object Z in $\mathcal{C}^H$ to $\alpha^* \text{hom}(Y, Z)$. For the left-hand side, the right adjoint is the functor that sends Z to $\text{hom}(\alpha^* Y, \alpha^* Z)$. The result follows from the fact that there is a natural isomorphism

$$\text{hom}(\alpha^* Y, \alpha^* Z) \simeq \alpha^* \text{hom}(Y, Z).$$

□

Proposition 3.3.18. *For every strong fixed cofibrant operad $\mathcal{O}$ in $\text{Fun}(I, \text{sSet})^G$, for every strong fixed cofibration $u: A \rightarrow B$ in $\text{Coll}(\text{Fun}(I, \text{sSet})^G)$ and for every pushout square*

$$\begin{array}{ccc} F(A) & \longrightarrow & \mathcal{O} \\ F(u) \downarrow & & \downarrow f \\ F(B) & \longrightarrow & \mathcal{P}, \end{array}$$

the morphism $U(f)$ is a strong fixed cofibration. In particular, $\mathcal{P}$ is also strong fixed cofibrant.

Proof. We recall first from [PS18a, Proposition 5.2] the construction of the Σ_n -morphism $f(n): \mathcal{O}(n) \rightarrow \mathcal{P}(n)$. The morphism $f(n)$ is obtained as the transfinite composition

$$\mathcal{O} = \mathcal{P}_0(n) \xrightarrow{f_0(n)} \mathcal{P}_1(n) \xrightarrow{f_1(n)} \mathcal{P}_2(n) \xrightarrow{f_2(n)} \dots$$

where every $f_k(n)$ is defined by induction as a pushout

$$\begin{array}{ccc} \bigsqcup_T \Sigma_n \times_{\text{Aut}(T)} u^*(T) & \longrightarrow & \mathcal{P}_k(n) \\ \bigsqcup_T \Sigma_n \times_{\text{Aut}(T)} \epsilon(T) \downarrow & & \downarrow f_k(n) \\ \bigsqcup_T \Sigma_n \times_{\text{Aut}(T)} u(T) & \longrightarrow & \mathcal{P}_{k+1}(n) \end{array}$$

where the coproducts run over all isomorphism classes of admissible marked trees with n leaves and k marked vertices, and where the $\text{Aut}(T)$ -morphism $\epsilon(T): u^*(T) \rightarrow u(T)$ is inductively defined as

$$\epsilon(T) := \epsilon(l) \square \bigsqcup_{i=1}^l \epsilon(T_i)$$

with l the arity of the root of T and with $\epsilon(l) \in \text{Fun}(I, \text{sSet})^{G \times \Sigma_l}$ defined as

$$\epsilon(l) = \begin{cases} u(l) & \text{if } l \text{ is marked;} \\ \eta_{\mathcal{O}}(l) & \text{if } l \text{ is not marked,} \end{cases}$$

with $\eta_{\mathcal{O}}: 1_{\mathcal{C}} \rightarrow \mathcal{O}$ the morphism given by the identity of $\mathcal{O}$.

Using Proposition 3.2.2, it is enough to prove that $\Sigma_n \times_{\text{Aut}(T)} \epsilon(T)$ is a strong fixed cofibration for every marked tree T . Let $\mathcal{P} = (U_1, \dots, U_j)$ be a partition of $\{1, \dots, n\}$ with $|\mathcal{P}_i| = n_i$. For every tuple $(f_1, \dots, f_j)$ of fixed cofibrations, we have to prove that

$$\left(\alpha_! (\epsilon(T)) \square \bigsqcup_{\mathcal{P}} f \right)_{\Sigma_{\mathcal{P}}}$$

is a fixed cofibration. Using Proposition 3.3.16, Proposition 3.3.17 and the fact that the conjugation of a subgroup of the form $\Sigma_{\mathcal{P}}$ in Σ_n is also of this form, it is enough to prove that

$$\left(\epsilon(T) \square \bigsqcup_{\mathcal{P}} f \right)_{\text{Aut}_{\mathcal{P}}(T)}$$

is a fixed cofibration, where the $\text{Aut}_{\mathcal{P}}(T)$ -action on $\bigsqcup_{\mathcal{P}} f$ is given by precomposing with the homomorphism $\text{Aut}_{\mathcal{P}}(T) \rightarrow \Sigma_{\mathcal{P}}$. We will prove the result by grafting induction on T as a partitioned tree. Suppose $T = t_m(T_1, \dots, T_m)$ such that we are in the situation of Construction 3.3.9. We have

$$\epsilon(T) = \epsilon(m) \square \bigsqcup_{i=1}^k \epsilon(T_i)^{\square^{m_i}}$$

and we can write $\epsilon(T) \square \bigsqcup_{\mathcal{P}} f$ as

$$\epsilon(m) \square \bigsqcup_{i=1}^k \left(\epsilon(T_i) \square \bigsqcup_{\mathcal{P}_i} f \right)^{\square^{m_i}}.$$

We deduce that

$$\left(\epsilon(T) \square \bigsqcup_{\mathcal{P}} f \right)_{\text{Aut}_{\mathcal{P}}(T)} \simeq \left(\epsilon(m) \square \bigsqcup_{i=1}^k \left(\left(\epsilon(T_i) \square \bigsqcup_{\mathcal{P}_i} f \right)_{\text{Aut}_{\mathcal{P}}(T_i)} \right)^{\square^{m_i}} \right)_{\prod_{i=1}^k \Sigma_{m_i}},$$

using the description of $\text{Aut}_{\mathcal{P}}(T)$ of Construction 3.3.9. Finally, we deduce that $(\epsilon(T) \square \square_{\mathcal{P}} f)_{\text{Aut}_{\mathcal{P}}(T)}$ is a fixed cofibration because $\epsilon(m)$ is a strong fixed cofibration and because $(\epsilon(T_i) \square \square_{\mathcal{P}_i} f)_{\text{Aut}(T_i)}$ is a fixed cofibration by induction. $\square$

Proposition 3.3.19. *For every strong fixed cofibrant operad $\mathcal{O}$ in $\text{Fun}(I, \text{sSet})^G$ and for every subgroup H of G , the H -fixed points functor $(-)^H$ given by the composition*

$$\text{Alg}_{\mathcal{O}}(\text{Fun}(I, \text{sSet})^G) \xrightarrow{U} \text{Fun}(I, \text{sSet})^G \xrightarrow{(-)^H} \text{Fun}(I, \text{sSet})$$

preserves cofibrant objects. Furthermore, for every cellular algebra F in $\text{Alg}_{\mathcal{O}}(\text{Fun}(I, \text{sSet})^G)$, the colimit of $U(F): I \rightarrow \text{sSet}^G$ commutes with H -fixed points.

Proof. Let $\mathcal{A}$ be a cellular algebra over $\mathcal{O}$. Using [BM03, Proposition 5.4], there is a morphism of operads $\mathcal{O} \rightarrow \mathcal{O}[A]$ such that $\mathcal{O}[A](0) = A$. Moreover, $\mathcal{O} \rightarrow \mathcal{O}[A]$ is a sequential colimit of cellular extension by strong fixed cofibrations and it follows that $\mathcal{O} \rightarrow \mathcal{O}[A]$ is a strong fixed cofibration. In particular, because $\mathcal{O}$ is strong fixed cofibrant, $\mathcal{O}[A]$ is also strong fixed cofibrant. Finally, because $A = \mathcal{O}[A](0)$, we have that A^H is cofibrant for every subgroup H of G and the colimit of A commutes with H -fixed point. Finally, if B is a retract of a cellular algebra A , then B^H is a retract of A^H and it follows that B^H is cofibrant. $\square$

Proposition 3.3.20. *For every graph Σ -cofibrant operad $\mathcal{O}$ in sSet^G , the fixed point functor*

$$\text{Alg}_{\mathcal{O}}(\text{sSet}^G) \xrightarrow{U} \text{sSet}^G \xrightarrow{(-)^H} \text{sSet}$$

preserves sifted homotopy colimits.

Proof. Suppose that I is a homotopy sifted category (in particular, I is sifted). As pointed out in [PS18a, Proposition 7.9], we can compute all sifted homotopy colimits by only considering homotopy sifted categories with finite coproducts. We can thus assume that I admits coproducts. For every n , using [Ste16], $\mathcal{O}_n$ is a cellular object in $\text{sSet}^{G \times \Sigma_n}$ for the graph model structure. In particular, $\mathcal{O}_n$ is also cellular when seen as a constant functor in $\text{Fun}(I, \text{sSet})^{G \times \Sigma_n}$ and this implies that $\mathcal{O}$ is strong fixed cofibrant as an operad in $\text{Fun}(I, \text{sSet})^G$. Finally, we have an equivalence $\text{Alg}_{\mathcal{O}}(\text{Fun}(I, \text{sSet})^G) \simeq \text{Fun}(I, \text{Alg}_{\mathcal{O}}(\text{sSet}^G))$ and the model structure on $\text{Alg}_{\mathcal{O}}(\text{Fun}(I, \text{sSet})^G)$ corresponds to the projective model structure on $\text{Fun}(I, \text{Alg}_{\mathcal{O}}(\text{sSet}^G))$. Consider now $F: I \rightarrow \text{Alg}_{\mathcal{O}}(\text{sSet}^G)$ a cellular functor. As F is strong fixed cofibrant, the functor $U(F)^H$ is cofibrant and $U(-)^H$ preserves the colimit of F because U preserves sifted colimits and because $(-)^H$ preserves the colimit of $U(F)$. The result now follows from the fact that homotopy colimits can always be computed as strict colimits of cellular functors. $\square$

The previous proposition applies in particular to algebras over an N_∞ -operad.

Corollary 3.3.21. *For every N_∞ -operad $\mathcal{O}$, the fixed point functor*

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{sSet}^G) \xrightarrow{U} \mathrm{sSet}^G \xrightarrow{(-)^H} \mathrm{sSet}$$

preserves sifted homotopy colimits.

In particular, we obtain the following result on the associated ∞ -categories.

Corollary 3.3.22. *For every N_∞ -operad $\mathcal{O}$, the forgetful functor*

$$U : \mathbf{Alg}_{\mathcal{O}} \rightarrow \mathcal{S}^G$$

preserves sifted colimits.

Proof. Using that colimits in a presheaf category can be computed component-wise, it is enough to show that the composition

$$U : \mathbf{Alg}_{\mathcal{O}} \xrightarrow{U} \mathcal{S}^G \xrightarrow{\mathrm{ev}_{G/H}} \mathcal{S}$$

preserves sifted colimits for every subgroup $H \subseteq G$. This functor is the right derived functor of the composition

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{sSet}^G) \xrightarrow{U} \mathrm{sSet}^G \xrightarrow{(-)^H} \mathrm{sSet}$$

and, combining [Lur09, Proposition 5.3.1.16], [Lur09, Corollary 5.5.8.17] and [Lur17, Theorem 1.3.4.24] with Corollary 3.3.21, the functor $U : \mathbf{Alg}_{\mathcal{O}} \rightarrow \mathcal{S}^G$ preserves sifted colimits. $\square$

Chapter 4

Homotopy theory of simplicial parametrized operads

This chapter is based on the preprint [Mar25].

4.1 Introduction

Operads were introduced by May [May72] to study infinite loop spaces, and they play an important role in encoding algebraic structures in different areas of homotopy theory. In this chapter, we are interested in coloured operads, which can be used to describe structures such as enriched categories or a ring together with a left module. Roughly speaking, a simplicial coloured operad $\mathcal{O}$ is the data of a set of colours along with a simplicial set $\mathcal{O}((x_1, \dots, x_n), x_0)$ of “multimorphisms from $(x_1, \dots, x_n)$ to x_0 ” for every list of colours $(x_0, \dots, x_n)$. These simplicial sets are equipped with actions of the corresponding symmetric groups that “permute the entries” along with identities and an associative composition. The homotopy theory of simplicial operads has been studied in the non-coloured case in the work of Berger and Moerdijk [BM03], and a model structure on simplicial coloured operads has been constructed later on independently by Cisinski and Moerdijk [CM13b] and by Robertson [Rob11]. Simplicial coloured operads provide one model for the theory of ∞ -operads [Lur17] that is known to be equivalent to the other models, combining the work of Cisinski and Moerdijk [CM11, CM13a, CM13b], Barwick [Bar18], and Chu, Haugseng, and Heuts [CHH18].

In recent years, equivariant homotopy theory has become an increasingly active area of research, spurred in part by Hill, Hopkins, and Ravenel’s resolution of the Kervaire invariant one problem [HHR16]. In particular, the theory of equivariant operads has gained particular attention through the development of N_∞ -operads by Blumberg and Hill, which generalise E_∞ -operads to

the equivariant setting. In contrast with the non-equivariant setting, where all E_∞ -operads are equivalent, N_∞ -operads are not all equivalent. Instead, they form a poset that can be described in terms of *G-indexing systems*, following work of Blumberg and Hill [BH15], Bonventre and Pereira [BP21], and Rubin [Rub21]. The study of the combinatorics of these *G-indexing systems* has recently become an active research area, for instance, with the work of Balchin, Barnes, and Roitzheim [BBR21] or Balchin, MacBrough, and Ormsby [BMO25]. The study of equivariant operads is not limited to N_∞ -operads, see for example, Hill's work on equivariant little discs operads [Hil22].

In [BH15], Blumberg and Hill defined the notion of *graph equivalences of G-operads*, which is part of a model structure on the category of simplicial *G-operads*, as proven by Bonventre and Pereira [BP21]. This graph model structure has then been extended to *G-simplicial coloured operads* by Bonventre and Pereira in [BP22c]. Moreover, other models for equivariant operads, such as *equivariant dendroidal sets* and *equivariant dendroidal spaces*, have been developed and compared by Bonventre and Pereira in [Per18, BP20, BP22a].

In the language of ∞ -categories, based on the theorem of Elmendorf [Elm83], equivariant homotopy theory can be formalised using the theory of *parametrized ∞ -categories*, as developed by Barwick, Dotto, Glasman, Nardin, and Shah in [BDG⁺16, Nar16, Sha17]. This theory provides proper categorical foundations for higher equivariant algebra, and it has known recent further developments, for instance with the work of Cnossen, Lenz, and Linskens [CLL25b, CLL25a, CLL25c] or with the work of Hilman [Hil24]. In particular, Lurie's theory of ∞ -operads [Lur17] admits a generalisation in the equivariant world through the theory of *parametrized ∞ -operads* of Nardin and Shah [NS22], see also work of Barkan, Haugseng, and Steinebrunner [BHS25]. In analogy with classical higher algebra, the theory of parametrized ∞ -operads is expected to play a central role in *higher equivariant algebra*. Progress in this direction has been made recently, for instance, with the work of Stewart [Ste25a, Ste25b, Ste25c].

The theory of equivariant operads has thus two facets: the classical theory of equivariant simplicial (or topological) operads and the theory of parametrized ∞ -operads. Both these approaches have been studied with different perspectives, and it is natural to expect them to be the two sides of the same coin. One way to approach a comparison between these two theories is to introduce an intermediary player that shares one feature with each of these theories and that can be compared with both of them. A good candidate is the theory of *simplicial parametrized operads* as introduced by Nardin and Shah in [NS22]. The theory of simplicial parametrized operads and the theory of equivariant simplicial operads have one feature in common: they are both defined as model categories and can hopefully be compared via a Quillen adjunction. On the other hand, the theory of parametrized simplicial operads shares with parametrized ∞ -operads that both these theories live in the parametrized world.

4.1.1 Main results

In this chapter, motivated by our overarching goal to provide a comparison of all models for equivariant operads, we construct a model structure on the category of simplicial (coloured) parametrized operads. In order to achieve this, we consider a generalisation of the theory of (coloured) symmetric operads that subsumes enriched categories, non-symmetric operads, and Nardin and Shah’s parametrized operads. Our theory is based on the notion of *F-multicategories* (see Subsection 4.2.1 for a definition), that is, a specialisation of Leinster’s generalised multicategories [Lei04]. If $\mathcal{V}$ is a cartesian category, then every *F*-multicategory $\mathbb{M}$ gives rise to a notion of (coloured) $\mathbb{M}$ -operads in $\mathcal{V}$ for which $\mathbb{M}$ plays the role played by the symmetric groupoid Σ_* for symmetric operads. Every $\mathbb{M}$ -operad in $\mathcal{V}$ has an underlying $\mathbb{M}$ -collection in $\mathcal{V}$, and we prove, under some assumptions on $\mathbb{M}$ and $\mathcal{V}$, the existence of a composition product $\circ_{\mathbb{M}}$ on the category $\text{Coll}_{\mathbb{M}}(\mathcal{V})$ of $\mathbb{M}$ -collections in $\mathcal{V}$ for which algebras are precisely $\mathbb{M}$ -operads in $\mathcal{V}$. More precisely, we prove the following theorem.

Theorem 4.A. (Corollary 4.3.20) *For every F-multicategory $\mathbb{M}$ that is target right fibrant in the sense of Definition 4.3.16 and for every cartesian closed category $\mathcal{V}$, the category $\text{Coll}_{\mathbb{M}}(\mathcal{V})$ of $\mathbb{M}$ -collections in $\mathcal{V}$ is endowed with the structure of a monoidal category for which the category of algebras is precisely the category of $\mathbb{M}$ -operads in $\mathcal{V}$.*

Using this generalisation of the composition product, and the fact that $\text{Coll}_{\mathbb{M}}(\text{sSet})$ can be endowed with a projective model structure, we prove the following theorem.

Theorem 4.B. (Theorem 4.4.1) *For every target right fibrant F-multicategory $\mathbb{M}$, the category $\text{sOp}_{\mathbb{M}}$ admits a model structure that is right lifted from $\text{Coll}_{\mathbb{M}}(\text{sSet})$ with respect to the free-forgetful adjunction*

$$F_{\mathbb{M}}: \text{Coll}_{\mathbb{M}}(\text{sSet}) \rightleftarrows \text{sOp}_{\mathbb{M}}: U_{\mathbb{M}}.$$

One main feature of our notion of generalised operads is that it provides a convenient way to encode coloured operads. For every $\mathbb{M}$ -set of colours A (in the sense of Definition 4.2.18), A -coloured operads are by definition $\mathbb{M}^A$ -operads for some *F*-multicategory $\mathbb{M}^A$. Using this description of coloured $\mathbb{M}$ -operads, we prove the following theorem.

Theorem 4.C. (Theorem 4.4.28) *For every F-multicategory $\mathbb{M}$ that is target right fibrant, the category of simplicial coloured $\mathbb{M}$ -operads admits a model structure for which the weak equivalences are the fully faithful essentially surjective morphisms and for which the fibrations are the isofibrations (See Definition 4.4.11 for all the terminology used here).*

We can specialise this theorem to obtain a model structure on Nardin and Shah’s simplicial parametrized operads.

Theorem 4.D. (Example 4.4.29) *For every category $\mathcal{O}$ and every orbital subcategory $\mathcal{T}$ of $\mathcal{O}$, the category $\text{sOp}_{(\mathcal{O}, \mathcal{T})}^{\text{col}}$ of simplicial coloured $(\mathcal{O}, \mathcal{T})$ -parametrized*

operads admits a model structure for which the weak equivalences are the fully faithful essentially surjective morphisms and for which the fibrations are the isofibrations (respectively defined in Definition 4.4.11).

4.1.2 Structure of the chapter

In Section 4.2, we define the notion of $\mathbf{F}$ -multicategories as well as our notion of generalised operads. In Section 4.3, we construct, for a target right fibrant $\mathbf{F}$ -multicategory $\mathbb{M}$, a convolution product on the category of $\mathbb{M}$ -collections for which the algebras are precisely the $\mathbb{M}$ -operads. In Section 4.4, for a target right fibrant $\mathbf{F}$ -multicategory $\mathbb{M}$, we construct a model structure on the category of simplicial (coloured) $\mathbb{M}$ -operads.

4.1.3 Related work

For a G -transfer system $\mathcal{T}$, the category $\mathrm{sOp}_{(\mathcal{O}_G, \mathcal{O}_T)}$ defined in Example 4.2.36 should be equivalent to the category of *simplicial $\mathcal{T}$ -partial genuine equivariant operads* of Bonventre and Pereira [BP21], however, such a comparison has not been made yet. In particular, we expect that the model structure on $\mathrm{sOp}_{(\mathcal{O}_G, \mathcal{O}_T)}$ that we obtain in Example 4.4.3 corresponds to their model structure.

This chapter is based on the idea that the theory of double categories can be used to implement operadic theories. This idea is not new, and such considerations have already been made by Pisani in [Pis22] and by Haugseng in [Hau22].

4.2 $\mathbb{M}$ -operads

4.2.1 $\mathbf{F}$ -multicategories

The goal of this section is to define the notion of $\mathbf{F}$ -multicategories as a specialisation of Leinster’s generalised multicategories [Lei04].

Definition 4.2.1 ([Ben85, Section 3]). Let $\mathcal{C}$ be a category. The *completion by finite coproducts* of $\mathcal{C}$ is the category $\mathbf{FC}$ whose objects are sequences $\underline{X} = (X_1, \dots, X_n)$ of objects of $\mathcal{C}$ and, given two such sequences $\underline{X} = (X_1, \dots, X_n)$ and $\underline{Y} = (Y_1, \dots, Y_m)$, whose set of morphisms $\mathbf{FC}(\underline{X}, \underline{Y})$ is

$$\coprod_{\alpha: \langle n \rangle \rightarrow \langle m \rangle} \prod_{i=1}^n \mathcal{C}(X_i, Y_{\alpha(i)}),$$

where $\langle n \rangle$ denotes the finite set $\{1, \dots, n\}$. The composition of morphisms in $\mathbf{FC}$ is naturally induced by the composition of maps of finite sets and by the composition of morphisms in $\mathcal{C}$.

Proposition 4.2.2. *For every category $\mathcal{C}$, the category $\mathbf{FC}$ admits finite coproducts and satisfies the following universal property: for every category $\mathcal{D}$ with finite*

coproducts, the canonical inclusion functor $\mathcal{C} \rightarrow \mathbf{FC}$ induces an equivalence of categories

$$\mathrm{Fun}_{\sqcup}(\mathbf{FC}, \mathcal{D}) \rightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

where $\mathrm{Fun}_{\sqcup}(\mathbf{FC}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathbf{FC}, \mathcal{D})$ spanned by these functors that preserve finite coproducts.

Proof. This proposition is classical. The coproducts in the category $\mathbf{FC}$ are given by the concatenation of sequences of objects in $\mathcal{C}$. $\square$

Remark 4.2.3. For every category $\mathcal{C}$, the category $(\mathbf{FC})^{\mathrm{op}}$ is the completion by finite products of $\mathcal{C}^{\mathrm{op}}$. In particular, for every functor $F: \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{V}$ where $\mathcal{V}$ is a category with finite products, F can be extended into a functor $\widetilde{F}: (\mathbf{FC})^{\mathrm{op}} \rightarrow \mathcal{V}$ that preserves finite products defined explicitly on objects by the formula

$$\widetilde{F}(X_1, \dots, X_n) = \prod_{i=1}^n F(X_i).$$

Proposition 4.2.4. *The functor $\mathbf{F}: \mathbf{Cat} \rightarrow \mathbf{Cat}$ is naturally endowed with the structure of a cartesian monad.*

Proof. For every category $\mathcal{C}$, we obtain a functor $\mu_{\mathbf{F}}: \mathbf{F}^2\mathcal{C} \rightarrow \mathbf{FC}$ induced by the identity $\mathrm{id}_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$ using Proposition 4.2.2. A direct verification shows that the natural transformation $\mu: \mathbf{F}^2\mathcal{C} \rightarrow \mathbf{FC}$ thus obtained, along with the inclusion functor $\mathcal{C} \hookrightarrow \mathbf{FC}$, induce the structure of a cartesian monad on $\mathbf{F}$. $\square$

Remark 4.2.5. Note that the structure of a monad on $\mathbf{FC}$ exists only because we chose a strict model for $\mathbf{FC}$. For a different model, we would only obtain the structure of a pseudo-monad on $\mathbf{F}$. In particular, the definition that we gave for $\mathbf{F}$ in Definition 2.4.1 is not strict enough to provide such a structure.

Specialising [Lei04, Definition 4.2.2] at the monad $\mathbf{F}$, we obtain a notion of $\mathbf{F}$ -multicategories as well as a notion of multifunctors between $\mathbf{F}$ -multicategories. We denote by $\mathbf{F}\text{-Multicat}$ the category of $\mathbf{F}$ -multicategories and multifunctors. Recall that an $\mathbf{F}$ -multicategory $\mathbb{M}$ is given by the following data:

1. a small category $\mathbb{M}_1$ of *multimorphisms* and a small category $\mathbb{M}_0$ of *objects*;
2. a *source functor* $s: \mathbb{M}_1 \rightarrow \mathbf{FM}_0$ and a *target functor* $t: \mathbb{M}_1 \rightarrow \mathbb{M}_0$;
3. an *identity functor* $1_{\mathbb{M}}: \mathbb{M}_0 \rightarrow \mathbb{M}_1$ and a *composition functor* $\odot: \mathbb{M}_1 \times_{\mathbf{FM}_0} \mathbb{M}_1 \rightarrow \mathbb{M}_1$.

These data are required to satisfy the following compatibilities:

1. the composition of $1_{\mathbb{M}}$ and s is the canonical inclusion $\mathbb{M}_0 \hookrightarrow \mathbf{FM}_0$ and $1_{\mathbb{M}}$ is a section of t ;

2. the composition of the composition functor $\odot: \mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1 \rightarrow \mathbb{M}_1$ with the target functor $t: \mathbb{M}_1 \rightarrow \mathbb{M}_0$ is the composition

$$\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1 \xrightarrow{p_1} \mathbb{M}_1 \xrightarrow{t} \mathbb{M}_0$$

and the composition of the functor $\odot: \mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1 \rightarrow \mathbb{M}_1$ with the source functor $s: \mathbb{M}_1 \rightarrow \mathbb{M}_0$ is the composition

$$\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1 \xrightarrow{p_2} \mathbb{F}\mathbb{M}_1 \xrightarrow{\mathbb{F}s} \mathbb{F}^2\mathbb{M}_0 \xrightarrow{\mu_{\mathbb{F}}} \mathbb{F}\mathbb{M}_0;$$

3. (associativity) the following diagram commutes

$$\begin{array}{ccc} \mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1 \times_{\mathbb{F}^2\mathbb{M}_0} \mathbb{F}^2\mathbb{M}_1 & \xrightarrow{\cong} & \mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}(\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1) \\ \odot \times \text{id}_{\mathbb{F}^2\mathbb{M}_1} \downarrow \text{id}_{\mathbb{F}^2\mathbb{M}_1} & & \downarrow \text{id}_{\mathbb{M}_1} \times \text{id}_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\odot \\ \mathbb{M}_1 \times_{\mathbb{F}^2\mathbb{M}_0} \mathbb{F}^2\mathbb{M}_1 & & \mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1 \\ \cong \downarrow & & \downarrow \odot \\ \mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1 & \xrightarrow{\quad \odot \quad} & \mathbb{M}_1 \end{array}$$

4. (unitality) the following diagrams commute

$$\begin{array}{ccc} \mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_0 & \xrightarrow{\text{id}_{\mathbb{M}_1} \times \text{id}_{\mathbb{M}_0} \mathbb{F}1_{\mathbb{M}}} & \mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1 \\ & \searrow \cong & \downarrow \odot \\ & & \mathbb{M}_1 \end{array}$$

$$\begin{array}{ccc} \mathbb{M}_0 \times_{\mathbb{M}_0} \mathbb{M}_1 & \xrightarrow{1_{\mathbb{M}} \times \iota_{\mathbb{M}_0} \iota_{\mathbb{M}_1}} & \mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1 \\ & \searrow \cong & \downarrow \odot \\ & & \mathbb{M}_1 \end{array}$$

with $\iota_{\mathbb{M}_0}$ and $\iota_{\mathbb{M}_1}$ the inclusions $\mathbb{M}_0 \hookrightarrow \mathbb{F}\mathbb{M}_0$ and $\mathbb{M}_1 \hookrightarrow \mathbb{F}\mathbb{M}_1$.

Recall from [Ehr63] the notion of a double category.

Definition 4.2.6. A *double category* is a category internal to $\widehat{\text{Cat}}$. Equivalently, a double category is an id-multicategory in the sense of [Lei04, Definition 4.2.2] where $\text{id}: \widehat{\text{Cat}} \rightarrow \widehat{\text{Cat}}$ is the identity monad. A double category is *small* if it is a category internal to Cat . We denote by DoubleCat the category of small double categories.

Remark 4.2.7. Note that Definition 4.2.6 is the definition of strict double categories. For the rest of this thesis, we mostly consider strict double categories, and we will always call them double categories. If a double category is not strict, then we call it a *weak double category*.

Proposition 4.2.8. *The functor $\mathbf{F}: \mathbf{Cat} \rightarrow \mathbf{Cat}$ is naturally endowed with the structure of a cartesian oplax morphism of monad $\mathbf{F}: (\mathbf{Cat}, \mathbf{F}) \rightarrow (\mathbf{Cat}, \text{id})$ in the sense of [Lei04, Section 6.7].*

Proof. The functor $\mathbf{F}$ is Cartesian and the structure of a colax functor on $\mathbf{F}$ is provided by the multiplication of $\mathbf{F}$. $\square$

Using [Lei04, Section 6.7] on the colax functor $\mathbf{F}: (\mathbf{Cat}, \mathbf{F}) \rightarrow (\mathbf{Cat}, \text{id})$, we obtain a functor

$$\mathbf{F}_*: \mathbf{F}\text{-Multicat} \rightarrow \mathbf{DoubleCat}$$

that sends a $\mathbf{F}$ -multicategory $\mathbb{M}$ to the small double category $\mathbf{FM}$ defined by $(\mathbf{FM})_1 = \mathbf{FM}_1$ and $(\mathbf{FM})_0 = \mathbf{FM}_0$ and where the source and target functors are, respectively, given by the composition $\mathbf{FM}_1 \xrightarrow{\mathbf{F}_s \mathbb{M}} \mathbf{F}^2 \mathbb{M}_0 \xrightarrow{\mu_{\mathbf{F}}} \mathbf{FM}_0$ and by $\mathbf{F} t_{\mathbb{M}}: \mathbf{FM}_1 \rightarrow \mathbf{FM}_0$, where the horizontal composition functor is given by $\mathbf{F} \odot: \mathbf{FM}_1 \times_{\mathbf{FM}_0} \mathbf{FM}_1 \rightarrow \mathbf{FM}_1$ using that $\mathbf{F}(\mathbf{FM}_1 \times_{\mathbf{FM}_0} \mathbb{M}_1) \simeq \mathbf{FM}_1 \times_{\mathbf{FM}_0} \mathbf{FM}_1$, and where the identity functor is $\mathbf{F} 1_{\mathbb{M}}: \mathbf{FM}_0 \rightarrow \mathbf{FM}_1$.

4.2.2 M-operads

In this section, we define our generalisation of operads based on the notion of $\mathbf{F}$ -multicategories.

Remark 4.2.9. Recall that every monoidal category $(\mathcal{C}, \otimes)$ can be seen as a weak double category with a trivial category of vertical morphisms and whose category of horizontal morphisms and squares is $\mathcal{C}$ itself. The horizontal composition is given by the monoidal structure of $(\mathcal{C}, \otimes)$.

We now recall from [GP99] the definition of lax functors between weak double categories. We only recall the definition when the target is a monoidal category and when the source is a double category.

Definition 4.2.10. Let $\mathbb{D}$ be a double category and let $(\mathcal{C}, \otimes_{\mathcal{C}})$ be a monoidal category. A *lax functor* $F: \mathbb{D} \rightarrow \mathcal{C}$ is a functor $F: \mathbb{D}_1 \rightarrow \mathcal{C}$ along with a natural transformation $\alpha: \otimes_{\mathcal{C}} \circ (F \times F) \Rightarrow F \circ \odot$ of functors $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \rightarrow \mathcal{C}$ and a natural transformation $\eta: 1_{\mathcal{C}} \Rightarrow F \circ 1_{\mathbb{D}}$ of functors $\mathbb{D}_0 \rightarrow \mathcal{C}$ satisfying the following compatibilities:

1. for every (f_1, f_2, f_3) in $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$, the following diagram commutes

$$\begin{array}{ccc} F(f_1) \otimes_{\mathcal{C}} F(f_2) \otimes_{\mathcal{C}} F(f_3) & \xrightarrow{\alpha \otimes \text{id}_{F(f_3)}} & F(f_1 \odot f_2) \otimes_{\mathcal{C}} F(f_3) \\ \text{id}_{F(f_1)} \otimes_{\mathcal{C}} \alpha \downarrow & & \downarrow \alpha \\ F(f_1) \otimes_{\mathcal{C}} F(f_2 \odot f_3) & \xrightarrow{\alpha} & F(f_1 \odot f_2 \odot f_3); \end{array}$$

2. for every X in $\mathbb{D}_0$ and every f in $\mathbb{D}_1$, the following triangles commute

$$\begin{array}{ccc}
 F(f) & \xrightarrow{\text{id}_{F(f)} \otimes_C \eta} & F(f) \otimes_C F(1_X) \\
 & \searrow = & \downarrow \alpha \\
 & & F(f \odot 1_X)
 \end{array}
 \qquad
 \begin{array}{ccc}
 F(f) & \xrightarrow{\eta \otimes_C \text{id}_{F(f)}} & F(1_X) \otimes_C F(f) \\
 & \searrow = & \downarrow \alpha \\
 & & F(1_X \odot f).
 \end{array}$$

A *vertical natural transformation* $\theta: F \Rightarrow G$ between two lax functors $F, G: \mathbb{D} \rightarrow \mathcal{C}$ is a natural transformation $\theta: F \Rightarrow G$ between the underlying functors $F, G: \mathbb{D}_1 \rightarrow \mathcal{C}$ satisfying the following conditions:

1. for every (f, g) in $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$, the following square commutes

$$\begin{array}{ccc}
 F(f) \otimes_C F(g) & \xrightarrow{\theta_f \otimes \theta_g} & G(f) \otimes_C G(g) \\
 \alpha \downarrow & & \downarrow \alpha \\
 F(f \odot g) & \xrightarrow{\theta_{f \odot g}} & G(f \odot g);
 \end{array}$$

2. for every X in $\mathbb{D}_0$, the following triangle commutes

$$\begin{array}{ccc}
 & & F(1_X) \\
 & \nearrow \eta & \downarrow \theta_{1_X} \\
 1_{\mathcal{C}} & & \\
 & \searrow \eta & \\
 & & G(1_X).
 \end{array}$$

We denote by $\text{Lax}(\mathbb{D}, \mathcal{C})$ the category of lax functors $F: \mathbb{D} \rightarrow \mathcal{C}$ and vertical natural transformations thus obtained.

Definition 4.2.11. Let $\mathbb{M}$ be an $\mathbf{F}$ -multicategory and let $\mathcal{V}$ be a category with finite products. An $\mathbb{M}$ -collection in $\mathcal{V}$ is a functor $S: \mathbb{M}_1^{\text{op}} \rightarrow \mathcal{V}$. An $\mathbb{M}$ -operad in $\mathcal{V}$ is a lax functor $\mathcal{O}: \mathbf{FM}^{\text{op}} \rightarrow \mathcal{V}$ in the sense of Definition 4.2.10 such that the underlying functor $\mathcal{O}: \mathbf{FM}_1^{\text{op}} \rightarrow \mathcal{V}$ preserves finite products. Explicitly, an $\mathbb{M}$ -operad $\mathcal{O}$ in $\mathcal{V}$ is the data of an $\mathbb{M}$ -collection $\mathcal{O}: \mathbb{M}_1^{\text{op}} \rightarrow \mathcal{V}$ in $\mathcal{V}$ along with:

1. identities given by a natural transformation $1_X: 1_{\mathcal{V}} \rightarrow \mathcal{O}(1_X)$ of functors $\mathbb{M}_0^{\text{op}} \rightarrow \mathcal{V}$ where the right-hand side is the composition

$$\mathbb{M}_0^{\text{op}} \xrightarrow{1_{\mathbb{M}}} \mathbb{M}_1^{\text{op}} \xrightarrow{\mathcal{O}} \mathcal{V},$$

and where the left-hand side is the functor constant on $1_{\mathcal{V}}$;

2. a composition law given by a natural transformation $\circ: \mathcal{O}(f) \times \mathcal{O}(f_1) \times \cdots \times \mathcal{O}(f_n) \rightarrow \mathcal{O}(f \odot (f_1, \dots, f_n))$ of functors $(\mathbb{M}_1 \times_{\mathbf{FM}_0} \mathbf{FM}_1)^{\text{op}} \rightarrow \mathcal{V}$ where the left-hand side is the composition

$$(\mathbb{M}_1 \times_{\mathbf{FM}_0} \mathbf{FM}_1)^{\text{op}} \longrightarrow (\mathbb{M}_1 \times \mathbf{FM}_1)^{\text{op}} \xrightarrow{\mathcal{O} \times \tilde{\mathcal{O}}} \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V},$$

with $\widetilde{\mathcal{O}}$ given by Remark 4.2.3, and where the right-hand side is the composition

$$(\mathbb{M}_1 \times_{\mathbb{FM}_0} \mathbb{FM}_1)^{\text{op}} \xrightarrow{\odot^{\text{op}}} \mathbb{M}_1^{\text{op}} \xrightarrow{\mathcal{O}} \mathcal{V}.$$

These data are required to satisfy the following compatibilities:

1. (associativity) for every $(f, (f_1, \dots, f_n), ((f_1^1, \dots, f_1^{m_1}), \dots, (f_n^1, \dots, f_n^{m_n})))$ in $\mathbb{M}_1 \times_{\mathbb{FM}_0} \mathbb{FM}_1 \times_{\mathbb{F}^2\mathbb{M}_0} \mathbb{F}^2\mathbb{M}_1$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}(f) \times \prod_{i=1}^n (\mathcal{O}(f_i) \times \prod_{j=1}^{m_j} \mathcal{O}(f_i^j)) & \xrightarrow{\cong} & \mathcal{O}(f) \times (\prod_{i=1}^n \mathcal{O}(f_i)) \times (\prod_{i=1}^n \prod_{j=1}^{m_i} \mathcal{O}(f_i^j)) \\ \text{id} \times \prod_{i=1}^n \odot \downarrow & & \downarrow \odot \times \text{id} \\ \mathcal{O}(f) \times \prod_{i=1}^n \mathcal{O}(f_i \odot (f_i^1, \dots, f_i^{m_i})) & & \mathcal{O}(f \odot (f_1, \dots, f_n)) \times (\prod_{i=1}^n \prod_{j=1}^{m_i} \mathcal{O}(f_i^j)) \\ \odot \downarrow & & \downarrow \odot \\ \mathcal{O}(f \odot (f_1 \odot (f_1^1, \dots, f_1^{m_1}), \dots, f_n \odot (f_n^1, \dots, f_n^{m_n}))) & \xrightarrow{\cong} & \mathcal{O}((f \odot (f_1, \dots, f_n)) \odot (f_1^1, \dots, f_n^{m_n})); \end{array}$$

2. (unitality) for every f in $\mathbb{M}_1$, if we denote by $(x_1, \dots, x_n)$ the source of f , then the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}(f) & \xrightarrow{(\text{id}, 1_{x_1}, \dots, 1_{x_n})} & \mathcal{O}(f) \times \mathcal{O}(1_{x_1}) \times \dots \times \mathcal{O}(1_{x_n}) \\ & \searrow = & \downarrow \odot \\ & & \mathcal{O}(f \odot (1_{x_1}, \dots, 1_{x_n})), \end{array}$$

and if we denote by x the target of f , then the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}(f) & \xrightarrow{(\text{id}, 1_x)} & \mathcal{O}(f) \times \mathcal{O}(1_x) \\ & \searrow = & \downarrow \odot \\ & & \mathcal{O}(1_x \odot f). \end{array}$$

A *morphism of $\mathbb{M}$ -operads* $\phi: \mathcal{O} \rightarrow \mathcal{P}$ in $\mathcal{V}$ is a vertical natural transformation $\phi: \mathcal{O} \Rightarrow \mathcal{P}$. Explicitly, a morphism $\phi: \mathcal{O} \rightarrow \mathcal{P}$ of $\mathbb{M}$ -operads in $\mathcal{V}$ is a natural transformation $\phi: \mathcal{O} \Rightarrow \mathcal{P}$ of functors $\mathbb{M}_1^{\text{op}} \rightarrow \mathcal{V}$ such that the following diagrams commute:

- 1.

$$\begin{array}{ccc} \mathcal{O}(f) \times \mathcal{O}(f_1) \times \dots \times \mathcal{O}(f_n) & \xrightarrow{\phi(f) \times \prod_{i=1}^n \phi(f_i)} & \mathcal{P}(f) \times \mathcal{P}(f_1) \times \dots \times \mathcal{P}(f_n) \\ \circ_{\mathcal{O}} \downarrow & & \downarrow \circ_{\mathcal{P}} \\ \mathcal{O}(f \odot (f_1, \dots, f_n)) & \xrightarrow{\phi(f \odot (f_1, \dots, f_n))} & \mathcal{P}(f \odot (f_1, \dots, f_n)); \end{array}$$

2.

$$\begin{array}{ccc}
 & & \mathcal{O}(1_X) \\
 1_{\mathcal{V}} & \begin{array}{c} \nearrow 1_X \\ \searrow 1_X \end{array} & \downarrow \phi(1_X) \\
 & & \mathcal{P}(1_X).
 \end{array}$$

In what follows, we denote by $\text{Op}_{\mathbb{M}}(\mathcal{V})$ the category of $\mathbb{M}$ -operads in $\mathcal{V}$ thus obtained. Note that, by definition, $\text{Op}_{\mathbb{M}}(\mathcal{V})$ is a full subcategory of $\text{Lax}(\mathbf{FM}^{\text{op}}, \mathcal{V})$.

Example 4.2.12. Consider the $\mathbf{F}$ -multicategory Σ_* whose category of multimorphisms is the symmetric groupoid $\Sigma_* = \bigsqcup_{n \geq 0} \mathbf{B} \Sigma_n$, whose category of objects is the final category $*$, whose source and target functors are the inclusion $\Sigma_* \hookrightarrow \mathbf{F}*$ and the unique functor $\Sigma_* \rightarrow *$ respectively, and whose composition functor $\odot: \Sigma_* \times_{\mathbf{F}^*} \mathbf{F} \Sigma_* \rightarrow \Sigma_*$ sends an object $(n, (k_1, \dots, k_n))$ to $k_1 + \dots + k_n$ and sends a morphism $(\delta, (\eta_1, \dots, \eta_n))$ with δ in Σ_n and η_i in Σ_{k_i} to the permutation

$$\delta_{k_1, \dots, k_n} \circ \left(\bigsqcup_{i=1}^k \eta_i \right),$$

with $\delta_{k_1, \dots, k_n}$ the block permutation. The category of Σ_* -operads in $\mathcal{V}$ is the category of symmetric operads in $\mathcal{V}$.

Example 4.2.13. Consider the $\mathbf{F}$ -multicategory $\mathbb{N}$ whose category of multimorphisms is the set $\mathbb{N}$ of non negative integers (seen as a discrete category), whose category of objects is $*$, whose source and target functors are the inclusion $\mathbb{N} \hookrightarrow \mathbf{F}*$ and the unique functor $\mathbb{N} \rightarrow *$ respectively, and whose composition functor $\mathbb{N} \times_{\mathbf{F}^*} \mathbf{F} \mathbb{N}$ sends an objects $(n, (k_1, \dots, k_n))$ to $k_1 + \dots + k_n$. The category of $\mathbb{N}$ -operads in $\mathcal{V}$ is the category of non-symmetric operads in $\mathcal{V}$.

Remark 4.2.14. Note that $\mathbb{N}$ is actually a classical multicategory seen here as a discrete $\mathbf{F}$ -multicategory.

Proposition 4.2.15. *For every multifunctor $\phi: \mathbb{M} \rightarrow \mathbb{N}$ between $\mathbf{F}$ -multicategories and for every functor $\psi: \mathcal{V} \rightarrow \mathcal{U}$ between categories with finite products that preserves finite products, we obtain by precomposition and post-composition functors $\phi^*: \text{Op}_{\mathbb{N}}(\mathcal{V}) \rightarrow \text{Op}_{\mathbb{M}}(\mathcal{V})$ and $\text{Op}_{\mathbb{N}}(\psi): \text{Op}_{\mathbb{N}}(\mathcal{V}) \rightarrow \text{Op}_{\mathbb{N}}(\mathcal{U})$. In particular, for every category with finite products $\mathcal{V}$, we obtain a functor*

$$\text{Op}_{(-)}(\mathcal{V}): \mathbf{F}\text{-Multicat}^{\text{op}} \rightarrow \widehat{\mathbf{Cat}}.$$

Proof. This follows directly from the description of $\mathbb{M}$ -operads in terms of lax functors. $\square$

4.2.3 Coloured $\mathbb{M}$ -operads

The goal of this section is to define the notion of coloured $\mathbb{M}$ -operads. Recall first the definition of the Grothendieck construction from [GR03].

Definition 4.2.16. Consider a functor $F: \mathcal{C}^{\text{op}} \rightarrow \widehat{\text{Cat}}$. The *Grothendieck construction* of F is the category $\int_{\mathcal{C}} F$ whose objects are pairs (X, x) with X in $\mathcal{C}$ and x in $F(X)$ and whose morphisms $(X, x) \rightarrow (Y, y)$ are pairs (f, ϕ) with $f: X \rightarrow Y$ a morphism in $\mathcal{C}$ and $\phi: x \rightarrow F(f)(y)$ a morphism in $F(X)$. We obtain, moreover, a functor $\pi_F: \int_{\mathcal{C}} F \rightarrow \mathcal{C}$ that sends (X, x) to X . If F is a presheaf of small sets on $\mathcal{C}$, the *category of elements* of F denoted by $\text{el}(F)$ is the Grothendieck construction of the composition

$$\mathcal{C}^{\text{op}} \xrightarrow{F} \text{Set} \hookrightarrow \widehat{\text{Cat}}$$

where the right-hand side functor sends a set to the corresponding discrete category.

Remark 4.2.17. Explicitly, the category of elements of a presheaf $F: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ is the small category whose objects are pairs (X, x) with X in $\mathcal{C}$ and x in $F(X)$ and where a morphism $(X, x) \rightarrow (Y, y)$ is a morphism $f: X \rightarrow Y$ in $\mathcal{C}$ such that $F(f)(y) = x$.

Definition 4.2.18. For every $\mathbf{F}$ -multicategory $\mathbb{M}$, the category of $\mathbb{M}$ -sets of colours is the category $\text{Psh}(\mathbb{M}_0)$ of presheaves of small sets on $\mathbb{M}_0$.

Construction 4.2.19. For every $\mathbf{F}$ -multicategory $\mathbb{M}$ and every functor $F: \mathcal{A} \rightarrow \mathbb{M}_0$ with $\mathcal{A}$ a small category, we define a new $\mathbf{F}$ -multicategory $F^*\mathbb{M}$ whose category of objects $(F^*\mathbb{M})_0$ is $\mathcal{A}$ and whose category of multimorphisms $(F^*\mathbb{M})_1$ is $\mathbb{M}_1 \times_{(\mathbb{M}_0 \times \mathbf{F}\mathbb{M}_0)} (\mathcal{A} \times \mathbf{F}\mathcal{A})$, whose composition functor

$$\odot_{F^*\mathbb{M}}: F^*\mathbb{M}_1 \times_{\mathbf{F}(F^*\mathbb{M}_0)} \mathbf{F}(F^*\mathbb{M}_1) \rightarrow F^*\mathbb{M}_1$$

is the functor

$$\left(\mathbb{M}_1 \times_{\mathbf{F}\mathbb{M}_0} \mathbf{F}\mathbb{M}_1 \right) \times_{(\mathbb{M}_0 \times \mathbf{F}\mathbb{M}_1 \times \mathbf{F}^2\mathbb{M}_0)} (\mathcal{A} \times \mathbf{F}\mathcal{A} \times \mathbf{F}^2\mathcal{A}) \xrightarrow{\circ \times \mu_{\mathbf{F}} \times \text{id}_{\mathcal{A}}} \mathbb{M}_1 \times_{(\mathbb{M}_0 \times \mathbf{F}\mathbb{M}_0)} (\mathcal{A} \times \mathbf{F}\mathcal{A})$$

using the isomorphism

$$F^*\mathbb{M}_1 \times_{\mathbf{F}(F^*\mathbb{M}_0)} \mathbf{F}(F^*\mathbb{M}_1) \simeq \left(\mathbb{M}_1 \times_{\mathbf{F}\mathbb{M}_0} \mathbf{F}\mathbb{M}_1 \right) \times_{(\mathbb{M}_0 \times \mathbf{F}\mathbb{M}_1 \times \mathbf{F}^2\mathbb{M}_0)} (\mathcal{A} \times \mathbf{F}\mathcal{A} \times \mathbf{F}^2\mathcal{A}),$$

and whose identity functor $1: F^*\mathbb{M}_0 \rightarrow F^*\mathbb{M}_1$ is the functor $\mathcal{A} \rightarrow \mathbb{M}_1 \times_{(\mathbb{M}_0 \times \mathbf{F}\mathbb{M}_0)} (\mathcal{A} \times \mathbf{F}\mathcal{A})$ given component-wise by the composition $\mathcal{A} \xrightarrow{F} \mathbb{M}_0 \xrightarrow{1} \mathbb{M}_1$, by the inclusion $\mathcal{A} \hookrightarrow \mathbf{F}\mathcal{A}$ and by the identity $\text{id}_{\mathcal{A}}$ respectively. In particular, we obtain a functor $(-)^*\mathbb{M}: \text{Cat}/_{\mathbb{M}_0} \rightarrow \mathbf{F}\text{-Multicat}$ that sends a functor $F: \mathcal{A} \rightarrow \mathbb{M}_0$ to the $\mathbf{F}$ -multicategory $F^*\mathbb{M}$ and that sends a commutative triangle of functors

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{H} & \mathcal{B} \\ & \searrow F & \swarrow G \\ & \mathbb{M}_0 & \end{array}$$

to the multifunctor $F^*\mathbb{M} \rightarrow G^*\mathbb{M}$ given on objects by H and on morphisms by the functor $(H^*\mathbb{M})_1 : (F^*\mathbb{M})_1 \rightarrow (G^*\mathbb{M})_1$ obtained by the following diagram

$$\begin{array}{ccc} (F^*\mathbb{M})_1 & \longrightarrow & \mathcal{A} \times \mathbf{F}\mathcal{A} \\ \downarrow (H^*\mathbb{M})_1 & & \downarrow H \times \mathbf{F}H \\ (G^*\mathbb{M})_1 & \longrightarrow & \mathcal{B} \times \mathbf{F}\mathcal{B} \\ \downarrow & & \downarrow G \times \mathbf{F}G \\ \mathbb{M}_1 & \xrightarrow{s \times t} & \mathbb{M}_0 \times \mathbf{F}\mathbb{M}_0. \end{array}$$

In what follows, we denote by $C_{\mathbb{M}}$ the composition $\text{Psh}(\mathbb{M}_0) \xrightarrow{\text{el}} \text{Cat}/_{\mathbb{M}_0} \xrightarrow{(-)^*\mathbb{M}} \mathbf{F}\text{-Multicat}$ and we denote by $\mathbb{M}^A$ the image of a $\mathbb{M}$ -set of colours A under the functor $C_{\mathbb{M}}$.

Definition 4.2.20. Let A be an $\mathbb{M}$ -set of colours. An A -coloured $\mathbb{M}$ -collection in $\mathcal{V}$ is an $\mathbb{M}^A$ -collection in $\mathcal{V}$. Similarly, an A -coloured $\mathbb{M}$ -operad in $\mathcal{V}$ is an $\mathbb{M}^A$ -operad in $\mathcal{V}$. We denote respectively by $\text{Coll}_{\mathbb{M}}^A(\mathcal{V})$ and $\text{Op}_{\mathbb{M}}^A(\mathcal{V})$ the categories of A -coloured collections and A -coloured operads in $\mathcal{V}$. Consider the functor $\Phi_{\mathbb{M}} : \text{Psh}(\mathbb{M}_0)^{\text{op}} \rightarrow \widehat{\text{Cat}}$ that sends A to $\text{Op}_{\mathbb{M}}^A(\mathcal{V})$, obtained as the composition

$$\text{Psh}(\mathbb{M}_0)^{\text{op}} \xrightarrow{C_{\mathbb{M}}} (\mathbf{F}\text{-Multicat})^{\text{op}} \xrightarrow{\text{Op}_{(-)}(\mathcal{V})} \widehat{\text{Cat}}.$$

We denote by $\text{Op}_{\mathbb{M}}^{\text{col}}(\mathcal{V})$ the Grothendieck construction of $\Phi_{\mathbb{M}}$. The category $\text{Op}_{\mathbb{M}}^{\text{col}}(\mathcal{V})$ is the category of *coloured $\mathbb{M}$ -operads in $\mathcal{V}$* .

Example 4.2.21. If we consider the $\mathbf{F}$ -multicategory Σ_* of Example 4.2.12, then the category $\text{Op}_{\Sigma_*}^{\text{col}}(\mathcal{V})$ is the category of coloured symmetric operads in $\mathcal{V}$.

Example 4.2.22. If we consider the $\mathbf{F}$ -multicategory $\mathbb{N}$ of Example 4.2.13, then the category $\text{Op}_{\mathbb{N}}^{\text{col}}(\mathcal{V})$ is the category of multicategories enriched in $\mathcal{V}$.

Construction 4.2.23. For every multifunctor $\phi : \mathbb{M} \rightarrow \mathbb{N}$, we define a natural transformation $(-)^*\phi : (-)^*\mathbb{M} \circ (\phi_0)^* \Rightarrow (-)^*\mathbb{N}$ with $(\phi_0)^* : \text{Psh}(\mathbb{N}_0) \rightarrow \text{Psh}(\mathbb{M}_0)$ the functor obtained by pulling back along ϕ_0 . For every functor $F : \mathcal{A} \rightarrow \mathbb{N}_0$, if we denote by G the functor $(\phi_0)^*F$, we have to construct a multifunctor $F^*\phi : G^*\mathbb{M} \rightarrow F^*\mathbb{N}$. At the level of categories of objects, the functor $(F^*\phi)_0 : \mathbb{M}_0 \times_{\mathbb{N}_0} \mathcal{A} \rightarrow \mathcal{A}$ is the canonical projection and at the level of multi-morphisms, the functor

$$(F^*\phi)_1 : \mathbb{M}_1 \times_{(\mathbb{M}_0 \times \mathbf{F}\mathbb{M}_0)} \left((\mathcal{A} \times_{\mathbb{N}_0} \mathbb{M}_0) \times \mathbf{F}(\mathcal{A} \times_{\mathbb{N}_0} \mathbb{M}_0) \right) \rightarrow \mathbb{N}_1 \times_{(\mathbb{N}_0 \times \mathbf{F}\mathbb{N}_0)} (\mathcal{A} \times \mathbf{F}\mathcal{A})$$

is given by $\phi_1 \times_{\text{id}_{(\mathbb{F}\mathbb{N}_0 \times \mathbb{N}_0)}} \text{id}_{\mathbf{F}\mathcal{C} \times \mathcal{C}}$ using the equivalence

$$\mathbb{M}_1 \times_{(\mathbb{M}_0 \times \mathbf{F}\mathbb{M}_0)} \left((\mathcal{A} \times_{\mathbb{N}_0} \mathbb{M}_0) \times \mathbf{F}(\mathcal{A} \times_{\mathbb{N}_0} \mathbb{M}_0) \right) \simeq \mathbb{M}_1 \times_{(\mathbb{N}_0 \times \mathbf{F}\mathbb{N}_0)} (\mathbf{F}\mathcal{A} \times \mathcal{A}).$$

In particular, for every multifunctor $\phi : \mathbb{M} \rightarrow \mathbb{N}$, we obtain a natural transformation $\Phi_{\phi} : \Phi_{\mathbb{N}} \Rightarrow \Phi_{\mathbb{M}} \circ (\phi_0)^*$ of functors $\text{Psh}(\mathbb{N}_0)^{\text{op}} \rightarrow \widehat{\text{Cat}}$ by precomposing

and post-composing $(-)^*\phi$ by el and $\text{Op}_{(-)}(\mathcal{V})$ respectively. We thus obtain a functor $\phi^*: \text{Op}_{\mathbb{M}}^{\text{col}}(\mathcal{V}) \rightarrow \text{Op}_{\mathbb{N}}^{\text{col}}(\mathcal{V})$ on the associated Grothendieck constructions using [HP15, Section 2.3].

Proposition 4.2.24. *The functor $(-)_0: \mathbf{F}\text{-Multicat} \rightarrow \mathbf{Cat}$ that sends an $\mathbf{F}$ -multicategory $\mathbb{M}$ to its category of objects $\mathbb{M}_0$ admits a left adjoint that sends a category $\mathcal{C}$ to the $\mathbf{F}$ -multicategory $\widetilde{\mathcal{C}}$ constant on $\mathcal{C}$. For every presheaf $A: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$, the category of A -coloured $\widetilde{\mathcal{C}}$ -operads in $\mathcal{V}$ is the category $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_{\mathcal{V}})^A$ of functors $F: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_{\mathcal{V}}$ (with $\text{Cat}_{\mathcal{V}}$ the category of small $\mathcal{V}$ -categories) such that the composition $\mathcal{C}^{\text{op}} \xrightarrow{F} \text{Cat}_{\mathcal{V}} \xrightarrow{\text{Ob}} \mathbf{Set}$ is A . Moreover, the category of coloured $\widetilde{\mathcal{C}}$ -operads is the category $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_{\mathcal{V}})$.*

Proof. Let $A: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ be a presheaf and let $\mathcal{O}$ be a A -coloured $\widetilde{\mathcal{C}}$ -operad in $\mathcal{V}$. The category $(\widetilde{\mathcal{C}}^A)_1$ is the pullback $\text{el}(X) \times_{\mathcal{C}} \text{el}(A)$. For every object X of $\mathcal{C}$, we can define a $\mathcal{V}$ -category $\mathcal{O}(X)$ with set of objects $A(X)$ and, given two objects x, y in $A(X)$, whose object in $\mathcal{V}$ of morphisms is $\mathcal{O}(X, x, y)$ where (X, x, y) is seen as an object of $\text{el}(A) \times_{\mathcal{C}} \text{el}(A)$. The composition and identities of the operad $\mathcal{O}$ respectively provide a composition and identities for the $\mathcal{V}$ -category $\mathcal{O}(X)$. This construction provides the promised equivalence between A -coloured $\widetilde{\mathcal{C}}$ -operads in $\mathcal{V}$ and $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_{\mathcal{V}})^A$. The last part of the proposition follows from the fact that the category $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_{\mathcal{V}})$ is the Grothendieck construction of the functor $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$ that sends A to $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_{\mathcal{V}})^A$. $\square$

Definition 4.2.25. For every $\mathbf{F}$ -multicategory $\mathbb{M}$, if we denote by $i: \widetilde{\mathbb{M}}_0 \rightarrow \mathbb{M}$ the counit of the adjunction $(-): \mathbf{Cat} \rightleftarrows \mathbf{F}\text{-Multicat}: (-)_0$ of Proposition 4.2.24, using Construction 4.2.23, we obtain a functor $i^*: \text{Op}_{\mathbb{M}}^{\text{col}}(\mathcal{V}) \rightarrow \text{Fun}(\mathbb{M}_0^{\text{op}}, \text{Cat}_{\mathcal{V}})$. For every coloured $\mathbb{M}$ -operad $\mathcal{O}$ in $\mathcal{V}$, the functor $i^*(\mathcal{O}): \mathbb{M}_0^{\text{op}} \rightarrow \text{Cat}_{\mathcal{V}}$ is the *underlying parametrized $\mathcal{V}$ -category* of $\mathcal{O}$.

4.2.4 Parametrized operads

In this section, we use the formalism of $\mathbb{M}$ -operads to define parametrized operads. We recall first the notion of orbital subcategories that we already introduced in Definition 2.4.3.

Definition 4.2.26 ([CLL25b, Definition 4.2.2]). Let $\mathcal{O}$ be a small category. An *orbital subcategory* $\mathcal{T}$ of $\mathcal{O}$ is a wide subcategory $\mathcal{T}$ of $\mathcal{O}$ such that the pullback in $\mathbf{F}\mathcal{O}$ of any morphism in $\mathbf{F}\mathcal{T} \subseteq \mathbf{F}\mathcal{O}$ exists and belongs to $\mathbf{F}\mathcal{T}$. An *orbital pair* is a pair $(\mathcal{O}, \mathcal{T})$ such that $\mathcal{O}$ is a small category and such that $\mathcal{T}$ is an orbital subcategory of $\mathcal{O}$. A *functor* $F: (\mathcal{O}_1, \mathcal{T}_1) \rightarrow (\mathcal{O}_2, \mathcal{T}_2)$ of orbital pairs is a functor $F: \mathcal{O}_1 \rightarrow \mathcal{O}_2$ such that $F(\mathcal{T}_1) \subseteq \mathcal{T}_2$ and such that $\mathbf{F}F$ preserves pullbacks of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow g \\ C & \longrightarrow & D \end{array}$$

where g belongs to $\mathcal{T}_1$. A category $\mathcal{O}$ is *orbital* if $\mathcal{O}$ is an orbital subcategory of itself.

Recall from Chapter 2 that orbital subcategories of $\mathcal{O}_G$ form a poset that is isomorphic to G -transfer systems.

Proposition 4.2.27 (Proposition 2.4.7). *The poset of orbital subcategories $\text{SubOrb}_{\mathcal{O}_G}$ of the orbit category $\mathcal{O}_G$ is equivalent to the poset $\mathbf{I}_G$ of G -indexing system. In particular, it follows from [BH15], [Rub21], [BP21] and [BBR21] that the poset $\text{SubOrb}_{\mathcal{O}_G}$ is also isomorphic to the poset of N_∞ -operads and of G -transfer systems.*

Example 4.2.28. For every category $\mathcal{C}$, the core of $\mathcal{C}$ is an orbital subcategory of $\mathcal{C}$. We thus obtain an orbital pair $(\mathcal{C}, \mathcal{C}^\simeq)$.

Example 4.2.29. The category of finite sets of cardinality less than n and surjective functions is an orbital category.

Definition 4.2.30. For every orbital pair $(\mathcal{O}, \mathcal{T})$, consider the subcategory $\Sigma_{(\mathcal{O}, \mathcal{T})}$ of $\text{Arr}(\mathbf{F}\mathcal{O})$ whose objects are morphisms $f: A \rightarrow B$ in $\mathbf{F}\mathcal{T}$ such that B belongs to $\mathcal{O}$ and where a morphism from $f: A \rightarrow B$ to $g: C \rightarrow D$ is a pullback square

$$\begin{array}{ccc} A & \xrightarrow{i} & C \\ f \downarrow & & \downarrow g \\ B & \xrightarrow{h} & D. \end{array}$$

in $\mathbf{F}\mathcal{O}$. We now define an $\mathbf{F}$ -multicategory $\underline{\Sigma}_{(\mathcal{O}, \mathcal{T})}$ whose category of multi-morphisms is $\Sigma_{(\mathcal{O}, \mathcal{T})}$, whose category of objects is $\mathcal{O}$, whose identity functor $1_{\underline{\Sigma}_{(\mathcal{O}, \mathcal{T})}}: \mathcal{O} \rightarrow \Sigma_{(\mathcal{O}, \mathcal{T})}$ sends an object X to the identity id_X , and whose composition functor $\odot: \Sigma_{(\mathcal{O}, \mathcal{T})} \times_{\mathbf{FS}} \mathbf{F}\Sigma_{(\mathcal{O}, \mathcal{T})}$ sends $(f, (f_1, \dots, f_n))$ to the composition $f \circ (f_1 \sqcup \dots \sqcup f_n)$ in the category $\mathbf{F}\mathcal{T}$. A (coloured) $(\mathcal{O}, \mathcal{T})$ -operad in $\mathcal{V}$ is a (coloured) $\underline{\Sigma}_{(\mathcal{O}, \mathcal{T})}$ -operad in $\mathcal{V}$ and we denote by $\text{Op}_{(\mathcal{O}, \mathcal{T})}(\mathcal{V})$ (respectively by $\text{Op}_{(\mathcal{O}, \mathcal{T})}^{\text{col}}(\mathcal{V})$) the category of $(\mathcal{O}, \mathcal{T})$ -operads (respectively coloured $(\mathcal{O}, \mathcal{T})$ -operads) in $\mathcal{V}$. If $\mathcal{O}$ is an orbital category, then we denote by $\text{Op}_{\mathcal{O}}(\mathcal{V})$ (respectively by $\text{Op}_{\mathcal{O}}^{\text{col}}(\mathcal{V})$) the category of $(\mathcal{O}, \mathcal{O})$ -operads in $\mathcal{V}$ (respectively coloured $(\mathcal{O}, \mathcal{O})$ -operads in $\mathcal{V}$).

Remark 4.2.31. If $\mathcal{O}$ is an orbital category, a coloured $(\mathcal{O}, \mathcal{O})$ -operad in $\mathbf{sSet}$ is a simplicial coloured $\mathcal{O}$ -operad in the sense of [NS22, Definition 2.5.4].

Remark 4.2.32. Note that it should be possible to extend Definition 4.2.30 to the setting of weak indexing systems in the sense of [Ste25d].

Proposition 4.2.33. *Every functor of orbital pairs $F: (\mathcal{O}_1, \mathcal{T}_1) \rightarrow (\mathcal{O}_2, \mathcal{T}_2)$ induces a multifunctor $F: \underline{\Sigma}_{(\mathcal{O}_1, \mathcal{T}_1)} \rightarrow \underline{\Sigma}_{(\mathcal{O}_2, \mathcal{T}_2)}$. In particular, every functor of orbital pairs $F: (\mathcal{O}_1, \mathcal{T}_1) \rightarrow (\mathcal{O}_2, \mathcal{T}_2)$ induces by restriction a functor $F^*: \text{Op}_{(\mathcal{O}_1, \mathcal{T}_1)}(\mathcal{V}) \rightarrow \text{Op}_{(\mathcal{O}_2, \mathcal{T}_2)}(\mathcal{V})$ as well as a functor $F^*: \text{Op}_{(\mathcal{O}_1, \mathcal{T}_1)}^{\text{col}}(\mathcal{V}) \rightarrow \text{Op}_{(\mathcal{O}_2, \mathcal{T}_2)}^{\text{col}}(\mathcal{V})$.*

Proof. The functor F induces a functor $\text{Arr}(\mathbf{F}F): \text{Arr}(\mathbf{F}\mathcal{O}_1) \rightarrow \text{Arr}(\mathbf{F}\mathcal{O}_2)$ and because $F\mathcal{T}_1 \subseteq \mathcal{T}_2$ and because $\mathbf{F}F$ preserves pullbacks of morphisms in $\mathcal{T}$, we deduce that $\text{Arr}(\mathbf{F}F)$ restricts to a functor $\Sigma_F: \Sigma_{(\mathcal{O}_1, \mathcal{T}_1)} \rightarrow \Sigma_{(\mathcal{O}_2, \mathcal{T}_2)}$. Thus, we obtain a multifunctor $\underline{\Sigma}_F: \underline{\Sigma}_{(\mathcal{O}_1, \mathcal{T}_1)} \rightarrow \underline{\Sigma}_{(\mathcal{O}_2, \mathcal{T}_2)}$. The proposition now follows from Construction 4.2.23. $\square$

Example 4.2.34. The final category $*$ is an orbital category and the $\mathbf{F}$ -multicategory $\underline{\Sigma}_*$ is the $\mathbf{F}$ -multicategory that we already defined in Example 4.2.12. In particular, the category $\mathrm{Op}_*^{\mathrm{col}}(\mathcal{V})$, is the category $\mathrm{Op}^{\mathrm{col}}(\mathcal{V})$ of coloured symmetric operads in $\mathcal{V}$.

Example 4.2.35. For every category $\mathcal{C}$, the category $\mathrm{Op}_{(\mathcal{C}, \mathcal{C}^{\approx})}^{\mathrm{col}}(\mathcal{V})$ is the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Op}^{\mathrm{col}}(\mathcal{V}))$. In particular, $\mathbf{B}G$ is an orbital category and the category of $\mathbf{B}G$ -operads in $\mathcal{V}$ is the category $\mathrm{Op}^{\mathrm{col}}(\mathcal{V})^G$ of G -objects in $\mathrm{Op}^{\mathrm{col}}(\mathcal{V})$.

Example 4.2.36. For every G -transfer system $\mathcal{T}$, it follows from Proposition 4.2.27 that we obtain a notion of $\mathcal{T}$ -incomplete coloured operads in $\mathcal{V}$ given by the category $\mathrm{Op}_{(\mathcal{O}_G, \mathcal{O}_{\mathcal{T}})}^{\mathrm{col}}(\mathcal{V})$. This category is expected to be equivalent to the category of *partial genuine equivariant operads* of Bonventre and Pereira [BP21]. However, such a comparison has not yet been made.

Proposition 4.2.37. *For every G -transfer system $\mathcal{T}$, the inclusion $\mathbf{B}G \hookrightarrow \mathcal{O}_G$ produces a functor $(\mathbf{B}G, \mathbf{B}G) \rightarrow (\mathcal{O}_G, \mathcal{O}_{\mathcal{T}})$ of orbital pairs. In particular, using Proposition 4.2.33, we obtain a restriction functor $\mathrm{Op}_{(\mathcal{O}_G, \mathcal{O}_{\mathcal{T}})}^{\mathrm{col}} \rightarrow \mathrm{Op}^{\mathrm{col}}(\mathcal{V})^G$.*

Proof. Note first that the groupoid $\mathbf{B}G$ is contained in every $\mathcal{O}_{\mathcal{T}}$ for every G -transfer system $\mathcal{T}$. Moreover, it is clear that the functor $\mathbf{F}\mathbf{B}G \hookrightarrow \mathbf{F}\mathcal{O}_G$ preserves pullbacks. It follows that for every G -transfer system $\mathcal{T}$, the inclusion $\mathbf{B}G \hookrightarrow \mathcal{O}_G$ provides a functor of orbital pairs $(\mathbf{B}G, \mathbf{B}G) \rightarrow (\mathcal{O}_G, \mathcal{O}_{\mathcal{T}})$. $\square$

4.3 Convolution product for double categories

The goal of this section is to construct a convolution product for double categories that generalises Day's convolution product. Note that similar considerations have already been made recently in [BMZ23]. Until the end of this section, $\mathbb{D}$ is a small double category, $(\mathcal{C}, \otimes_{\mathcal{C}})$ is a cocomplete closed monoidal category, and $\mathcal{V}$ is a cocomplete cartesian closed category.

4.3.1 Monoidal structure on spans

Definition 4.3.1 ([BP21, Definition 4.3]). If $\mathcal{B}$ and $\mathcal{C}$ are categories, then we denote by $\mathrm{Span}_{\mathbb{D}}(\mathcal{B}, \mathcal{C})$ the category with objects the spans of functors

$$\mathcal{B} \xleftarrow{k} \mathcal{A} \xrightarrow{X} \mathcal{C}$$

where $\mathcal{A}$ is a small category, with morphisms the following diagrams

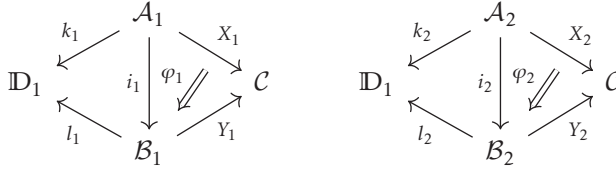
$$\begin{array}{ccccc} & & \mathcal{A} & & \\ & \swarrow k & \downarrow i & \searrow X & \\ \mathcal{B} & & & & \mathcal{C} \\ & \nwarrow l & \downarrow \varphi & \nearrow Y & \\ & & \mathcal{D} & & \end{array}$$

which we write as $(i, \varphi): (k_1, X_1) \rightarrow (k_2, X_2)$, and composition given in the obvious way.

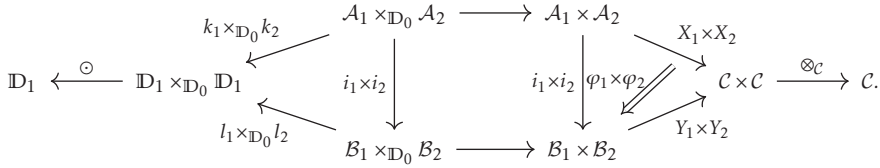
Proposition 4.3.2. *The category $\text{Span}_L(\mathbb{D}_1, \mathcal{C})$ is endowed with the structure of a monoidal category defined as follows: the tensor product $\otimes$ of two spans $\mathbb{D}_1 \xleftarrow{k_1} \mathcal{A}_1 \xrightarrow{X_1} \mathcal{C}$ and $\mathbb{D}_1 \xleftarrow{k_2} \mathcal{A}_2 \xrightarrow{X_2} \mathcal{C}$ is the span*

$$\mathbb{D}_1 \xleftarrow{k} \mathcal{A}_1 \times_{\mathbb{D}_0} \mathcal{A}_2 \xrightarrow{X} \mathcal{C}$$

where the left-hand morphism is the composition $\mathcal{A}_1 \times_{\mathbb{D}_0} \mathcal{A}_2 \xrightarrow{k_1 \times_{\mathbb{D}_0} k_2} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\odot} \mathbb{D}_1$ and where the right-hand side morphism is the composition $\mathcal{A}_1 \times_{\mathbb{D}_0} \mathcal{A}_2 \xrightarrow{X_1 \times X_2} \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes_{\mathcal{C}}} \mathcal{C}$. The tensor product $\otimes$ of a pair of morphisms of spans



is the morphisms of spans $(i_1 \times i_2, \varphi_1 \times \varphi_2)$ obtained by considering the diagram



The unit of $\text{Span}_L(\mathbb{D}_1, \mathcal{C})$ is the span

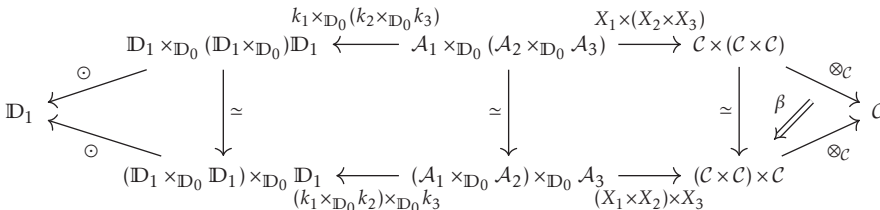
$$\mathbb{D}_1 \xleftarrow{1_{\mathbb{D}}} \mathbb{D}_0 \xrightarrow{1_{\mathcal{C}}} \mathcal{C}$$

where the right-hand side functor is constant on the unit $1_{\mathcal{C}}$ of $\mathcal{C}$.

Proof. We have to construct the associators and unitors of the tensor product $\otimes$. If we consider three spans $S_1 = \mathbb{D}_1 \xleftarrow{k_1} \mathcal{A}_1 \xrightarrow{X_1} \mathcal{C}$, $S_2 = \mathbb{D}_1 \xleftarrow{k_2} \mathcal{A}_2 \xrightarrow{X_2} \mathcal{C}$ and $S_3 = \mathbb{D}_1 \xleftarrow{k_3} \mathcal{A}_3 \xrightarrow{X_3} \mathcal{C}$, their tensor product $S_1 \otimes (S_2 \otimes S_3)$ is the span obtained as the composition of

$$\mathbb{D}_1 \xleftarrow{\odot} \mathbb{D}_1 \times (\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1) \xleftarrow{\quad} \mathcal{A}_1 \times_{\mathbb{D}_0} (\mathcal{A}_2 \times_{\mathbb{D}_0} \mathcal{A}_3) \xrightarrow{\quad} \mathcal{C} \times (\mathcal{C} \times \mathcal{C}) \xrightarrow{\otimes} \mathcal{C}$$

and the iterated tensor product $(S_1 \otimes S_2) \otimes S_3$ is also obtained in a similar way. It follows that the associator $\alpha_{S_1, S_2, S_3}: S_1 \otimes (S_2 \otimes S_3) \simeq (S_1 \otimes S_2) \otimes S_3$ can be provided by the following diagram



where β is the associator of $\mathcal{C}$. The pentagon axiom can be verified using the observation that it is verified both in $\mathcal{C}$ and for pullbacks over a given category. For every span $S = \mathbb{D}_1 \xleftarrow{k} \mathcal{A} \xrightarrow{X} \mathcal{C}$, the right unitor $\eta_S: S \otimes 1 \simeq S$ is given by the diagram

$$\begin{array}{ccccccc}
 & & \mathcal{A} \times_{\mathbb{D}_0} \mathbb{D}_0 & \longrightarrow & \mathcal{C} \times * & \longrightarrow & \mathcal{C} \times \mathcal{C} \\
 & \swarrow & \downarrow \simeq & & \downarrow \simeq & \searrow \rho & \downarrow \otimes_{\mathcal{C}} \\
 \mathbb{D}_1 & & \mathcal{A} & \xrightarrow{X} & \mathcal{C} & \xrightarrow{\text{id}_{\mathcal{C}}} & \mathcal{C} \\
 & \nwarrow k & & & & & \uparrow \text{id}_{\mathcal{C}} \\
 & & & & & & \mathcal{C}
 \end{array}$$

where ρ is the right unitor of $\mathcal{C}$. The left unitor is given by a similar diagram, and the triangle identities are easily verified since they are already verified in $\mathcal{C}$ and for pullbacks over a given category. $\square$

Remark 4.3.3. Note that the tensor product $\otimes$ on $\text{Span}_{\mathbb{L}}(\mathbb{D}_1, \mathcal{C})$ exists only if $\mathbb{D}$ is a strict double category.

4.3.2 Left Kan extension along morphisms of cocartesian fibrations

The results of this subsection are mostly technical and will be used in Subsection 4.3.3.

Notation 4.3.4. Consider a functor $G: \mathcal{A} \rightarrow \mathcal{B}$ over $\mathcal{C}$

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{G} & \mathcal{B} \\
 & \searrow F & \swarrow H \\
 & \mathcal{C} &
 \end{array}$$

Recall that, for X in $\mathcal{B}$, the category $G_{/X}$ is obtained as the following pullback

$$\begin{array}{ccc}
 G_{/X} & \longrightarrow & \mathcal{B}_{/X} \\
 \downarrow & & \downarrow \\
 \mathcal{A} & \longrightarrow & \mathcal{B}.
 \end{array}$$

For every object X of $\mathcal{B}$, we can consider the fibre category $H^{-1}(HX) \hookrightarrow \mathcal{B}$ and we thus obtain a subcategory of $(H^{-1}(HX))_{/X} \hookrightarrow \mathcal{B}_{/X}$. We denote by $G_{/HX}$ the subcategory of $G_{/X}$ obtained as the pullback

$$\begin{array}{ccc}
 G_{/HX} & \longrightarrow & (H^{-1}(HX))_{/X} \\
 \downarrow & & \downarrow \\
 G_{/X} & \longrightarrow & \mathcal{B}_{/X} \\
 \downarrow & & \downarrow \\
 \mathcal{A} & \xrightarrow{G} & \mathcal{B}.
 \end{array}$$

Proposition 4.3.5 ([Hau22, Lemma 3.1.11]). *If $G: \mathcal{A} \rightarrow \mathcal{B}$ is a morphism of cocartesian fibrations*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{G} & \mathcal{B} \\ & \searrow F \quad \swarrow H & \\ & \mathcal{C} & \end{array}$$

then the subcategory $G_{/HX}$ is a reflexive subcategory of $G_{/X}$. In particular, the inclusion $G_{/HX} \hookrightarrow G_{/X}$ is final.

Corollary 4.3.6. *Consider a morphism of left fibrations*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{G} & \mathcal{B} \\ & \searrow F \quad \swarrow H & \\ & \mathcal{C} & \end{array}$$

such that G is an isofibration. For every object X of $\mathcal{B}$, the inclusion $G^{-1}(X) \hookrightarrow G_{/X}$ is a final functor.

Proof. The functor H is a left fibration, and it follows that $H^{-1}(HX)$ is a groupoid. In particular, the category $(H^{-1}(HX))_{/X}$ is contractible. We now obtain the diagram

$$\begin{array}{ccc} G^{-1}(X) & \longrightarrow & * \\ \downarrow & & \downarrow \simeq \\ G_{/HX} & \longrightarrow & (H^{-1}(HX))_{/X} \\ \downarrow & & \downarrow \\ G_{/X} & \longrightarrow & \mathcal{B}_{/X}. \end{array}$$

Finally, because G is an isofibration, the functor $G_{/HX} \rightarrow (H^{-1}(HX))_{/X}$ is also an isofibration and $G^{-1}(X) \rightarrow G_{/HX}$ is an equivalence as it is the pullback of an equivalence along an isofibration. $\square$

4.3.3 Convolution product for double categories

Definition 4.3.7. A double category $\mathbb{D}$ is *target left fibrant* if the target functor $t: \mathbb{D}_1 \rightarrow \mathbb{D}_0$ is a left fibration and if the horizontal composition functor $\odot: \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \rightarrow \mathbb{D}_1$ is an isofibration. If $\mathbb{D}$ is a target left fibrant double category, then a functor $F: \mathcal{A} \rightarrow \mathbb{D}_1$ is a *target left fibration* if it is an isofibration and if the composition $\mathcal{A} \xrightarrow{F} \mathbb{D}_1 \xrightarrow{t} \mathbb{D}_0$ is a left fibration.

For the rest of this section, $\mathbb{D}$ is a small target left fibrant double category.

Definition 4.3.8. The category $\text{Span}_L^t(\mathbb{D}_1, \mathcal{C})$ is the full subcategory of $\text{Span}_L(\mathbb{D}_1, \mathcal{C})$ spanned by the spans $\mathbb{D}_1 \xleftarrow{k} \mathcal{A} \xrightarrow{X} \mathcal{C}$ such that k is a target left fibration.

Proposition 4.3.9. The tensor product $\otimes$ on $\text{Span}_L(\mathbb{D}_1, \mathcal{C})$ restricts to a monoidal structure on the subcategory $\text{Span}_L^t(\mathbb{D}_1, \mathcal{C})$.

Proof. Consider two spans $\mathbb{D}_1 \xleftarrow{k} \mathcal{A} \xrightarrow{X} \mathcal{C}$ and $\mathbb{D}_1 \xleftarrow{l} \mathcal{B} \xrightarrow{Y} \mathcal{C}$ such that k and l are target left fibrations. The left leg of their tensor product is the following composition

$$\mathcal{A} \times_{\mathbb{D}_0} \mathcal{B} \xrightarrow{k \times_{\mathbb{D}_0} l} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\odot} \mathbb{D}_1$$

and its composition with $t: \mathbb{D}_1 \rightarrow \mathbb{D}_0$ is the composition

$$\mathcal{A} \times_{\mathbb{D}_0} \mathcal{B} \xrightarrow{p_2} \mathcal{B} \xrightarrow{l} \mathbb{D}_1 \xrightarrow{t} \mathbb{D}_0$$

that is a left fibration as a composition of left fibrations (using that $\mathcal{A} \times_{\mathbb{D}_0} \mathcal{B} \rightarrow \mathcal{B}$ is a pullback of $t \circ k$). $\square$

Proposition 4.3.10. The functor

$$\text{Lan}: \text{Span}_L(\mathbb{D}_1, \mathcal{C}) \rightarrow \text{Fun}(\mathbb{D}_1, \mathcal{C})$$

that sends a span $\mathbb{D}_1 \xleftarrow{k} \mathcal{A} \xrightarrow{X} \mathcal{C}$ to the left Kan extension of X along k is a left adjoint of the functor $v: \text{Fun}(\mathbb{D}_1, \mathcal{C}) \rightarrow \text{Span}_L(\mathbb{D}_1, \mathcal{C})$ that sends a functor $X: \mathbb{D}_1 \rightarrow \mathcal{C}$ to the span $\mathbb{D}_1 \xleftarrow{\text{id}_{\mathbb{D}_1}} \mathbb{D}_1 \xrightarrow{X} \mathcal{C}$. Moreover, this adjunction restricts to the adjunction

$$\text{Lan}: \text{Span}_L^t(\mathbb{D}_1, \mathcal{C}) \rightleftarrows \text{Fun}(\mathbb{D}_1, \mathcal{C}): v.$$

Proof. The first part of the proposition is [BP21, Remark 4.5]. The second part of the statement follows from the fact that the identity of $\mathbb{D}_1$ is a target left fibration. $\square$

Proposition 4.3.11 ([Kel74, Theorem 3.1]). For every reflexive adjunction $L: \mathcal{D} \rightleftarrows \mathcal{E}: R$ such that the category $\mathcal{D}$ is endowed with the structure of a monoidal category $(\mathcal{D}, \otimes_{\mathcal{D}})$, if for all A and B in $\mathcal{D}$, the morphism

$$L(\eta_A \otimes_{\mathcal{D}} \eta_B): L(A \otimes B) \rightarrow L(RLA \otimes_{\mathcal{D}} RLB)$$

is an isomorphism, then $\mathcal{E}$ is naturally endowed with the structure of a monoidal category where the tensor product of two objects X and Y of $\mathcal{E}$ is defined by $X \otimes_{\mathcal{E}} Y := L(R(X) \otimes_{\mathcal{D}} R(Y))$ and where the unit of $\mathcal{E}$ is the image of the unit of $\mathcal{D}$ under L . Moreover, the category of algebras in $\mathcal{E}$ over $\otimes_{\mathcal{E}}$ is the full subcategory of the category of algebras in $\mathcal{D}$ over $\otimes_{\mathcal{D}}$ spanned by the algebras whose underlying object belongs to $\mathcal{E}$. Furthermore, if $X_1, \dots, X_n$ are objects of $\mathcal{E}$, then we have $X_1 \otimes_{\mathcal{E}} \dots \otimes_{\mathcal{E}} X_n \simeq L(R(X) \otimes_{\mathcal{D}} \dots \otimes_{\mathcal{D}} R(X_n))$.

Proposition 4.3.12. *For all functors $F: \mathcal{A}_1 \rightarrow \mathcal{C}$, $G: \mathcal{B}_1 \rightarrow \mathcal{C}$, $H: \mathcal{A}_1 \rightarrow \mathcal{B}_1$ and $L: \mathcal{A}_2 \rightarrow \mathcal{B}_2$ such that $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are small categories, if we denote by $L_H F$ and $L_L G$ the left Kan extension of F along H and G along L respectively, then the left Kan extension of $\otimes_{\mathcal{C}} \circ (F \times G)$ along $H \times L$ is $\otimes_{\mathcal{C}} \circ (L_H F \times L_L G)$.*

Proof. This follows directly from the pointwise formula for left Kan extensions using that $\otimes_{\mathcal{C}}$ preserves colimits in both variables. $\square$

Proposition 4.3.13. *The adjunction*

$$\text{Lan}: \text{Span}_{\mathbb{L}}^t(\mathbb{D}_1, \mathcal{C}) \rightleftarrows \text{Fun}(\mathbb{D}_1, \mathcal{C}): v$$

satisfies the conditions of Proposition 4.3.11.

Proof. Consider two spans $\mathbb{D}_1 \xleftarrow{F} \mathcal{A} \xrightarrow{X} \mathcal{C}$ and $\mathbb{D}_1 \xleftarrow{G} \mathcal{B} \xrightarrow{Y} \mathcal{C}$ such that F and G are target left fibrations, and consider their associated left Kan extensions

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{X} & \mathcal{C} \\ F \downarrow & \searrow \varphi & \nearrow L_F X \\ \mathbb{D}_1 & & \end{array} \quad \begin{array}{ccc} \mathcal{B} & \xrightarrow{Y} & \mathcal{C} \\ G \downarrow & \searrow \psi & \nearrow L_G Y \\ \mathbb{D}_1 & & \end{array}$$

We have to show that the composite of the following diagram

$$\begin{array}{ccccc} \mathcal{A} \times_{\mathbb{D}_0} \mathcal{B} & \longrightarrow & \mathcal{A} \times \mathcal{B} & & \\ F \times_{\mathbb{D}_0} G \downarrow & & F \times G \downarrow & \searrow \varphi \times \psi & \nearrow X \times Y \\ \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 & \longrightarrow & \mathbb{D}_1 \times \mathbb{D}_1 & \xrightarrow{(L_F X) \times (L_G Y)} & \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes_{\mathcal{C}}} \mathcal{C} \end{array} \quad (E)$$

is a left Kan extension diagram. Using Proposition 4.3.12 and the pointwise formula for left Kan extension, it is enough to show that the functor

$$\Phi: (F \times_{\mathbb{D}_0} G)_{/(X,Y)} \rightarrow (F \times G)_{/(X,Y)}$$

is final. The left-hand side square of the diagram (E) is a pullback, and we thus obtain the following square

$$\begin{array}{ccc} (F \times_{\mathbb{D}_0} G)^{-1}(X, Y) & \xrightarrow{\sim} & (F \times G)^{-1}(X, Y) \\ \downarrow & & \downarrow \\ (F \times_{\mathbb{D}_0} G)_{/(X,Y)} & \xrightarrow{\Phi} & (F \times G)_{/(X,Y)}. \end{array} \quad (F)$$

The functor $F \times G$ is an isofibration as a product of isofibrations, and it follows that $F \times_{\mathbb{D}_0} G$ is also an isofibration as the pullback of an isofibration.

Moreover, $F \times G$ can be seen as a morphism of left fibrations as follows

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{B} & \xrightarrow{F \times G} & \mathbb{D}_1 \times \mathbb{D}_1 \\ & \searrow (toF) \times (toG) \quad \swarrow t \times t & \\ & \mathbb{D}_0 \times \mathbb{D}_0 & \end{array}$$

and the functor $F \times_{\mathbb{D}_0} G$ can be seen as a morphism of left fibrations as follows

$$\begin{array}{ccc} \mathcal{A} \times_{\mathbb{D}_0} \mathcal{B} & \xrightarrow{F \times_{\mathbb{D}_0} G} & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \\ & \searrow toG \circ p_2 \quad \swarrow top_2 & \\ & \mathbb{D}_0 & \end{array}$$

Finally, by Corollary 4.3.6, the vertical inclusions of the square (F) are final functors, and it follows that Φ is also final. $\square$

Corollary 4.3.14. *The category $\text{Fun}(\mathbb{D}_1, \mathcal{C})$ is naturally endowed with the structure of a closed monoidal category for which the tensor product $\otimes_{\mathbb{D}}$ of two functors $F, G: \mathbb{D}_1 \rightarrow \mathcal{C}$ is given by the following left Kan extension*

$$\begin{array}{ccccc} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 & \xrightarrow{F \times G} & \mathcal{C} \times \mathcal{C} & \xrightarrow{\otimes_{\mathcal{C}}} & \mathcal{C} \\ \circlearrowleft \downarrow & \swarrow & \searrow & \nearrow & \\ \mathbb{D}_1 & & & \nearrow F \otimes_{\mathbb{D}} G & \end{array}$$

and whose unit is the left Kan extension

$$\begin{array}{ccc} \mathbb{D}_0 & \longrightarrow & * \xrightarrow{1_{\mathcal{C}}} \mathcal{C} \\ 1_{\mathbb{D}} \downarrow & \swarrow & \nearrow \\ \mathbb{D}_1 & & \end{array}$$

Moreover, the category of algebras over $\otimes_{\mathbb{D}}$ is the category $\text{Lax}(\mathbb{D}, \mathcal{C})$. Furthermore, for all functors $F_1, \dots, F_n: \mathbb{D}_1 \rightarrow \mathcal{C}$, the tensor product $F_1 \otimes_{\mathbb{D}} \dots \otimes_{\mathbb{D}} F_n$ is the left Kan extension

$$\begin{array}{ccc} \mathbb{D}_n & \longrightarrow & \mathcal{C}^{\times n} \xrightarrow{\otimes_{\mathcal{C}}^n} \mathcal{C} \\ \circlearrowleft^n \downarrow & \swarrow & \nearrow \\ \mathbb{D}_1 & & \nearrow F_1 \otimes_{\mathbb{D}} \dots \otimes_{\mathbb{D}} F_n \end{array}$$

where $\mathbb{D}_n$ is the pullback $\mathbb{D}_1 \times_{\mathbb{D}_0} \dots \times_{\mathbb{D}_0} \mathbb{D}_1$ with n occurrences of $\mathbb{D}_1$.

Proof. The existence of the monoidal structure follows from the combination of Proposition 4.3.13 and Proposition 4.3.11. We show now that $(\text{Fun}(\mathbb{D}_1, \mathcal{C}), \otimes_{\mathbb{D}})$ is a closed monoidal category. We will show that $(\text{Fun}(\mathbb{D}_1, \mathcal{C}), \otimes_{\mathbb{D}})$ is left closed;

the right closedness can be shown with a symmetric proof. Let F be a functor $F: \mathbb{D}_1 \rightarrow \mathcal{C}$. We have to show that the functor

$$(-) \otimes_{\mathbb{D}} F: \text{Fun}(\mathbb{D}_1, \mathcal{C}) \rightarrow \text{Fun}(\mathbb{D}_1, \mathcal{C})$$

admits a right adjoint. This functor can be viewed as the following composition

$$\text{Fun}(\mathbb{D}_1, \mathcal{C}) \xrightarrow{\otimes_{\mathcal{C}} \circ ((-) \times F)} \text{Fun}(\mathbb{D}_1 \times \mathbb{D}_1, \mathcal{C}) \longrightarrow \text{Fun}(\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1, \mathcal{C}) \xrightarrow{\odot_!} \text{Fun}(\mathbb{D}_1, \mathcal{C}).$$

The last functor is a left adjoint of $\odot^*$ and the second functor admits a right adjoint given by taking right Kan extensions along the functor $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \rightarrow \mathbb{D}_1 \times \mathbb{D}_1$. It remains to show that the first functor admits a right adjoint. As the monoidal category $\mathcal{C}$ is closed, this functor is a left adjoint of the functor $\text{Fun}(\mathbb{D}_1 \times \mathbb{D}_1, \mathcal{C}) \rightarrow \text{Fun}(\mathbb{D}_1, \mathcal{C})$ that sends a functor $H: \mathbb{D}_1 \times \mathbb{D}_1 \rightarrow \mathcal{C}$ to the functor $\mathbb{D}_1 \rightarrow \mathcal{C}$ that sends an object X of $\mathbb{D}_1$ to the end

$$\int_{Y \in \mathbb{D}_1} \text{hom}_{\mathcal{C}}(F(Y), H(X, Y)).$$

We still have to show that the category of algebras over $(\text{Fun}(\mathbb{D}_1, \mathcal{C}), \otimes_{\mathbb{D}})$ is the category $\text{Lax}(\mathbb{D}, \mathcal{C})$. Using again Proposition 4.3.11, it is enough to show that the full subcategory of algebras in $\text{Span}_{\mathbb{L}}(\mathbb{D}_1, \mathcal{C})$ spanned by the algebras

whose underlying spans have the form $\mathbb{D}_1 \xleftarrow{\text{id}_{\mathbb{D}_1}} \mathbb{D}_1 \xrightarrow{X} \mathcal{C}$ is equivalent to $\text{Lax}(\mathbb{D}, \mathcal{C})$. Consider now a span $S = \mathbb{D}_1 \xleftarrow{\text{id}_{\mathbb{D}_1}} \mathbb{D}_1 \xrightarrow{X} \mathcal{C}$ with a structure of algebra $(S, \mu_S, 1_S)$. The multiplication $\mu_S: S \otimes S \rightarrow S$ is given by a diagram

$$\begin{array}{ccccc} & & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 & & \\ & \swarrow \odot & \downarrow \odot & \searrow \otimes_{\mathcal{C}} \circ (S \times S) & \\ \mathbb{D}_1 & & \mathbb{D}_1 & & \mathcal{C} \\ & \swarrow \text{id}_{\mathbb{D}_1} & & \searrow S & \\ & & \mathbb{D}_1 & & \end{array}$$

α

and the unit $1_S: 1 \rightarrow S$ is given by a diagram

$$\begin{array}{ccccc} & & \mathbb{D}_0 & & \\ & \swarrow 1_{\mathbb{D}} & \downarrow 1_{\mathbb{D}} & \searrow 1_{\mathcal{C}} & \\ \mathbb{D}_1 & & \mathbb{D}_1 & & \mathcal{C} \\ & \swarrow \text{id}_{\mathbb{D}_1} & & \searrow S & \\ & & \mathbb{D}_1 & & \end{array}$$

β

and it is clear that the natural transformations α and β endow $S: \mathbb{D}_1 \rightarrow \mathcal{C}$ with the structure of a lax functor from $\mathbb{D}$ to $\mathcal{C}$. If we consider now two such algebras $(S, \mu_S, 1_S)$ and $(T, \mu_T, 1_T)$ in $\text{Span}_{\mathbb{L}}(\mathbb{D}_1, \mathcal{C})$, a morphism of algebras $(S, \mu_S, 1_S) \rightarrow (T, \mu_T, 1_T)$ is given by a diagram

$$\begin{array}{ccccc} & & \mathbb{D}_1 & & \\ & \swarrow \text{id}_{\mathbb{D}_1} & \downarrow \text{id}_{\mathbb{D}_1} & \searrow S & \\ \mathbb{D}_1 & & \mathbb{D}_1 & & \mathcal{C} \\ & \swarrow \text{id}_{\mathbb{D}_1} & & \searrow T & \\ & & \mathbb{D}_1 & & \end{array}$$

ϕ

and the natural transformation $\phi: S \rightarrow T$ thus obtained produces a vertical natural transformation between the corresponding lax functors. This provides the equivalence between $\text{Alg}(\text{Fun}(\mathbb{D}_1, \mathcal{C}), \otimes_{\mathbb{D}})$ and $\text{Lax}(\mathbb{D}, \mathcal{C})$. $\square$

Remark 4.3.15. Similarly, a convolution product on $\text{Fun}(\mathbb{D}_1^{\text{op}}, \mathcal{C})$ can be constructed for other classes of double categories, such as framed double categories. In this chapter, we chose to concentrate only on target left fibrant double categories, as this class of double categories provides all the examples we are interested in.

4.3.4 Composition product for F-multicategories

In this section, we use the convolution product for double categories that we constructed in the previous section to define a composition product for F-multicategories.

Definition 4.3.16. An F-multicategory $\mathbb{M}$ is *target right fibrant* if $t: \mathbb{M}_1 \rightarrow \mathbb{M}_0$ is a right fibration and if $\odot: \mathbb{M}_1 \times_{\mathbb{M}_0} \mathbb{F}\mathbb{M}_1$ is an isofibration.

Proposition 4.3.17. For every orbital pair $(\mathcal{O}, \mathcal{T})$, the F-multicategory $\underline{\Sigma}_{(\mathcal{O}, \mathcal{T})}$ is a target right fibrant F-multicategory.

Proof. We prove first that the target functor $t: \Sigma_{(\mathcal{O}, \mathcal{T})} \rightarrow \mathcal{O}$ is a right fibration. If we consider any morphism $f: A \rightarrow B$ in $\mathcal{T}$ such that B belongs to $\mathcal{O}$ and any morphism $g: C \rightarrow B$ in $\mathcal{O}$, every pullback square

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

provides a cartesian lift of $g: C \rightarrow t(f)$ in $\Sigma_{(\mathcal{O}, \mathcal{T})}$. Since, by definition, morphisms in $\Sigma_{(\mathcal{O}, \mathcal{T})}$ are pullback squares, this shows that t is a right fibration. We now prove that $\odot: \Sigma_{(\mathcal{O}, \mathcal{T})} \times_{\mathbb{F}\mathcal{O}} \mathbb{F}\Sigma_{(\mathcal{O}, \mathcal{T})} \rightarrow \Sigma_{(\mathcal{O}, \mathcal{T})}$ is an isofibration. If we consider any isomorphism

$$\begin{array}{ccc} A & \xrightarrow{\cong} & B \\ \downarrow g & & \downarrow i \\ & & C \\ \downarrow & & \downarrow j \\ D & \xrightarrow{\cong} & E, \end{array} \tag{L}$$

in $\Sigma_{(\mathcal{O}, \mathcal{T})}$, then the choice of any pullback square

$$\begin{array}{ccc} F & \xrightarrow{\cong} & C \\ k \downarrow & & \downarrow j \\ D & \xrightarrow{\cong} & E \end{array}$$

provides a lift of (L) given by the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\simeq} & B \\
 \downarrow & & \downarrow i \\
 F & \xrightarrow{\simeq} & C \\
 \downarrow k & & \downarrow j \\
 D & \xrightarrow[\simeq]{} & E,
 \end{array}
 \quad g \left(\begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \right)$$

seen as an isomorphism in $\Sigma_{(\mathcal{O}, \mathcal{T})} \times_{\mathbf{F}\mathcal{O}} \mathbf{F}\Sigma_{(\mathcal{O}, \mathcal{T})}$. $\square$

Proposition 4.3.18. *The functor $\mathbf{F}$ preserves cartesian fibrations, left fibrations, and isofibrations. In particular, if $\mathbb{M}$ is a target right fibrant $\mathbf{F}$ -multicategory, then $\mathbf{F}\mathbb{M}^{\text{op}}$ is a target left fibrant double category.*

Proof. The first part of the statement is analogous to the proof of [BP21, Lemma 2.20]. Suppose now that $\mathbb{M}$ is a target right fibrant $\mathbf{F}$ -multicategory. The horizontal composition $\mathbf{F}\mathbb{M}_1 \times_{\mathbf{F}\mathbb{M}_0} \mathbf{F}\mathbb{M}_1 \rightarrow \mathbf{F}\mathbb{M}_1$ of $\mathbf{F}\mathbb{M}_1$ is given by $\mathbf{F}\odot$ with $\odot: \mathbb{M}_1 \times_{\mathbf{F}\mathbb{M}_0} \mathbb{M}_1 \rightarrow \mathbb{M}_1$ the composition functor of $\mathbb{M}$, using the isomorphism

$$\mathbf{F}\mathbb{M}_1 \times_{\mathbf{F}\mathbb{M}_0} \mathbf{F}\mathbb{M}_1 \simeq \mathbf{F}(\mathbb{M}_1 \times_{\mathbf{F}\mathbb{M}_0} \mathbb{M}_1).$$

Moreover, the target functor $\mathbf{F}\mathbb{M}_1 \rightarrow \mathbf{F}\mathbb{M}_0$ is the image under $\mathbf{F}$ of the target functor $t: \mathbb{M}_1 \rightarrow \mathbb{M}_0$. Finally, the result follows from the fact that the opposite functor of an isofibration is an isofibration and from the fact that the opposite functor of a right fibration is a left fibration. $\square$

Proposition 4.3.19. *If the following diagram*

$$\begin{array}{ccc}
 \mathcal{A}^{\text{op}} & \xrightarrow{F} & \mathcal{V} \\
 G^{\text{op}} \downarrow & \eta \swarrow & \nearrow L_G F \\
 \mathcal{B}^{\text{op}} & &
 \end{array}$$

exhibits $L_G F$ as the left Kan extension of F along G^{op} , then the following diagram

$$\begin{array}{ccc}
 \mathbf{F}\mathcal{A}^{\text{op}} & \xrightarrow{(\mathbf{F}F^{\text{op}})^{\text{op}}} & (\mathbf{F}\mathcal{V}^{\text{op}})^{\text{op}} \xrightarrow{\widetilde{\text{id}}_{\mathcal{V}}} \mathcal{V} \\
 \mathbf{F}G^{\text{op}} \downarrow & \mathbf{F}\eta \swarrow & \nearrow L_G F \\
 \mathbf{F}\mathcal{B}^{\text{op}} & & \xrightarrow{\widetilde{L_G F}}
 \end{array}$$

is a left Kan extension diagram, with $\widetilde{(-)}$ as defined in Remark 4.2.3.

Proof. As suggested by [BP21, Remark 2.25], the proof of this proposition is the same as the proof of [BP21, Lemma 2.21]. $\square$

Corollary 4.3.20. *For every target right fibrant $\mathbf{F}$ -multicategory, we obtain a monoidal structure $(\text{Coll}_{\mathbb{M}}(\mathcal{V}), \circ_{\mathbb{M}})$ on the category $\text{Coll}_{\mathbb{M}}(\mathcal{V})$ where $\circ_{\mathbb{M}}$ is defined on two $\mathbb{M}$ -collections $S, T: \mathbb{M}_1^{\text{op}} \rightarrow \mathcal{V}$ as the functor $S \circ_{\mathbb{M}} T$ given by the left Kan extension*

$$\begin{array}{ccc}
 (\mathbb{M}_1 \times_{\mathbf{FM}_0} \mathbf{FM}_1)^{\text{op}} & \xrightarrow{S \times \tilde{T}} \mathcal{V} \times \mathcal{V} & \xrightarrow{\times} \mathcal{V} \\
 \circ_{\mathbb{M}}^{\text{op}} \downarrow & \swarrow & \nearrow S \circ_{\mathbb{M}} T \\
 \mathbb{M}_1^{\text{op}} & &
 \end{array} \quad (\text{A})$$

Moreover, the category of algebras in $(\text{Coll}_{\mathbb{M}}(\mathcal{V}), \circ_{\mathbb{M}})$ is $\text{Op}_{\mathbb{M}}(\mathcal{V})$.

Proof. The category $\text{Coll}_{\mathbb{M}}(\mathcal{V})$ can be identified with the full subcategory of $\text{Fun}(\mathbf{FM}_1^{\text{op}}, \mathcal{V})$ spanned by the functors $F: \mathbf{FM}_1^{\text{op}} \rightarrow \mathcal{V}$ that preserve finite products. It is now enough to show that for all functors $S, T: \mathbb{M}_1^{\text{op}} \rightarrow \mathcal{V}$, if we denote by $\tilde{S}, \tilde{T}: \mathbf{FM}_1^{\text{op}} \rightarrow \mathcal{V}$ the associated functors that preserve finite products, then the functor $\tilde{S} \otimes_{\mathbf{FM}^{\text{op}}} \tilde{T}$ is isomorphic to $\widetilde{S \circ_{\mathbb{M}} T}$. The category $\mathbf{FM}_1 \times_{\mathbf{FM}_0} \mathbf{FM}_1$ is equivalent to $\mathbf{F}(\mathbb{M}_1 \times_{\mathbf{FM}_0} \mathbf{FM}_1)$ and, through this equivalence, the functor $(\mathbf{FM}_1 \times_{\mathbf{FM}_0} \mathbf{FM}_1)^{\text{op}} \rightarrow \mathbf{FM}_1^{\text{op}} \times \mathbf{FM}_1^{\text{op}} \xrightarrow{\tilde{S} \times \tilde{T}} \mathcal{V} \times \mathcal{V} \xrightarrow{\times} \mathcal{V}$ is isomorphic to the composition of the image of $(\mathbb{M}_1 \times_{\mathbf{FM}_0} \mathbf{FM}_1)^{\text{op}} \rightarrow \mathbb{M}_1^{\text{op}} \times \mathbf{FM}_1^{\text{op}} \rightarrow \mathcal{V} \times \mathcal{V} \xrightarrow{\times} \mathcal{V}$ under $\mathbf{F}((-)^{\text{op}})^{\text{op}}$ with $\widetilde{\text{id}_{\mathcal{V}}}: \mathbf{F}(\mathcal{V}^{\text{op}})^{\text{op}} \rightarrow \mathcal{V}$. The first part of the corollary now follows from Proposition 4.3.19 applied to the diagram (A). The second part of the corollary follows from the fact that the inclusion $\text{Coll}_{\mathbb{M}}(\mathcal{V}) \hookrightarrow \text{Fun}(\mathbf{FM}_1^{\text{op}}, \mathcal{V})$ is monoidal and from Corollary 4.3.14 using the description of $\mathbb{M}$ -operads in terms of lax functors. $\square$

Proposition 4.3.21. *For every target right fibrant $\mathbf{F}$ -multicategory $\mathbb{M}$ and for every right fibration $F: \mathcal{A} \rightarrow \mathbb{M}_0$, the $\mathbf{F}$ -multicategory $F^*\mathbb{M}$ is also target right fibrant. In particular, for every $\mathbb{M}$ -set of colours A , the $\mathbf{F}$ -multicategory $\mathbb{M}^A$ is target right fibrant if $\mathbb{M}$ is target right fibrant.*

Proof. We show first that $t: F^*\mathbb{M}_1 \rightarrow \mathcal{A}$ is a right fibration. The functor t is by definition the projection $p_2: \mathbb{M}_1 \times_{(\mathbb{M}_0 \times \mathbf{FM}_0)} (\mathcal{A} \times \mathbf{FA}) \rightarrow \mathcal{A}$ that is a pullback of the functor $\mathbb{M}_1 \times_{\mathbf{FM}_0} \mathbf{FA} \rightarrow \mathbb{M}_0$. The functor $\mathbb{M}_1 \times_{\mathbf{FM}_0} \mathbf{FA} \rightarrow \mathbb{M}_0$ is the composition

$$\mathbb{M}_1 \times_{\mathbf{FM}_0} \mathbf{FA} \rightarrow \mathbb{M}_1 \xrightarrow{t_{\mathbb{M}}} \mathbb{M}_0,$$

where the first functor is a right fibration as a pullback of $\mathbf{FA} \xrightarrow{\mathbf{FF}} \mathbf{FM}_0$ and where $t_{\mathbb{M}}$ is, by assumptions a right fibration. It follows that $\mathbb{M}_1 \times_{\mathbf{FM}_0} \mathbf{FA} \rightarrow \mathbb{M}_0$ and t are also right fibrations. We now have to show that the functor

$$(\mathbb{M}_1 \times_{\mathbf{FM}_0} \mathbf{FM}_1) \times_{(\mathbb{M}_0 \times \mathbf{FM}_0 \times \mathbf{F}^2 \mathbb{M}_0)} (\mathcal{A} \times \mathbf{FA} \times \mathbf{F}^2 \mathcal{A}) \rightarrow \mathbb{M}_1 \times_{(\mathbb{M}_0 \times \mathbf{FM}_0)} (\mathcal{A} \times \mathbf{FA})$$

is an isofibration. Note that we have an isomorphism

$$\mathbb{M}_1 \times_{(\mathbb{M}_0 \times \mathbf{FM}_0)} (\mathcal{A} \times \mathbf{FA}) \simeq \mathbb{M}_1 \times_{(\mathbb{M}_0 \times \mathbf{F}^2 \mathbb{M}_0)} (\mathcal{A} \times \mathbf{F}^2 \mathcal{A})$$

and that we have a pullback square

$$\begin{array}{ccc}
 (\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1) \times_{(\mathbb{M}_0 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}^2 \mathbb{M}_0)} (\mathcal{A} \times \mathbb{F}\mathcal{A} \times \mathbb{F}^2 \mathcal{A}) & \longrightarrow & \mathbb{F}\mathcal{A} \\
 \downarrow G & & \downarrow \mathbb{F}\mathbb{F} \\
 (\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1) \times_{(\mathbb{M}_0 \times_{\mathbb{F}^2 \mathbb{M}_0})} (\mathcal{A} \times \mathbb{F}^2 \mathcal{A}) & \longrightarrow & \mathbb{F}\mathbb{M}_0.
 \end{array}$$

The functor $\odot_{\mathbb{F}^*\mathbb{M}}$ is the composition of G with $\odot_{\mathbb{M}} \times_{\text{id}_{(\mathbb{M}_0 \times_{\mathbb{F}^2 \mathbb{M}_0)}} (\mathcal{A} \times \mathbb{F}^2 \mathcal{A})$ and the result follows from the fact that G is an isofibration as the pullback of an isofibration (using that $\mathbb{F}$ preserves isofibrations) and from the fact that $\odot_{\mathbb{M}}$ is an isofibration. $\square$

Corollary 4.3.22. *For every target right fibrant $\mathbb{F}$ -multicategory $\mathbb{M}$ and every $\mathbb{M}$ -set of colours $\mathcal{A}$, the category of $\mathcal{A}$ -coloured $\mathbb{M}$ -collections in $\mathcal{V}$ is endowed with the structure of a monoidal category for which the associated category of algebras is the category of $\mathcal{A}$ -coloured $\mathbb{M}$ -operads in $\mathcal{V}$.*

Proof. This follows directly from Corollary 4.3.20 and Proposition 4.3.21. $\square$

4.3.5 Base change adjunctions

In this section, we investigate when a multifunctor $\phi: \mathbb{M} \rightarrow \mathbb{N}$ between target right fibrant $\mathbb{F}$ -multicategories induces base change adjunctions between (coloured) $\mathbb{M}$ -operads and $\mathbb{N}$ -operads in $\mathcal{V}$. In particular, we study base change adjunctions associated with changes of colours.

Proposition 4.3.23. *Let $\phi: \mathbb{D} \rightarrow \mathbb{E}$ be a double functor between target left fibrant double categories. The functor $(\phi_1)^*: \text{Fun}(\mathbb{E}_1, \mathcal{C}) \rightarrow \text{Fun}(\mathbb{D}_1, \mathcal{C})$ is naturally endowed with the structure of a lax functor.*

Proof. The following diagram

$$\begin{array}{ccccc}
 \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 & \xrightarrow{\phi_1 \times \phi_0} & \mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 & \xrightarrow{S \times T} & \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes_{\mathcal{C}}} \mathcal{C} \\
 \downarrow \circ & & \downarrow \circ & \swarrow & \nearrow S \otimes_{\mathbb{D}} T \\
 \mathbb{D}_1 & \xrightarrow{\phi_1} & \mathbb{E}_1 & &
 \end{array}$$

induces a morphism $(S \circ \phi_1) \otimes_{\mathbb{D}} (T \circ \phi_1) \rightarrow (S \otimes_{\mathbb{D}} T) \circ \phi_1$ by the universal property of left Kan extension and the following diagram

$$\begin{array}{ccccc}
 \mathbb{D}_0 & \xrightarrow{\phi_0} & \mathbb{E}_0 & \xrightarrow{1_{\mathcal{C}}} & \mathcal{C} \\
 1_{\mathbb{D}} \downarrow & & 1_{\mathbb{E}} \downarrow & \swarrow & \nearrow 1_{\text{Fun}(\mathbb{D}_1, \mathcal{C})} \\
 \mathbb{D}_1 & \xrightarrow{\phi_1} & \mathbb{E}_1 & &
 \end{array}$$

induces a morphism $1_{\text{Fun}(\mathbb{D}_1, \mathcal{C})} \rightarrow (\phi^*)(1_{\text{Fun}(\mathbb{E}_1, \mathcal{C})})$. We thus obtain the structure of a lax functors on $(\phi_1)^*$. $\square$

Corollary 4.3.24. *Let $\phi: \mathbb{M} \rightarrow \mathbb{N}$ be a functor between target right fibrant $\mathbf{F}$ -multicategories. The functor $(\phi_1)^*: \text{Coll}_{\mathbb{N}}(\mathcal{V}) \rightarrow \text{Coll}_{\mathbb{M}}(\mathcal{V})$ is naturally endowed with the structure of a lax functor. In particular, the functor $(\phi_1)_!: \text{Coll}_{\mathbb{M}}(\mathcal{V}) \rightarrow \text{Coll}_{\mathbb{N}}(\mathcal{V})$ is naturally endowed with the structure of an oplax functor.*

Proof. This follows directly from Proposition 4.3.23 using that $(\phi_1)^*$ is the restriction of

$$(\mathbf{F}\phi_1^{\text{op}})^*: \text{Fun}(\mathbf{F}\mathbb{N}_1^{\text{op}}, \mathcal{V}) \rightarrow \text{Fun}(\mathbf{F}\mathbb{M}_1^{\text{op}}, \mathcal{C})$$

to $\text{Coll}_{\mathbb{N}}(\mathcal{V}) \subseteq \text{Fun}(\mathbf{F}\mathbb{N}_1^{\text{op}}, \mathcal{V})$ and $\text{Coll}_{\mathbb{M}}(\mathcal{V}) \subseteq \text{Fun}(\mathbf{F}\mathbb{M}_1^{\text{op}}, \mathcal{V})$. $\square$

Proposition 4.3.25. *For every multifunctor $\phi: \mathbb{M} \rightarrow \mathbb{N}$, if for every $\mathbb{N}$ -set of colours X , the functor $(\phi^X)^*: \text{Op}_{\mathbb{N}}^X(\mathcal{V}) \rightarrow \text{Op}_{\mathbb{M}}^{(\phi_0)^*X}(\mathcal{V})$ admits a right adjoint (respectively a left adjoint), then the functor $\phi^*: \text{Op}_{\mathbb{N}}^{\text{col}}(\mathcal{V}) \rightarrow \text{Op}_{\mathbb{M}}^{\text{col}}(\mathcal{V})$ admits a right adjoint (respectively a left adjoint).*

Proof. Suppose first that for every $\mathbb{N}$ -set of colours X , $(\phi^X)^*: \text{Op}_{\mathbb{N}}^X(\mathcal{V}) \rightarrow \text{Op}_{\mathbb{M}}^{(\phi_0)^*X}(\mathcal{V})$ admits a right adjoint. If we consider the adjunction

$$(\phi_0)^*: \text{Psh}(\mathbb{N}_0) \rightleftarrows \text{Psh}(\mathbb{M}_0): (\phi_0)_*,$$

we have that $\Phi_\phi: \Phi_{\mathbb{N}} \Rightarrow \Phi_{\mathbb{M}} \circ (\phi_0)^*$ is a left morphism from $\Phi_{\mathbb{N}}$ to $\Phi_{\mathbb{M}}$ in the sense of [HP15, Section 2.3] and we thus obtain the promised adjunction as described in [HP15, Section 2.3]. Suppose now that for every $\mathbb{N}$ -set of colours X , $(\phi^X)^*: \text{Op}_{\mathbb{N}}^X(\mathcal{V}) \rightarrow \text{Op}_{\mathbb{M}}^{(\phi_0)^*X}(\mathcal{V})$ admits a left adjoint. If we consider the adjunction

$$(\phi_0)_!: \text{Psh}(\mathbb{M}_0) \rightleftarrows \text{Psh}(\mathbb{N}_0): (\phi_0)^*,$$

then the natural transformation $\Phi_\phi: \Phi_{\mathbb{N}} \Rightarrow \Phi_{\mathbb{M}} \circ (\phi_0)^*$ produces a right morphism from $\Phi_{\mathbb{M}} \circ (\phi_0)^*$ to Φ_ϕ and we obtain the promised adjunction using [HP15, Remark 2.3.5]. $\square$

Proposition 4.3.26. *For every multifunctor $\phi: \mathbb{M} \rightarrow \mathbb{N}$ between target right fibrant $\mathbf{F}$ -multicategories such that ϕ_0 is an isomorphism and such that ϕ_1 is an isofibration, the functor*

$$(\phi_1)_!: (\text{Coll}_{\mathbb{M}}(\mathcal{V}), \circ_{\mathbb{M}}) \rightarrow (\text{Coll}_{\mathbb{N}}(\mathcal{V}), \circ_{\mathbb{N}})$$

is strong monoidal.

Proof. Note first that because ϕ_0 is an isomorphism, the morphism

$$(\phi_1)_!(1_{\text{Coll}_{\mathbb{M}}(\mathcal{V})}) \rightarrow 1_{\text{Coll}_{\mathbb{N}}(\mathcal{V})}$$

is an isomorphism. Consider now $S, T: \mathbb{M}_1^{\text{op}} \rightarrow \mathcal{V}$ two $\mathbb{M}$ -collections in $\mathcal{V}$. We have to show that the canonical natural transformation $(\phi_1)_!(S \circ_{\mathbb{M}} T) \Rightarrow$

$((\phi_1)_! S) \circ_{\mathbb{N}} ((\phi_1)_! T)$ is an isomorphism. It is enough to show that the pasting of the following diagram

$$\begin{array}{ccccc}
 (\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1)^{\text{op}} & \longrightarrow & (\mathbb{M}_1 \times \mathbb{F}\mathbb{M}_1)^{\text{op}} & \xrightarrow{S \times \bar{T}} & \mathcal{V} \times \mathcal{V} \xrightarrow{\times} \mathcal{V} \\
 \downarrow & & \downarrow & \swarrow & \nearrow \\
 (\mathbb{N}_1 \times_{\mathbb{F}\mathbb{N}_0} \mathbb{F}\mathbb{N}_1)^{\text{op}} & \longrightarrow & (\mathbb{N}_1 \times \mathbb{F}\mathbb{N}_1)^{\text{op}} & & \\
 \downarrow \circ & \swarrow & & & \\
 \mathbb{N}_1^{\text{op}} & & & \searrow & \\
 & & & & \mathcal{V}
 \end{array}$$

is a left Kan extension. By definition, the diagram

$$\begin{array}{ccccc}
 (\mathbb{N}_1 \times_{\mathbb{F}\mathbb{N}_0} \mathbb{F}\mathbb{N}_1)^{\text{op}} & \longrightarrow & (\mathbb{N}_1 \times \mathbb{F}\mathbb{N}_1)^{\text{op}} & \xrightarrow{S \times \bar{T}} & \mathcal{V} \times \mathcal{V} \xrightarrow{\times} \mathcal{V} \\
 \downarrow \circ & \swarrow & & & \\
 \mathbb{N}_1^{\text{op}} & & & \searrow & \\
 & & & & \mathcal{V}
 \end{array}$$

$S \circ_{\mathbb{N}} T$

is a left Kan extension, and we have to check that the diagram

$$\begin{array}{ccccc}
 (\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1)^{\text{op}} & \longrightarrow & \mathbb{M}_1 \times \mathbb{F}\mathbb{M}_1 & \xrightarrow{S \times \bar{T}} & \mathcal{V} \times \mathcal{V} \xrightarrow{\times} \mathcal{V} \\
 \downarrow & & \downarrow & \swarrow & \nearrow \\
 (\mathbb{N}_1 \times_{\mathbb{F}\mathbb{N}_0} \mathbb{F}\mathbb{N}_1)^{\text{op}} & \longrightarrow & \mathbb{N}_1 \times \mathbb{F}\mathbb{N}_1 & &
 \end{array}$$

is also a left Kan extension diagram. Finally, using the pointwise formula for left Kan extension, it is enough to show that the functor

$$\Phi: (\phi_1 \times_{\phi_0} \mathbb{F}\phi_1)^{\text{op}}_{/(f, (f_1, \dots, f_n))} \rightarrow (\phi_1 \times \mathbb{F}\phi_1)^{\text{op}}_{/(f, (f_1, \dots, f_n))}$$

is final. Because ϕ_1 is an isofibration, both the functors $(\phi_1 \times_{\phi_0} \mathbb{F}\phi_1)^{\text{op}}$ and $(\phi_1 \times \mathbb{F}\phi_1)^{\text{op}}$ are isofibrations and, moreover, $(\phi_1 \times_{\phi_0} \mathbb{F}\phi_1)^{\text{op}}$ is a morphism of left fibrations as depicted by the diagram

$$\begin{array}{ccc}
 (\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1)^{\text{op}} & \xrightarrow{(\phi_1 \times_{\phi_0} \mathbb{F}\phi_1)^{\text{op}}} & (\mathbb{N}_1 \times_{\mathbb{F}\mathbb{N}_0} \mathbb{F}\mathbb{N}_1)^{\text{op}} \\
 \searrow \text{top}_1 & & \swarrow \text{top}_1 \\
 & \mathbb{M}_0^{\text{op}} &
 \end{array}$$

and the morphism $(\phi_1 \times \mathbb{F}\phi_1)^{\text{op}}$ is also a morphism of left fibrations as depicted

by the following diagram

$$\begin{array}{ccc}
 (\mathbb{M}_1 \times \mathbf{F}\mathbb{M}_1)^{\text{op}} & \xrightarrow{(\phi_1 \times \mathbf{F}\phi_1)^{\text{op}}} & (\mathbb{N}_1 \times \mathbf{F}\mathbb{N}_1)^{\text{op}} \\
 & \searrow (t \times \mathbf{F}t)^{\text{op}} \quad \swarrow (t \times \mathbf{F}t)^{\text{op}} & \\
 & (\mathbb{M}_0 \times \mathbf{F}\mathbb{M}_0)^{\text{op}}. &
 \end{array}$$

It now follows from Corollary 4.3.6 that the vertical functors of the square

$$\begin{array}{ccc}
 \left((\phi_1 \times_{\phi_0} \mathbf{F}\phi_1)^{\text{op}} \right)^{-1} (f, (f_1, \dots, f_n)) & \xrightarrow{\simeq} & \left((\phi_1 \times \mathbf{F}\phi_1)^{\text{op}} \right)^{-1} (f, (f_1, \dots, f_n)) \\
 \downarrow & & \downarrow \\
 (\phi_1 \times_{\phi_0} \mathbf{F}\phi_1)^{\text{op}}_{/(f, (f_1, \dots, f_n))} & \longrightarrow & (\phi_1 \times \mathbf{F}\phi_1)^{\text{op}}_{/(f, (f_1, \dots, f_n))}
 \end{array}$$

are final and, because the upper horizontal functor is an equivalence, it follows that the bottom horizontal functor is also final. $\square$

Proposition 4.3.27. *The category $\mathbf{F}\text{-Multicat}$ admits finite coproducts. Moreover, for all target right fibrant $\mathbf{F}$ -multicategories $\mathbb{M}$ and $\mathbb{N}$, $\mathbb{M} \sqcup \mathbb{N}$ is target right fibrant and the category $\text{Op}_{\mathbb{M} \sqcup \mathbb{N}}(\mathcal{V})$ is equivalent to the category $\text{Op}_{\mathbb{M}}(\mathcal{V}) \times \text{Op}_{\mathbb{N}}(\mathcal{V})$.*

Proof. Let $\mathbb{M}$ and $\mathbb{N}$ be two $\mathbf{F}$ -multicategories. We define $(\mathbb{M} \sqcup \mathbb{N})_0$ and $(\mathbb{M} \sqcup \mathbb{N})_1$ respectively by $\mathbb{M}_0 \sqcup \mathbb{N}_0$ and $\mathbb{M}_1 \sqcup \mathbb{N}_1$, the target functor is $t_{\mathbb{M}} \sqcup t_{\mathbb{N}}: \mathbb{M}_1 \sqcup \mathbb{N}_1 \rightarrow \mathbb{M}_0 \sqcup \mathbb{N}_0$, the identity functor is $1_{\mathbb{M}} \sqcup 1_{\mathbb{N}}: \mathbb{M}_0 \sqcup \mathbb{N}_0 \rightarrow \mathbb{M}_1 \sqcup \mathbb{N}_1$, the source functor is the composition

$$\mathbb{M}_1 \sqcup \mathbb{N}_1 \xrightarrow{s_{\mathbb{M}} \sqcup s_{\mathbb{N}}} \mathbf{F}(\mathbb{M}_0) \sqcup \mathbf{F}(\mathbb{N}_0) \xrightarrow{(A, B)} \mathbf{F}(\mathbb{M}_0 \sqcup \mathbb{N}_0)$$

where the right-hand side functor is the canonical functor given by the universal property of the coproduct, and finally, the composition is given by

$$(\mathbb{M}_1 \times_{\mathbf{F}\mathbb{M}_0} \mathbf{F}\mathbb{M}_1) \sqcup (\mathbb{N}_1 \times_{\mathbf{F}\mathbb{N}_0} \mathbf{F}\mathbb{N}_1) \xrightarrow{\odot_{\mathbb{M} \sqcup \mathbb{N}}} \mathbb{M}_1 \sqcup \mathbb{N}_1$$

using the isomorphism

$$(\mathbb{M}_1 \sqcup \mathbb{N}_1) \times_{\mathbf{F}(\mathbb{M}_0 \sqcup \mathbb{N}_0)} \mathbf{F}(\mathbb{M}_1 \sqcup \mathbb{N}_1) \simeq (\mathbb{M}_1 \times_{\mathbf{F}\mathbb{M}_0} \mathbf{F}\mathbb{M}_1) \sqcup (\mathbb{N}_1 \times_{\mathbf{F}\mathbb{N}_0} \mathbf{F}\mathbb{N}_1).$$

where the first isomorphism follows from the universality of coproducts in Cat and from the fact that $\mathbf{F}$ preserves pullbacks. It is clear that $\mathbb{M} \sqcup \mathbb{N}$ is target right fibrant using that both $\mathbb{M}$ and $\mathbb{N}$ are target right fibrant and that coproducts of right fibrations and isofibrations are respectively right fibrations and isofibrations. The last part of the statement follows from the fact that the equivalence $\text{Coll}_{\mathbb{M} \sqcup \mathbb{N}}(\mathcal{V}) \simeq \text{Coll}_{\mathbb{M}}(\mathcal{V}) \times \text{Coll}_{\mathbb{N}}(\mathcal{V})$ is monoidal. $\square$

Notation 4.3.28. If $\mathbb{M}$ is a target right fibrant $\mathbf{F}$ -multicategory and if $i: A \rightarrow B$ is a morphism of $\mathbb{M}$ -sets of colours, then we denote by i^* the functor $(\mathbb{M}^i)^*: \text{Op}_{\mathbb{M}}^B(\mathcal{V}) \rightarrow \text{Op}_{\mathbb{M}}^A(\mathcal{V})$.

Construction 4.3.29. Suppose that $\mathbb{M}$ is a target right fibrant $\mathbf{F}$ -multicategory and let $i: A \hookrightarrow B$ be a monomorphism of $\mathbb{M}$ -sets of colours. If we write i as an inclusion of summand $i: A \hookrightarrow A \sqcup C$, then the multifunctor $\mathbb{M}^i: \mathbb{M}^A \rightarrow \mathbb{M}^B$ is the composition

$$\mathbb{M}^A \hookrightarrow \mathbb{M}^A \sqcup \mathbb{M}^C \xrightarrow{(\mathbb{M}^{i_A}, \mathbb{M}^{i_C})} \mathbb{M}^{A \sqcup C}$$

where $i_A: A \hookrightarrow A \sqcup C$ and $i_C: C \hookrightarrow A \sqcup C$ are the canonical inclusions. The multifunctor

$$(\mathbb{M}^{i_A}, \mathbb{M}^{i_C}): \mathbb{M}^A \sqcup \mathbb{M}^C \rightarrow \mathbb{M}^{A \sqcup C}$$

satisfies the conditions of Proposition 4.3.26 and it follows that the functor $i^*: \mathrm{Op}_{\mathbb{M}}^B(\mathcal{V}) \rightarrow \mathrm{Op}_{\mathbb{M}}^A(\mathcal{V})$ admits a left adjoint $i_!: \mathrm{Op}_{\mathbb{M}}^A(\mathcal{V}) \rightarrow \mathrm{Op}_{\mathbb{M}}^B(\mathcal{V})$ given by the composition

$$\mathrm{Op}_{\mathbb{M}}^A(\mathcal{V}) \longrightarrow \mathrm{Op}_{\mathbb{M}}^A(\mathcal{V}) \times \mathrm{Op}_{\mathbb{M}}^C(\mathcal{V}) \xrightarrow{(\mathbb{M}_1^{i_A}, \mathbb{M}_1^{i_C})_!} \mathrm{Op}_{\mathbb{M}}^B(\mathcal{V})$$

where the first functor is given component-wise by the identity of $\mathrm{Op}_{\mathbb{M}}^A(\mathcal{V})$ and by the functor constant on the initial object of $\mathrm{Op}_{\mathbb{M}}^C(\mathcal{V})$.

Corollary 4.3.30. *For every monomorphism $i: A \hookrightarrow B$ of $\mathbb{M}$ -sets of colours, the adjunction*

$$i_!: \mathrm{Op}_{\mathbb{M}}^A(\mathcal{V}) \rightleftarrows \mathrm{Op}_{\mathbb{M}}^B(\mathcal{V}): i^*$$

is coreflexive.

Proof. Using the previous construction, we can write

$$i_!: \mathrm{Op}_{\mathbb{M}}^A(\mathcal{V}) \rightleftarrows \mathrm{Op}_{\mathbb{M}}^B(\mathcal{V}): i^*$$

as a composition of two adjunctions

$$\mathrm{Op}_{\mathbb{M}}^A(\mathcal{V}) \rightleftarrows \mathrm{Op}_{\mathbb{M}}^A(\mathcal{V}) \times \mathrm{Op}_{\mathbb{M}}^C(\mathcal{V}) \xrightleftharpoons[(\mathbb{M}_1^{i_A}, \mathbb{M}_1^{i_C})^*]{(\mathbb{M}_1^{i_A}, \mathbb{M}_1^{i_C})_!} \mathrm{Op}_{\mathbb{M}}^B(\mathcal{V})$$

and it is enough to show that both these adjunctions are coreflexive. It is clear that the left-hand side adjunction is coreflexive. For the right-hand side adjunction, this follows from the fact that the underlying monoidal adjunction

$$(\mathbb{M}_1^{i_A}, \mathbb{M}_1^{i_C})_!: \mathrm{Coll}_{\mathbb{M}}^A(\mathcal{V}) \times \mathrm{Coll}_{\mathbb{M}}^C(\mathcal{V}) \rightleftarrows \mathrm{Coll}_{\mathbb{M}}^B(\mathcal{V}): (\mathbb{M}_1^{i_A}, \mathbb{M}_1^{i_C})^*$$

is coreflexive as the functor $(\mathbb{M}_1^{i_A}, \mathbb{M}_1^{i_C})_!$ is fully faithful. $\square$

Proposition 4.3.31. *For every multifunctor $\phi: \mathbb{M} \rightarrow \mathbb{N}$ between target right fibrant $\mathbf{F}$ -multicategories such that ϕ_0 is an isomorphism and for every monomorphism $i: A \hookrightarrow B$ of $\mathbb{N}$ -sets of colours, the mate*

$$\begin{array}{ccc}
 \mathrm{Op}_{\mathbb{N}}^B(\mathcal{V}) & \xrightarrow{(\phi^B)^*} & \mathrm{Op}_{\mathbb{M}}^B(\mathcal{V}) \\
 i_! \uparrow & \swarrow & \uparrow i_! \\
 \mathrm{Op}_{\mathbb{N}}^A(\mathcal{V}) & \xrightarrow{(\phi^A)^*} & \mathrm{Op}_{\mathbb{M}}^A(\mathcal{V})
 \end{array} \tag{A}$$

associated with the commutative square

$$\begin{array}{ccc}
 \mathrm{Op}_{\mathbb{N}}^B(\mathcal{V}) & \xrightarrow{(\phi^B)^*} & \mathrm{Op}_{\mathbb{M}}^B(\mathcal{V}) \\
 i^* \downarrow & & \downarrow i^* \\
 \mathrm{Op}_{\mathbb{N}}^A(\mathcal{V}) & \xrightarrow{(\phi^A)^*} & \mathrm{Op}_{\mathbb{M}}^A(\mathcal{V})
 \end{array} \tag{B}$$

is an isomorphism.

Proof. If we write $i: A \hookrightarrow B$ as an inclusion of the form $i: A \hookrightarrow A \sqcup C$, the square (B) can be decomposed into a composition of two squares as follows

$$\begin{array}{ccc}
 \mathrm{Op}_{\mathbb{N}}^{A \sqcup C}(\mathcal{V}) & \xrightarrow{(\phi^{A \sqcup C})^*} & \mathrm{Op}_{\mathbb{M}}^{A \sqcup C}(\mathcal{V}) \\
 \downarrow & & \downarrow \\
 \mathrm{Op}_{\mathbb{N}}^A(\mathcal{V}) \times \mathrm{Op}_{\mathbb{N}}^C(\mathcal{V}) & \longrightarrow & \mathrm{Op}_{\mathbb{M}}^A(\mathcal{V}) \times \mathrm{Op}_{\mathbb{M}}^C(\mathcal{V}) \\
 \downarrow & & \downarrow \\
 \mathrm{Op}_{\mathbb{N}}^A(\mathcal{V}) & \xrightarrow{(\phi^A)^*} & \mathrm{Op}_{\mathbb{M}}^A(\mathcal{V})
 \end{array} \tag{C}$$

and the mate (A) is the pasting of the diagram

$$\begin{array}{ccc}
 \mathrm{Op}_{\mathbb{N}}^{A \sqcup C}(\mathcal{V}) & \xrightarrow{(\phi^{A \sqcup C})^*} & \mathrm{Op}_{\mathbb{M}}^{A \sqcup C}(\mathcal{V}) \\
 \uparrow & \swarrow & \uparrow \\
 \mathrm{Op}_{\mathbb{N}}^A(\mathcal{V}) \times \mathrm{Op}_{\mathbb{N}}^C(\mathcal{V}) & \longrightarrow & \mathrm{Op}_{\mathbb{M}}^A(\mathcal{V}) \times \mathrm{Op}_{\mathbb{M}}^C(\mathcal{V}) \\
 \uparrow & \swarrow & \uparrow \\
 \mathrm{Op}_{\mathbb{N}}^A(\mathcal{V}) & \xrightarrow{(\phi^A)^*} & \mathrm{Op}_{\mathbb{M}}^A(\mathcal{V})
 \end{array} \tag{D}$$

It is now enough to show that the two mates of the diagram (D) are isomorphisms. For the upper square, using that the vertical adjunctions are provided by monoidal adjunctions, it is enough to show that the following mate

$$\begin{array}{ccc}
 \mathrm{Fun}\left(\left(\mathbb{N}_1^{A \sqcup C}\right)^{\mathrm{op}}, \mathcal{V}\right) & \xrightarrow{(\phi_1^{A \sqcup C})^*} & \mathrm{Fun}\left(\left(\mathbb{M}_1^{A \sqcup C}\right)^{\mathrm{op}}, \mathcal{V}\right) \\
 \uparrow & \swarrow & \uparrow \\
 \mathrm{Fun}\left(\left(\mathbb{N}_1^A \sqcup \mathbb{N}_1^C\right)^{\mathrm{op}}, \mathcal{V}\right) & \longrightarrow & \mathrm{Fun}\left(\left(\mathbb{M}_1^A \sqcup \mathbb{M}_1^C\right)^{\mathrm{op}}, \mathcal{V}\right)
 \end{array}$$

is an isomorphism. We have now to show that for every $(\mu, (b_1, \dots, b_n), b_0)$ in $\mathbb{M}_1^{A \sqcup C}$, the functor

$$(\mathbb{M}_1^{i_A}, \mathbb{M}_1^{i_C})_{/(\mu, (b_1, \dots, b_n), b_0)} \rightarrow (\mathbb{N}_1^{i_A}, \mathbb{N}_1^{i_C})_{/(\phi_1(\mu), (b_1, \dots, b_n), b_0)}$$

is final. Using that both $(\mathbb{M}_1^{i_A}, \mathbb{M}_1^{i_C})$ and $(\mathbb{N}_1^{i_A}, \mathbb{N}_1^{i_C})$ are cocartesian fibrations, it is enough to show that the upper horizontal functor of the square

$$\begin{array}{ccc} (\mathbb{M}_1^{i_A}, \mathbb{M}_1^{i_C})^{-1}(\mu, (b_1, \dots, b_n), b_0) & \longrightarrow & (\mathbb{N}_1^{i_A}, \mathbb{N}_1^{i_C})^{-1}(\phi_1(\mu), (b_1, \dots, b_n), b_0) \\ \downarrow & & \downarrow \\ (\mathbb{M}_1^{i_A}, \mathbb{M}_1^{i_C})_{/(\mu, (b_1, \dots, b_n), b_0)} & \longrightarrow & (\mathbb{N}_1^{i_A}, \mathbb{N}_1^{i_C})_{/(\phi_1(\mu), (b_1, \dots, b_n), b_0)} \end{array}$$

is an equivalence. This follows from the fact that the categories

$$(\mathbb{M}_1^{i_A}, \mathbb{M}_1^{i_C})^{-1}(\mu, (b_1, \dots, b_n), b_0) \text{ and } (\mathbb{N}_1^{i_A}, \mathbb{N}_1^{i_C})^{-1}(\phi_1(\mu), (b_1, \dots, b_n), b_0)$$

are both empty if there exist i and j such that $b_i \in C$ and $b_j \in A$ and both a singleton if all b_i belong to A or if all b_i belong to C . $\square$

4.3.6 A Monad for $\mathbb{M}$ -operads

In what follows, $\mathbb{M}$ is a target right fibrant $\mathbf{F}$ -multicategory.

Proposition 4.3.32. *The monoidal category $(\text{Coll}_{\mathbb{M}}(\mathcal{V}), \circ_{\mathbb{M}})$ is left closed.*

Proof. The proof of this proposition is analogous to the proof that the convolution product is closed that we gave for Corollary 4.3.14. Let T be an $\mathbb{M}$ -collection in $\mathcal{V}$. We have to show that the functor

$$(-) \circ_{\mathbb{M}} T : \text{Coll}_{\mathbb{M}}(\mathcal{V}) \rightarrow \text{Coll}_{\mathbb{M}}(\mathcal{V})$$

admits a right adjoint. This functor can be obtained as the composition of the three following functors:

1. $\times_{\mathcal{V}} \circ ((-) \times \tilde{T}) : \text{Fun}(\mathbb{M}_1^{\text{op}}, \mathcal{V}) \longrightarrow \text{Fun}((\mathbb{M}_1 \times \mathbf{F}\mathbb{M}_1)^{\text{op}}, \mathcal{V})$;
2. $L^* : \text{Fun}((\mathbb{M}_1 \times \mathbf{F}\mathbb{M}_1)^{\text{op}}, \mathcal{V}) \longrightarrow \text{Fun}((\mathbb{M}_1 \times_{\mathbf{F}\mathbb{M}_0} \mathbf{F}\mathbb{M}_1)^{\text{op}}, \mathcal{V})$ with L the canonical functor $\mathbb{M}_1 \times_{\mathbf{F}\mathbb{M}_0} \mathbf{F}\mathbb{M}_1 \rightarrow \mathbb{M}_1 \times \mathbf{F}\mathbb{M}_1$;
3. $\odot_! : \text{Fun}((\mathbb{M}_1 \times_{\mathbf{F}\mathbb{M}_0} \mathbf{F}\mathbb{M}_1)^{\text{op}}, \mathcal{V}) \longrightarrow \text{Fun}(\mathbb{M}_1^{\text{op}}, \mathcal{V})$.

The last functor is a left adjoint of $\odot^*$ and the second functor admits a right adjoint given by taking right Kan extensions. It remains to show that the first functor admits a right adjoint. As the category $\mathcal{V}$ is cartesian closed, this functor is a left adjoint of the functor $\text{Fun}((\mathbb{M}_1 \times \mathbf{F}\mathbb{M}_1)^{\text{op}}, \mathcal{V}) \rightarrow \text{Fun}(\mathbb{M}_1^{\text{op}}, \mathcal{V})$ that sends a functor $H : (\mathbb{M}_1 \times \mathbf{F}\mathbb{M}_1)^{\text{op}} \rightarrow \mathcal{V}$ to the functor $\mathbb{M}_1^{\text{op}} \rightarrow \mathcal{V}$ that sends an object X of $\mathbb{M}_1$ to the end

$$\int_{Y \in \mathbf{F}\mathbb{M}_1^{\text{op}}} \text{hom}_{\mathcal{V}}(\tilde{T}(Y), H(X, Y)).$$

$\square$

Proposition 4.3.33. *The subcategory $\text{Coll}_{\mathbb{M}}(\mathcal{V})$ of $\text{Fun}(\mathbf{FM}^{\text{op}}, \mathcal{V})$ is closed under sifted colimits. In particular, for every $\mathbb{M}$ -collection T in $\mathcal{V}$, the functor $\circ_{\mathbb{M}}$ preserves sifted colimits in both variables.*

Proof. For every sifted category I and every functor $F: I \rightarrow \text{Coll}_{\mathbb{M}}(\mathcal{V})$, the colimit of F in the category $\text{Fun}(\mathbf{FM}^{\text{op}}, \mathcal{V})$ is given by the composition

$$\mathbf{FM}^{\text{op}} \rightarrow \text{Fun}(I, \mathcal{V}) \xrightarrow{\text{colim}} \mathcal{V}.$$

The first functor preserves products as F factors by definition through the inclusion $\text{Coll}_{\mathbb{M}}(\mathcal{V}) \hookrightarrow \text{Fun}(\mathbf{FM}^{\text{op}}, \mathcal{V})$ and the second functor preserves finite products because I is sifted and because $\mathcal{V}$ is closed. The composition product $\circ_{\mathbb{M}}$ is the restriction to $\text{Coll}_{\mathbb{M}}(\mathcal{V})$ of the convolution product $\otimes_{\mathbb{M}}$ on $\text{Fun}(\mathbf{FM}_1^{\text{op}}, \mathcal{V})$ and, as the convolution product $\otimes_{\mathbb{M}}$ is closed, it follows that $\circ_{\mathbb{M}}$ preserves sifted colimits in both variables. $\square$

Proposition 4.3.34 ([Kel80, Theorem 23.3]). *For every left closed cocomplete monoidal category $(\mathcal{C}, \otimes)$ such that for all X in $\mathcal{C}$, the functor $X \otimes (-): \mathcal{C} \rightarrow \mathcal{C}$ preserves filtered colimits, the category of algebras in $\mathcal{C}$ is the category of algebras over a monad $\mathbf{T}$ on $\mathcal{C}$.*

We recall the construction of the functor $\mathbf{T}: \mathcal{C} \rightarrow \mathcal{C}$. We first define a functor $\mathbf{T}_n: \mathcal{C} \rightarrow \mathcal{C}$ by induction on n as follows:

1. the functor $\mathbf{T}_0$ is constant on the unit I of $\mathcal{C}$;
2. for all X in $\mathcal{C}$, we define $\mathbf{T}_{n+1}X$ as $I \sqcup (X \otimes \mathbf{T}_n X)$;

We construct now by induction a natural transformation $i_n: \mathbf{T}_n \rightarrow \mathbf{T}_{n+1}$ as follows:

1. for all X in $\mathcal{C}$, the morphism i_0 is the canonical morphism $I \rightarrow I \sqcup X$;
2. for all X in $\mathcal{C}$, the morphism $i_{n+1}: \mathbf{T}_{n+1} \rightarrow \mathbf{T}_{n+2}$ is the morphism

$$\text{id}_I \sqcup (\text{id}_X \otimes i_n): I \sqcup (X \otimes \mathbf{T}_n X) \rightarrow I \sqcup (X \otimes \mathbf{T}_{n+1}(X)).$$

Finally, for all X in $\mathcal{C}$, we define the image of X under $\mathbf{T}$ as the colimit of the diagram

$$\mathbf{T}_0 X \xrightarrow{i_0} \mathbf{T}_1 X \xrightarrow{i_1} \mathbf{T}_2 X \xrightarrow{i_2} \dots$$

Corollary 4.3.35. *For every target right fibrant $\mathbf{F}$ -multicategory $\mathbb{M}$, the category of $\mathbb{M}$ -operads in $\mathcal{V}$ is the category of algebras over a monad $\mathbf{T}$ in $\text{Coll}_{\mathbb{M}}(\mathcal{V})$, that we denote by $\mathbf{T}_{\mathbb{M}}$.*

Proof. This follows directly from Corollary 4.3.20, Proposition 4.3.32, Proposition 4.3.33, and Proposition 4.3.34. $\square$

Proposition 4.3.36. *The monad $\mathbf{T}_{\mathbb{M}}$ preserves sifted colimits.*

Proof. It is enough to show that for every non-negative integer n , the functor $(\mathbf{T}_{\mathbb{M}})_n$ preserves sifted colimits. The functor $(\mathbf{T}_{\mathbb{M}})_0$ is constant and preserves sifted colimits as sifted categories are connected. Suppose now by induction that $(\mathbf{T}_{\mathbb{M}})_n$ preserves sifted colimits. The functor $(\mathbf{T}_{\mathbb{M}})_{n+1}$ is defined as the composition

$$\mathrm{Coll}_{\mathbb{M}}(\mathcal{V}) \xrightarrow{\Delta} \mathrm{Coll}_{\mathbb{M}}(\mathcal{V}) \times \mathrm{Coll}_{\mathbb{M}}(\mathcal{V}) \xrightarrow{\mathrm{id}_{\mathcal{C}} \times (\mathbf{T}_{\mathbb{M}})_n} \mathrm{Coll}_{\mathbb{M}}(\mathcal{V}) \times \mathrm{Coll}_{\mathbb{M}}(\mathcal{V}) \xrightarrow{\circ_{\mathbb{M}}} \mathrm{Coll}_{\mathbb{M}}(\mathcal{V})$$

and preserves sifted colimits as $\circ_{\mathbb{M}}$ preserves sifted colimits in each variable and both $\mathrm{id}_{\mathcal{C}}$ and $(\mathbf{T}_{\mathbb{M}})_n$ preserve sifted colimits. $\square$

Corollary 4.3.37. *If $\mathcal{V}$ is bicomplete, then the category $\mathrm{Op}_{\mathbb{M}}(\mathcal{V})$ is also bicomplete.*

Proof. The category $\mathrm{Fun}(\mathbb{M}_1^{\mathrm{op}}, \mathcal{V})$ is bicomplete as the category $\mathcal{V}$ is bicomplete and the result now follows from the fact that $\mathrm{Op}_{\mathbb{M}}(\mathcal{V})$ is the category of algebras over a monad on $\mathrm{Fun}(\mathbb{M}_1^{\mathrm{op}}, \mathcal{V})$ that preserves sifted colimits. $\square$

4.4 Homotopy theory of simplicial coloured $\mathbb{M}$ -operads

In what follows, $\mathbb{M}$ is a target right fibrant $\mathbf{F}$ -multicategory. A *simplicial $\mathbb{M}$ -operad* (respectively a *simplicial coloured $\mathbb{M}$ -operad*) is an $\mathbb{M}$ -operad (respectively a coloured $\mathbb{M}$ -operad) in $\mathbf{sSet}$. We will denote by $\mathrm{sOp}_{\mathbb{M}}$, $\mathrm{sOp}_{\mathbb{M}}^A$, and $\mathrm{sOp}_{\mathbb{M}}^{\mathrm{col}}$ the categories of simplicial $\mathbb{M}$ -operads, simplicial A -coloured $\mathbb{M}$ -operads, and simplicial coloured $\mathbb{M}$ -operads respectively. We will also denote by $\mathrm{Op}_{\mathbb{M}}$ (respectively $\mathrm{Op}_{\mathbb{M}}^{\mathrm{col}}$) the category of $\mathbb{M}$ -operads in $\mathbf{Set}$ (respectively coloured $\mathbb{M}$ -operads in $\mathbf{Set}$). In this section, our aim is to establish model structures on $\mathrm{sOp}_{\mathbb{M}}$ and $\mathrm{sOp}_{\mathbb{M}}^{\mathrm{col}}$, extending the classical model structures on simplicial (coloured) operads.

4.4.1 Model structure on simplicial $\mathbb{M}$ -operads

In this subsection, we construct the model structure on $\mathrm{sOp}_{\mathbb{M}}$.

Theorem 4.4.1. *There exists a model structure on the category of simplicial $\mathbb{M}$ -operads, right-induced by the free-forgetful adjunction*

$$F_{\mathbb{M}}: \mathrm{Coll}_{\mathbb{M}}(\mathbf{sSet}) \rightleftarrows \mathrm{sOp}_{\mathbb{M}}: U_{\mathbb{M}}$$

from the projective model structure on $\mathrm{Coll}_{\mathbb{M}}(\mathbf{sSet})$.

Proof. The proof is analogous to [BM03, Theorem 3.2], and uses the transfer principle of [BM03, Section 2.5]. Note first that $U_{\mathbb{M}}$ preserves filtered colimits because $\mathbf{T}_{\mathbb{M}}$ does, and that $\mathrm{sOp}_{\mathbb{M}}$ is bicomplete by Corollary 4.3.37. By Proposition 4.2.15, the functor Ex^{∞} induces a fibrant replacement functor on $\mathrm{sOp}_{\mathbb{M}}$ by post-composition. Finally, the interval $\Delta^0 \sqcup \Delta^0 \rightarrow \Delta^1 \xrightarrow{\sim} \Delta^0$ induces a functorial path object $\mathcal{O} \xrightarrow{\sim} \mathcal{O}^{\Delta^1} \rightarrow \mathcal{O} \times \mathcal{O}$ for fibrant operads in $\mathrm{sOp}_{\mathbb{M}}$ using Proposition 4.2.15. $\square$

Corollary 4.4.2. *For every $\mathbb{M}$ -set of colours A , using Corollary 4.3.22, we obtain a cofibrantly generated model structure on the category of simplicial A -coloured $\mathbb{M}$ -operads $\mathbf{sOp}_{\mathbb{M}}^A$.*

Example 4.4.3. For every orbital pair $(\mathcal{O}, \mathcal{T})$, we thus obtain a model structure on the category $\mathbf{sOp}_{(\mathcal{O}, \mathcal{T})}$ of simplicial $(\mathcal{O}, \mathcal{T})$ -operads. Note that in the case of the orbital pair $(\mathcal{O}_G, \mathcal{O}_T)$ provided by a G -transfer system $\mathcal{T}$ associated with a finite group G , this model structure is expected to correspond to the model structure constructed on *partial genuine equivariant operads* in [BP21, Theorem II]. However, as we already pointed out in Example 4.2.36, it is not known whether these two categories are equivalent.

Example 4.4.4. The $\mathbf{F}$ -multicategory $\mathbb{N}$ is target right fibrant, and we thus obtain a model structure on the category of simplicial non-symmetric operads.

Proposition 4.4.5. *For every multifunctor $\phi: \mathbb{M} \rightarrow \mathbb{N}$ between target fibrant $\mathbf{F}$ -multicategories, the functor $\phi^*: \mathbf{sOp}_{\mathbb{N}} \rightarrow \mathbf{sOp}_{\mathbb{M}}$ admits a left adjoint $\phi_!: \mathbf{sOp}_{\mathbb{M}} \rightarrow \mathbf{sOp}_{\mathbb{N}}$. In particular, the functor $\phi^*: \mathbf{sOp}_{\mathbb{N}}^{\text{col}} \rightarrow \mathbf{sOp}_{\mathbb{M}}^{\text{col}}$ also admits a left adjoint $\phi_!: \mathbf{sOp}_{\mathbb{M}}^{\text{col}} \rightarrow \mathbf{sOp}_{\mathbb{N}}^{\text{col}}$.*

Proof. The category $\text{Coll}_{\mathbb{M}}(\mathbf{sSet})$ is locally presentable as it is a category of presheaves. Moreover, the monad $\mathbf{T}_{\mathbb{M}}$ preserves sifted colimits, and it follows that the category $\mathbf{sOp}_{\mathbb{M}}$ is also locally presentable. Finally, the functor ϕ^* preserves limits and filtered colimits and the result now follows from the adjoint functor theorem [AR94, Theorem 1.66]. The second part of the statement follows directly from Proposition 4.3.25. $\square$

Proposition 4.4.6. *For every multifunctor $\phi: \mathbb{M} \rightarrow \mathbb{N}$ between target fibrant $\mathbf{F}$ -multicategories, the adjunction*

$$\phi_!: \mathbf{sOp}_{\mathbb{M}} \rightleftarrows \mathbf{sOp}_{\mathbb{N}}: \phi^*$$

is a Quillen adjunction.

Proof. It is clear that ϕ^* is a right Quillen functor as weak equivalences and fibrations in simplicial $\mathbb{M}$ -operads are the morphisms which respectively forget to weak equivalences and fibrations of $\mathbb{M}$ -collections in $\mathbf{sSet}$. $\square$

4.4.2 Model structure on simplicial coloured $\mathbb{M}$ -operads

In this subsection, we construct the model structure on $\mathbf{sOp}_{\mathbb{M}}^{\text{col}}$. The first step is to show that $\mathbf{sOp}_{\mathbb{M}}^{\text{col}}$ is bicomplete.

Definition 4.4.7. The strict 2-category Adj is the wide subcategory of $\widehat{\text{Cat}}$ spanned by the functors $F: \mathcal{C} \rightarrow \mathcal{D}$ that admit a left adjoint. If $F: \mathcal{C}^{\text{op}} \rightarrow \text{Adj}$ is a functor, we can in particular consider its Grothendieck construction $\Gamma_F: \int_{\mathcal{C}} F \rightarrow \mathcal{C}$.

Proposition 4.4.8 ([HP15, Proposition 2.4.4]). *For every functor $F: \mathcal{C}^{\text{op}} \rightarrow \text{Adj}$ such that $\mathcal{C}$ is bicomplete and such that $F(X)$ is bicomplete for every X in $\mathcal{C}$, the category $\int_{\mathcal{C}} F$ is also bicomplete.*

Proposition 4.4.9. *The category $\mathbf{sOp}_M^{\text{col}}$ is bicomplete.*

Proof. The result follows from Proposition 4.4.8, Corollary 4.3.37 and Proposition 4.4.5 using that $\mathbf{Psh}(\mathbb{M}_0)$ is bicomplete. $\square$

Proposition 4.4.10. *The category $\mathbf{sOp}_M^{\text{col}}$ is equivalent to the subcategory of $\mathbf{Fun}(\Delta^{\text{op}}, \mathbf{Op}_M^{\text{col}})$ spanned by the simplicial objects which are constant on colours.*

Proof. Note first that we have an equivalence

$$\mathbf{sOp}_M \simeq \mathbf{Fun}(\Delta^{\text{op}}, \mathbf{Op}_M)$$

for every $\mathbf{F}$ -multicategory M . This equivalence is functorial in M and it follows that the category $\mathbf{sOp}_M^{\text{col}}$ is equivalent to the Grothendieck construction of the functor $\mathbf{Psh}(\mathbb{M}_0)^{\text{op}} \rightarrow \overline{\mathbf{Cat}}$ that sends A to $\mathbf{Fun}(\Delta^{\text{op}}, \mathbf{Op}_M^A)$. Finally, it is clear that the Grothendieck construction of the latter functor can be identified with the subcategory of $\mathbf{Fun}(\Delta^{\text{op}}, \mathbf{Op}_M^{\text{col}})$ spanned by these functors $\mathcal{O}: \Delta^{\text{op}} \rightarrow \mathbf{Op}_M^{\text{col}}$ which are constant on colours. $\square$

If we denote by $\mathbf{sCat}$ the category of simplicial categories, then we have an adjunction

$$i_!: \mathbf{Fun}(\mathbb{M}_0^{\text{op}}, \mathbf{sCat}) \rightleftarrows \mathbf{sOp}_M^{\text{col}}: i^*$$

provided by Definition 4.2.25 and Proposition 4.4.5 and we denote by

$$(i_X)_!: \mathbf{sCat} \rightleftarrows \mathbf{sOp}_M^{\text{col}}: (i_X)^*$$

the composition of adjunctions

$$\mathbf{sCat} \begin{array}{c} \xrightarrow{\text{hom}(-, X) \times (-)} \\ \xleftarrow{\text{ev}_X} \end{array} \mathbf{Fun}(\mathbb{M}_0^{\text{op}}, \mathbf{sCat}) \begin{array}{c} \xrightarrow{i_!} \\ \xleftarrow{i^*} \end{array} \mathbf{sOp}_M^{\text{col}}.$$

Recall that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between simplicial categories is *essentially surjective* (respectively an *isofibration*) if $\pi_0 F$ is essentially surjective (respectively if $\pi_0 F$ is an isofibration). We can now define the following classes of morphisms in $\mathbf{sOp}_M^{\text{col}}$.

Definition 4.4.11. We say that a morphism of simplicial coloured M -operads $(f, \phi): (A, \mathcal{O}) \rightarrow (B, \mathcal{P})$ is:

1. *fully-faithful* if $\phi: \mathcal{O} \rightarrow f^* \mathcal{P}$ is a weak equivalence in $\mathbf{sOp}_M^A$;
2. *essentially surjective* if for all X in $\mathbb{M}_0$, $(i_X)^*(f, \phi)$ is essentially surjective in $\mathbf{sCat}$;
3. a *weak equivalence* if it is fully-faithful and essentially surjective;
4. a *local fibration* if the morphism $\phi: \mathcal{O} \rightarrow f^* \mathcal{P}$ is a fibration in $\mathbf{sOp}_M^A$;
5. an *isofibration* if it is a local fibration and if for every X in $\mathbb{M}_0$, $(i_X)^*(f, \phi)$ is an isofibration in $\mathbf{sCat}$.

We say that a simplicial coloured $\mathbb{M}$ -operad $(A, \mathcal{O})$ is *fibrant* if the unique morphism $(A, \mathcal{O}) \rightarrow *$ is an isofibration, where $*$ is the final simplicial coloured $\mathbb{M}$ -operad. Equivalently, a simplicial coloured $\mathbb{M}$ -operad $(A, \mathcal{O})$ is fibrant if $\mathcal{O}$ is fibrant in $\text{sOp}_{\mathbb{M}}^A$.

We now want to show that the classes of weak equivalences and isofibrations are part of a model structure on $\text{sOp}_{\mathbb{M}}$. This model structure is cofibrantly generated and we introduce its sets of generators.

Notation 4.4.12. Recall that FIM_0 can be identified with the full subcategory of $\text{Psh}(\mathbb{M}_0)$ spanned by finite coproducts of representables. For every $\mathbb{M}$ -set of colours A that belongs to FIM_0 , we denote respectively by I_A and J_A the sets of generating cofibrations and generating trivial cofibrations of the model structure on $\text{sOp}_{\mathbb{M}}^A$ given by Corollary 4.4.2. Consider the following two classes of functors in sCat :

- (C_2) the unique functor $\emptyset \rightarrow *$ where $\emptyset$ and $*$ are respectively the initial category and the final category;
- (A_2) the functors $* \rightarrow H$ which are fully faithful and essentially surjective, where H is a simplicial category with set of objects $\{0, 1\}$, with the property that H_n is countable for each $n \geq 0$ and that H is cofibrant in $\text{sCat}_{\{0,1\}}$.

We now consider the following classes of morphisms in $\text{sOp}_{\mathbb{M}}^{\text{col}}$:

- (J_1) the union of all the classes J_A in $\text{sOp}_{\mathbb{M}}^{\text{col}}$ for all A in FIM_0 ;
- (J_2) the union of the images of A_2 in $\text{sOp}_{\mathbb{M}}^{\text{col}}$ under all the functors $(i_X)_!$ for all X in $\mathbb{M}_0$;
- (J) the union of J_1 and J_2 ;
- (I_1) the union of all the classes I_A in $\text{sOp}_{\mathbb{M}}^{\text{col}}$ for all A in FIM_0 ;
- (I_2) the union of all the images of C_2 in $\text{sOp}_{\mathbb{M}}^{\text{col}}$ under all the functors $(i_X)_!$ for all X in $\mathbb{M}_0$;
- (I) the union of I_1 and I_2 .

Proposition 4.4.13. *A morphism $(f, \phi): (A, \mathcal{O}) \rightarrow (B, \mathcal{P})$ of simplicial coloured $\mathbb{M}$ -operads is a local weak equivalence (respectively a local fibration) if and only if for every morphism of $\mathbb{M}$ -sets of colours $g: C \rightarrow A$ with C in FIM_0 , the morphism $g^*\phi: g^*\mathcal{O} \rightarrow g^*f^*\mathcal{P}$ is a weak equivalence (respectively a fibration) in $\text{sOp}_{\mathbb{M}}^C$.*

Proof. We only show the result for local weak equivalences; the proof for local fibrations is analogous. Consider a morphism $(f, \phi): (A, \mathcal{O}) \rightarrow (B, \mathcal{P})$ of simplicial coloured $\mathbb{M}$ -operads. Suppose first that for every $g: C \rightarrow A$ with C in FIM_0 , the morphism $g^*\phi$ is a weak equivalence. Let us show that (f, ϕ) is a local weak equivalence. We have to show that, for every $(\mu, (a_1, \dots, a_n), a_0)$ in $\mathbb{M}^A$, the map

$$\phi(\mu, (a_1, \dots, a_n), a_0): \mathcal{O}(\mu, (a_1, \dots, a_n), a_0) \rightarrow \mathcal{P}(\mu, (f(a_1), \dots, f(a_n)), f(a_0))$$

is a weak equivalence in $\mathbf{sSet}$. Consider the object $C := (a_0, \dots, a_n)$ of $\mathbf{FM}_0$ seen as an $\mathbb{M}$ -set of colours. We have a canonical morphism $g: C \rightarrow A$ using the Yoneda Lemma and the fact that every a_i belongs to A . By assumptions $g^*(\phi)$ is a weak equivalence and it follows directly that $\phi(\mu, (a_1, \dots, a_n), a_0)$ is also a weak equivalence. Suppose now that (f, ϕ) is a local weak equivalence. The morphism ϕ is a weak equivalence in $\mathbf{sOp}_{\mathbb{M}}^A$ and it follows that for every morphism of $\mathbb{M}$ -sets of colours $g: C \rightarrow A$, the morphism $g^*(\phi)$ is a weak equivalence in $\mathbf{sOp}_{\mathbb{M}}^C$, because g^* preserves weak equivalences. $\square$

Proposition 4.4.14. *A morphism of simplicial coloured $\mathbb{M}$ -operads is a local fibration if and only if it has the right lifting property with respect to J_1 .*

Proof. Let $(f, \phi): (A, \mathcal{O}) \rightarrow (B, \mathcal{P})$ be a morphism of simplicial coloured $\mathbb{M}$ -operads. By definition of J_1 , we have to show that (f, ϕ) is a local fibration if and only if it has the right lifting property with respect to J_C for every C in $\mathbf{FM}_0$. Suppose first that (f, ϕ) has the right lifting property with respect to J_C for every C in $\mathbf{FM}_0$. Using Proposition 4.4.13, we have to show that for every C in $\mathbf{FM}_0$ and every morphism $\iota: C \rightarrow A$, the morphism $\iota^*(\phi)$ is a fibration in $\mathbf{sOp}_{\mathbb{M}}^C$. For every $g: \mathcal{Q} \rightarrow \mathcal{R}$ in J_C consider any commutative square

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{\psi} & \iota^* \mathcal{O} \\ g \downarrow & & \downarrow \iota^* \phi \\ \mathcal{R} & \longrightarrow & \iota^* f^* \mathcal{P} \end{array} \quad (C)$$

in $\mathbf{sOp}_{\mathbb{M}}^C$. If we now consider the associated square

$$\begin{array}{ccc} (C, \mathcal{Q}) & \xrightarrow{(\iota, \psi)} & (A, \mathcal{O}) \\ (\text{id}_C, g) \downarrow & & \downarrow (f, \phi) \\ (C, \mathcal{R}) & \longrightarrow & (B, \mathcal{P}) \end{array}$$

in $\mathbf{sOp}_{\mathbb{M}}^{\text{col}}$, we obtain the existence of a lifting map $(\iota, \eta): (C, \mathcal{R}) \rightarrow (A, \mathcal{O})$ and it follows that $\eta: \mathcal{R} \rightarrow \iota^* \mathcal{O}$ produces a lift for the square (C). In particular, $\iota^*(\phi)$ has the right lifting property with respect to J_C in $\mathbf{sOp}_{\mathbb{M}}^C$ and is thus a fibration in $\mathbf{sOp}_{\mathbb{M}}^C$. Suppose now that (f, ϕ) is a local fibration and consider a square

$$\begin{array}{ccc} (C, \mathcal{Q}) & \xrightarrow{(\iota, \psi)} & (A, \mathcal{O}) \\ (\text{id}_C, g) \downarrow & & \downarrow (f, \phi) \\ (C, \mathcal{R}) & \longrightarrow & (B, \mathcal{P}) \end{array}$$

with g in J_C . Obtaining a lift for this square is equivalent as obtaining a lift for the square

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{\psi} & \iota^* \mathcal{O} \\ \downarrow & & \downarrow \iota^* \phi \\ \mathcal{R} & \longrightarrow & \iota^* f^* \mathcal{P} \end{array} \quad (4.1)$$

in $\mathrm{sOp}_{\mathbb{M}}^C$. The result now follows from the fact that $\iota^*\phi$ is a fibration, as $\iota^*: \mathrm{sOp}_{\mathbb{M}}^A \rightarrow \mathrm{sOp}_{\mathbb{M}}^C$ is a right Quillen functor. $\square$

Proposition 4.4.15. *A morphism of simplicial coloured $\mathbb{M}$ -operads is a local trivial fibration if and only if it has the right lifting property with respect to I_1 .*

Proof. The proof is analogous to the proof of Proposition 4.4.14. $\square$

Proposition 4.4.16. *A morphism of simplicial coloured $\mathbb{M}$ -operads is an isofibration if and only if it has the right lifting property with respect to the set J .*

Proof. A morphism of simplicial coloured $\mathbb{M}$ -operads $(f, \phi): (A, \mathcal{O}) \rightarrow (B, \mathcal{P})$ is an isofibration if and only if it is a local fibration and for every X in $\mathbb{M}_0$, $(i_X)^*(f, \phi)$ is an isofibration in sCat . The result now follows from Proposition 4.4.14 and from the fact that isofibrations in sCat are precisely the functors with the right lifting property with respect to A_2 in sCat , as proven in [Ber07a]. $\square$

Proposition 4.4.17. *A morphism of simplicial $\mathbb{M}$ -operads has the right lifting property with respect to I if and only if it is an isofibration and a weak equivalence.*

Proof. The proof of this proposition is analogous to the proof of Proposition 4.4.16 and uses Proposition 4.4.15 as well as the fact that essentially surjective isofibrations in sCat are precisely the functors with the right lifting property with respect to A_2 in sCat , as proven in [Ber07a]. $\square$

Our goal is now to prove that the domains of morphisms in J and I are small relative to J -cells and I -cells, respectively.

Proposition 4.4.18. *For every multifunctor $\phi: \mathbb{M} \rightarrow \mathbb{N}$ such that $\phi_0: \mathbb{M}_0 \rightarrow \mathbb{N}_0$ is an isomorphism, the functor $\phi^*: \mathrm{sOp}_{\mathbb{N}}^{\mathrm{col}} \rightarrow \mathrm{sOp}_{\mathbb{M}}^{\mathrm{col}}$ preserves filtered colimits of diagrams $F: I \rightarrow \mathrm{sOp}_{\mathbb{N}}^{\mathrm{col}}$ such that, for every $f: i \rightarrow j$ in I , the morphism $F(f): F(i) \rightarrow F(j)$ is a monomorphism on colours.*

Proof. If we denote by $(A_i, \mathcal{O}_i)$ the image of i under F , the colimit of F is the simplicial coloured $\mathbb{N}$ -operad $(A, \mathcal{O})$ with $A = \mathrm{colim}_{i \in I} A_i$ and $\mathcal{O} = \mathrm{colim}_i (\alpha_i)_! \mathcal{O}_i$ where $\alpha_i: A_i \rightarrow A$ is the coprojection. We now have that

$$\phi^*(A, \mathcal{O}) = \left(A, (\phi^A)^* (\mathrm{colim}_i (\alpha_i)_! \mathcal{O}_i) \right)$$

and $(\phi^A)^* (\mathrm{colim}_i (\alpha_i)_! \mathcal{O}_i) \simeq \mathrm{colim}_i ((\phi^A)^* (\alpha_i)_! \mathcal{O}_i)$ because $(\phi^A)^*$ preserves filtered colimits. Finally, using Proposition 4.3.31, we have that the canonical morphism $(\phi^X)^* (\alpha_i)_! \rightarrow (\alpha_i)_! (\phi^X)^*$ is an isomorphism and the result follows. $\square$

Remark 4.4.19. Note that the morphisms in I and J are monomorphisms on colours. Because pushout of morphisms of simplicial coloured $\mathbb{M}$ -operads that are monomorphisms on colours are also monomorphisms on colours, it follows that I -cells and J -cells are sequential colimits of morphisms that are monomorphisms on colours. It follows that Proposition 4.4.18 applies to I -cells and J -cells.

Proposition 4.4.20. *Let $F: \mathcal{C}^{\text{op}} \rightarrow \text{AdjCat}$ be a functor such that $\mathcal{C}$ and $F(X)$ admit sequential colimits for every X in $\mathcal{C}$, and such that, for every morphism $f: X \rightarrow Y$ in $\mathcal{C}$, the functor $f^*: F(Y) \rightarrow F(X)$ preserves sequential colimits. If (Y, y) is an object of $\int_{\mathcal{C}} F$ such that Y is small in $\mathcal{C}$ and such that y is small in $F(Y)$, then (Y, y) is small relative to the sequential colimits*

$$(X_0, x_0) \xrightarrow{(f_0, \phi_0)} (X_1, x_1) \xrightarrow{(f_1, \phi_1)} \dots \quad (\text{A})$$

such that the adjunction

$$(\alpha_i)_!: F(X_i) \rightleftarrows F(\text{colim}_i X_i): (\alpha_i)^*$$

associated with the coprojection $\alpha_i: X_i \rightarrow \text{colim}_i X_i$ is coreflexive.

Proof. We have to show that every morphism $(g, \psi): (Y, y) \rightarrow \text{colim}_i (X_i, x_i)$ factors through a morphism $(g', \psi'): (Y, y) \rightarrow (X_k, x_k)$ for some k . If we denote by (X, x) the colimit of (A), then, using [HP15, Proposition 2.4.4], we have that

$$(X, x) = (\text{colim}_i X_i, \text{colim}_i (\alpha_i)_! x_i)$$

with $\alpha_i: X_i \rightarrow \text{colim}_i X_i$ the coprojection. By assumptions, Y is small in $\mathcal{C}$ and it follows that $g: Y \rightarrow \text{colim}_i X_i$ factors through a morphism $g'': Y \rightarrow X_m$ for some m . The object y is small in $F(Y)$, and using that g^* preserves sequential colimits, the morphism $\psi: y \rightarrow \text{colim}_i g^*(\alpha_i)_! x_i$ factors through a morphism $\psi'': y \rightarrow g^*(\alpha_l)_! x_l$ for some l . If we denote by k the maximum of m and l , then we obtain that g factors through a morphism $g': Y \rightarrow X_k$ and that ψ factors through a morphism $\psi': y \rightarrow g^*(\alpha_k)_! x_k$. Moreover, as we have that $g = \alpha_k \circ g'$, we obtain that ψ actually factors through a morphism $\psi': y \rightarrow (g')^*(\alpha_k)^*(\alpha_k)_! x_k \simeq (g')^* x_k$ using that $((\alpha_k)_!, (\alpha_k)^*)$ is coreflexive by assumptions. It now follows that (g, ψ) factors through a morphism $(g', \psi'): (y, Y) \rightarrow (X_k, x_k)$. $\square$

Proposition 4.4.21. *The domains of morphisms in I and J are small relative to I -cells and J -cells respectively.*

Proof. The domains of the morphisms in C_2 and A_2 are small in sCat and, because the functor $(i_X)^*: \text{sOp}_{\mathbb{M}}^{\text{col}} \rightarrow \text{Fun}(\mathbb{M}_0^{\text{op}}, \text{sCat})$ preserves sequential colimits of morphisms that are monomorphisms on colours, the functor $(i_X)_!$ preserves small objects relative to this class of sequential colimits. Using Remark 4.4.19, the domain of the morphisms in I_2 and J_2 in $\text{sOp}_{\mathbb{M}}^{\text{col}}$ are small relative to I -cells and J -cells respectively. We now show that, for every A in FM_0 , the domains of morphisms in I_A and J_A are small relative to I -cells and J -cells. By assumption, A is small in $\text{Psh}(\mathbb{M}_0)$ and we already know that the domains of maps in I_A and J_A are small in $\text{sOp}_{\mathbb{M}}^A$. The result now follows from Remark 4.4.19, Proposition 4.4.18, and Corollary 4.3.30. $\square$

Proposition 4.4.22. *Let $(A, \mathcal{O})$ be a fibrant simplicial coloured $\mathbb{M}$ -operad and consider $(\mu, (a_1, \dots, a_n), a_0)$ and $(\mu, (b_1, \dots, b_n), b_0)$ two objects of $\mathbb{M}^A$ associated with the same multimorphism μ in $\mathbb{M}$ with source $(X_1, \dots, X_n)$ and target X_0 . Suppose*

$a_i \sim b_i$ in $(i_{X_i})^*(A, \mathcal{O})$ for all $i = 0, \dots, n$. The Kan complexes $\mathcal{O}(\mu, (a_1, \dots, a_n), a_0)$ and $\mathcal{O}(\mu, (b_1, \dots, b_n), b_0)$ are homotopy equivalent. Moreover, any morphism of coloured $\mathbb{M}$ -operads $(f, \phi): (A, \mathcal{O}) \rightarrow (B, \mathcal{P})$ induces functorially a homotopy equivalence $\mathcal{P}(\mu, (f(a_1), \dots, f(a_n)), f(a_0)) \rightarrow \mathcal{P}(\mu, (f(b_1), \dots, f(b_n)), f(b_0))$.

Proof. By assumptions, we have maps of simplicial sets $\alpha_i: \Delta^0 \rightarrow \mathcal{O}(1_{X_i}, b_i, a_i)$ and $\beta_i: \Delta^0 \rightarrow \mathcal{O}(1_{X_i}, a_i, b_i)$ for all $i = 1, \dots, n$ that exhibit the equivalences $a_i \sim b_i$ in $(i_{X_i})^*(A, \mathcal{O})$. We thus obtain a map of simplicial sets $\theta: \mathcal{O}(\mu, (a_1, \dots, a_n), a_0) \rightarrow \mathcal{O}(\mu, (b_1, \dots, b_n), b_0)$ given by the composition of the map

$$\alpha': \mathcal{O}(\mu, (a_1, \dots, a_n), a_0) \rightarrow \prod_{i=1}^n \mathcal{O}(1_{X_i}, b_i, a_i) \times \mathcal{O}(\mu, (a_1, \dots, a_n), a_0) \times \mathcal{O}(1_{X_0}, a_0, b_0)$$

with the map

$$\circ: \prod_{i=1}^n \mathcal{O}(1_{X_i}, b_i, a_i) \times \mathcal{O}(\mu, (a_1, \dots, a_n), a_0) \times \mathcal{O}(1_{X_0}, a_0, b_0) \rightarrow \mathcal{O}(\mu, (b_1, \dots, b_n), b_0)$$

where $\alpha' = ((\alpha_i), \text{id}_{\mathcal{O}(\mu, (a_1, \dots, a_n), a_0)}, \beta_0)$. Similarly, we also obtain a map

$$\eta: \mathcal{O}(\mu, (b_1, \dots, b_n), b_0) \rightarrow \mathcal{O}(\mu, (a_1, \dots, a_n), a_0)$$

given by the composition of

$$\beta': \mathcal{O}(\mu, (b_1, \dots, b_n), b) \rightarrow \prod_{i=1}^n \mathcal{O}(1_{X_i}, a_i, b_i) \times \mathcal{O}(\mu, (b_1, \dots, b_n), b_0) \times \mathcal{O}(1_{X_0}, b_0, a_0)$$

with the map

$$\circ: \prod_{i=1}^n \mathcal{O}(1_{X_i}, a_i, b_i) \times \mathcal{O}(\mu, (b_1, \dots, b_n), b_0) \times \mathcal{O}(1_{X_0}, b_0, a_0) \rightarrow \mathcal{O}(\mu, (a_1, \dots, a_n), a_0)$$

where $\beta' = ((\beta_i), \text{id}, \alpha_0)$. Using that α_0 and β_0 and α_i and β_i form equivalences, we deduce that θ and η form the desired homotopy equivalence. The second part of the statement follows from the fact that the images of α_i and β_i under (f, ϕ) provide by the same construction the promised homotopy equivalence. $\square$

Proposition 4.4.23. *The class of weak equivalences in $\text{sOp}_{\mathbb{M}}^{\text{col}}$ satisfies the two-out-of-three property.*

Proof. Using that the functor Ex^∞ induces a fibrant replacement functor on $\text{sOp}_{\mathbb{M}}^{\text{col}}$, it is enough to show the result for fibrant simplicial coloured $\mathbb{M}$ -operads. Consider now two morphisms of coloured simplicial $\mathbb{M}$ -operads

$$(A, \mathcal{O}) \xrightarrow{(f, \phi)} (B, \mathcal{P}) \xrightarrow{(g, \psi)} (C, \mathcal{Q}).$$

The two-out-of-three property is true for essentially surjective functors as it is already proven in the category $\mathbf{sCat}$ and we must now show that if two out of three of the morphisms (f, ϕ) , (g, ψ) and $(g, \psi) \circ (f, \phi)$ are weak equivalences, then the third is a local weak equivalence. If (f, ϕ) and (g, ψ) are both weak equivalences, then $f^*(\psi) \circ \phi$ is a local weak equivalence in $\mathbf{sOp}_M^A$ because f^* preserves weak equivalences. Similarly, if the composition $(g, \psi) \circ (f, \phi)$ is a weak equivalence and if (g, ψ) is a weak equivalence, then using the two-out-of-three for weak equivalences in $\mathbf{sOp}_M^A$, we deduce that (f, ψ) is a local weak equivalence. Finally, we have to show that if both the composition $(g, \psi) \circ (f, \phi)$ and (f, ϕ) are weak equivalences, then (g, ψ) is a local weak equivalence. Using again the two-out-of-three of weak equivalences in $\mathbf{sOp}_M^A$, we deduce that $f^*(\psi)$ is a weak equivalence in $\mathbf{sOp}_M^A$ and we want to show that ψ is a weak equivalence in $\mathbf{sOp}_M^B$. For every $(\mu, (b_1, \dots, b_n), b_0)$ in M^B , we want to show that $\psi(\mu, (b_1, \dots, b_n), b_0): \mathcal{P}(\mu, (b_1, \dots, b_n), b_0) \rightarrow \mathcal{Q}(\mu, (g(b_1), \dots, g(b_n)), g(b_0))$ is a weak equivalence in $\mathbf{sOp}_M^B$. Because $i^*(f, \phi)$ is essentially surjective, we can find colours $a_0, \dots, a_n$ in A such that $b_i \sim f(a_i)$ for every $i = 0, \dots, n$. Finally, using Proposition 4.4.22, we obtain a commutative square

$$\begin{array}{ccc} \mathcal{P}(\mu, (b_1, \dots, b_n), b_0) & \xrightarrow{\simeq} & \mathcal{P}(\mu, (f(a_1), \dots, f(a_n)), f(a_0)) \\ \psi(\mu, (b_1, \dots, b_n), b_0) \downarrow & & \downarrow \simeq \\ \mathcal{Q}(\mu, (g(b_1), \dots, g(b_n)), g(b_0)) & \xrightarrow{\simeq} & \mathcal{Q}(\mu, (gf(a_1), \dots, gf(a_n)), gf(a_0)) \end{array}$$

where the horizontal arrows are homotopy equivalences and where the right vertical arrow is a weak equivalence because $f^*\psi$ is a weak equivalence. Finally, we deduce that ψ is a weak equivalence, and, in particular, that (g, ψ) is a local weak equivalence. $\square$

Proposition 4.4.24. *The class of weak equivalences in $\mathbf{sOp}_M^{\text{col}}$ is closed under retracts.*

Proof. We show that both local weak equivalences and essentially surjective morphisms are closed under retracts. For essentially surjective morphisms, this follows from the fact that this property is verified in the category $\mathbf{sCat}$. Suppose now that a morphism $(f, \phi): (A, \mathcal{O}) \rightarrow (B, \mathcal{P})$ is a retract of a local weak equivalence $(g, \psi): (C, \mathcal{Q}) \rightarrow (D, \mathcal{R})$. If we consider the associated diagram

$$\begin{array}{ccccc} (A, \mathcal{O}) & \longrightarrow & (C, \mathcal{Q}) & \longrightarrow & (A, \mathcal{O}) \\ (f, \phi) \downarrow & & \downarrow (g, \psi) & & \downarrow (f, \phi) \\ (B, \mathcal{P}) & \longrightarrow & (D, \mathcal{R}) & \xrightarrow{(h, \theta)} & (B, \mathcal{P}), \end{array}$$

we deduce that ϕ is a retract of $h^*\psi$ and the result follows from the fact that $h^*\psi$ is a weak equivalence in $\mathbf{sOp}_M^B$ and from the fact that weak equivalences in $\mathbf{sOp}_M^B$ are closed under retracts. $\square$

Proposition 4.4.25. *For every sequence*

$$(A_0, \mathcal{O}_0) \xrightarrow{(f_0, \phi_0)} (A_1, \mathcal{O}_1) \xrightarrow{(f_1, \phi_1)} \dots \quad (\text{B})$$

of morphisms in $\text{sOp}_{\mathbb{M}}^{\text{col}}$ that are monomorphisms on colours and weak equivalences, the canonical morphism $\alpha_0: (A_0, \mathcal{O}_0) \rightarrow \text{colim}_i (A_i, \mathcal{O}_i)$ is a weak equivalence and a monomorphism on colours.

Proof. A sequential colimit of monomorphisms in $\text{Psh}(\mathbb{M}_0)$ is a monomorphism, and the second part of the statement follows. We now show that the canonical morphism $\alpha_0: (A_0, \mathcal{O}_0) \rightarrow \text{colim}_i (A_i, \mathcal{O}_i)$ is a weak equivalence. Using Proposition 4.4.18 on the functor

$$i^*: \text{sOp}_{\mathbb{M}}^{\text{col}} \rightarrow \text{Fun}(\mathbb{M}_0^{\text{op}}, \text{sCat})$$

and using the fact that weak equivalences in sCat are closed under sequential colimits, we already know that $\alpha_0: (A_0, \mathcal{O}_0) \rightarrow \text{colim}_i (A_i, \mathcal{O}_i)$ is essentially surjective, and we must prove that it is also fully faithful. The colimit of the sequence (B) is the simplicial $\mathbb{M}$ -operad

$$(\text{colim}_i A_i, \text{colim}_i (\alpha_i)_! \mathcal{O}_i)$$

and we have to prove that the canonical morphism

$$\phi_0: \mathcal{O}_0 \rightarrow (\alpha_0)^*(\text{colim}_i (\alpha_i)_! \mathcal{O}_i)$$

is a weak equivalence in $\text{sOp}_{\mathbb{M}}^{A_0}$. If we denote by $\alpha_{i,j}: A_i \rightarrow A_j$ the morphism provided by the sequence (B) whenever $j > i$, we have that $\alpha_j \circ \alpha_{i,j} = \alpha_i$ and using that $((\alpha_i)_!, (\alpha_i)^*)$ is coreflexive, we obtain that

$$(\alpha_0)^*(\text{colim}_i (\alpha_i)_! \mathcal{O}_i) \simeq \text{colim}_i (\alpha_0)^*(\alpha_i)_! \mathcal{O}_i \simeq \text{colim}_i \alpha_{0,i}^* \mathcal{O}_i.$$

Finally, the morphism ϕ is the canonical morphism $\mathcal{O}_0 \rightarrow \text{colim}_i \alpha_{0,i}^* \mathcal{O}_i$ associated with the sequential colimit

$$\mathcal{O}_0 = \text{colim}_i \alpha_{0,0}^* \mathcal{O}_0 \longrightarrow \text{colim}_i \alpha_{0,1}^* \mathcal{O}_1 \longrightarrow \dots$$

in $\text{sOp}_{\mathbb{M}}^{A_0}$. This morphism is a weak equivalence, as it is a sequential colimit of weak equivalences in $\text{sOp}_{\mathbb{M}}^{A_0}$. $\square$

Proposition 4.4.26. *Let $u: K \rightarrow H$ be a fully faithful functor between small categories such that the set of objects of H is $\{0, 1\}$, while the only object of K is 0 (such that $u(0) = 0$). For every presheaf $F: \mathbb{M}_0^{\text{op}} \rightarrow \text{Set}$ and every pushout square*

$$\begin{array}{ccc} i_!(F \times K) & \xrightarrow{v} & (A, \mathcal{O}) \\ i_!(\text{id}_F \times u) \downarrow & & \downarrow (f, \phi) \\ i_!(F \times H) & \longrightarrow & (B, \mathcal{P}) \end{array}$$

in the category $\text{Op}_{\mathbb{M}}^{\text{col}}$, the morphism (f, ϕ) is fully faithful. If moreover u is an equivalence, then (f, ϕ) is also an equivalence.

Proof. The proof of this proposition is similar to the proof of [CM13b, Lemma 1.29]. Note first that we see here both H and K as the objects of $\text{Fun}(\mathbb{M}_0^{\text{op}}, \text{sCat})$ given by the associated constant functors. The proof of this proposition relies on an explicit description of the $\mathbb{M}$ -operad $(B, \mathcal{P})$. We denote by w the parameterized functor $F \times K \rightarrow i^*(A, \mathcal{O})$ associated with u by adjunction and, for every X in $\mathbb{M}_0$ and x in $F(X)$, we denote by $w_x: K \rightarrow i^*(A, \mathcal{O})(X)$ the associated functor. For every X in $\mathbb{M}_0$ and every x in $F(X)$, we denote by x_0 the image in $A(X)$ of x under w_x . The $\mathbb{M}$ -set of colours B is the $\mathbb{M}$ -set $A \sqcup F$ and for all X in $\mathbb{M}_0$ and x in $F(X)$, we denote by x_1 the corresponding element of $B(X)$ through the inclusion $F(X) \hookrightarrow A(X) \sqcup F(X) = B(X)$. Consider now a multimorphism μ of $\mathbb{M}$ whose source and target are respectively $(X_1, \dots, X_n)$ and X_0 . For every $(\mu, (b_1, \dots, b_n), b_0)$ in $\mathbb{M}^B$, we define

$$c_i = \begin{cases} b_i & \text{if } b_i \in A(X_i); \\ x_0 & \text{if } b_i = x \in F(X_i). \end{cases}$$

We now define $\mathcal{P}(\mu, (b_1, \dots, b_n), b_0)$ as follows:

1. Suppose that c_0 belongs to $A(X_0)$. We denote by $\mathcal{Q}(\mu, (b_1, \dots, b_n), b_0)$ the set

$$\mathcal{O}(\mu, (c_1, \dots, c_n), c_0) \times \prod_{i, b_i \in F(X_i), i \neq 0} H(1, 0).$$

We can consider on $\mathcal{Q}(\mu, (b_1, \dots, b_n), b_0)$ the equivalence relation generated by

$$(f, (h_1 \circ g_1, \dots, h_m \circ g_m)) \sim (f \circ (l_1, \dots, l_n), (g_1, \dots, g_m))$$

where $(f, (g_1, \dots, g_m))$ belongs to $\mathcal{Q}(\mu, (b_1, \dots, b_n), b_0)$, where $h_i \in K(0, 0)$ for $i = 1, \dots, m$ with $m = |\{b_i \mid b_i \in A(X_i)\}|$ and where l_i is defined by

$$l_i = \begin{cases} 1_{b_i} & \text{if } b_i \in A(X_i); \\ w_{b_i}(h_i) & \text{if } b_i \in F(X_i). \end{cases}$$

We define the set $\mathcal{P}(\mu, (b_1, \dots, b_n), b_0)$ as the quotient of $\mathcal{Q}(\mu, (b_1, \dots, b_n), b_0)$ by this equivalence relation;

2. Suppose that c_0 belongs to $F(X_0)$. We denote by $\mathcal{Q}(\mu, (b_1, \dots, b_n), b_0)$ the set

$$H(0, 1) \times \mathcal{O}(\mu, (c_1, \dots, c_n), c_0) \times \prod_{i, b_i \in F(X_i), i \neq 0} H(1, 0).$$

We can consider on $\mathcal{Q}(\mu, (b_1, \dots, b_n), b_0)$ the equivalence relation generated by

$$(g_0 \circ h_0, f, (h_1 \circ g_1, \dots, h_m \circ g_m)) \sim (g_0, l_0 \circ f \circ (l_1, \dots, l_n), (g_1, \dots, g_m))$$

where $(g_0, f, (g_1, \dots, g_m))$ belongs to $\mathcal{Q}(\mu, (b_1, \dots, b_n), b_0)$, where $h_i \in K(0, 0)$ for $i = 1, \dots, m$ and where $h_0 \in H(0, 1)$ with $m = |\{b_i \mid b_i \in A(X_i)\}|$ and where l_i is defined by

$$l_i = \begin{cases} 1_{b_i} & \text{if } b_i \in A(X_i); \\ w_{b_i}(h_i) & \text{if } b_i \in F(X_i); \end{cases}$$

We define the set $\mathcal{P}(\mu, (b_1, \dots, b_n), b_0)$ as the quotient of $\mathcal{Q}(\mu, (b_1, \dots, b_n), b_0)$ by this equivalence relation.

We describe the structure of B -coloured $\mathbb{M}$ -operad of $\mathcal{P}$. We only explain how to compose operations corresponding to case (2); the other possible compositions can be defined in a similar way. Consider composable operations $(\lambda_0, (\lambda_1, \dots, \lambda_n))$ in $\mathbb{M}_1^B \times_{\mathbb{FM}_0^B} \mathbb{FM}_1^B$ such that the source of λ_0 is the sequence $(b_1, \dots, b_n)$ of elements of B . For all $i = 0, \dots, n$, let p_i be an element of $\mathcal{P}(\lambda_i)$ represented by $(g_i^0, f_i, g_i^1, \dots, g_i^{m_i})$. We define $p_0 \circ (p_1, \dots, p_n)$ as the operation of $\mathcal{P}$ that is represented by

$$(g_0^0, f_0 \circ (k_1 \circ l_1, \dots, k_n \circ l_n) \circ (f_1, \dots, f_n), (g_1^1, \dots, g_1^{m_1}, g_2^1, \dots, g_n^{m_n}))$$

where l_i and k_i are respectively defined for $i = 1, \dots, n$ by

$$l_i = \begin{cases} g_0^i & \text{if } b_i \in F(X_i) \\ 1_{b_i} & \text{if } b_i \in A(X_i) \end{cases} \quad \text{and} \quad k_i = \begin{cases} g_i^0 & \text{if } b_i \in F(X_i) \\ 1_{b_i} & \text{if } b_i \in A(X_i). \end{cases}$$

The morphism of simplicial coloured $\mathbb{M}$ -operads $(f, \phi): \mathcal{O} \rightarrow \mathcal{P}$ can now be defined on colours by the inclusion $f: A \hookrightarrow A \sqcup F = B$ and by the identity $\phi: \mathcal{O} \rightarrow i^*(\mathcal{P}) = \mathcal{O}$. In particular, the morphism (f, ϕ) is fully faithful. Note also that if $u: K \rightarrow H$ is an equivalence of categories, (f, ϕ) is moreover an equivalence. To conclude the proof, we must show that (f, ϕ) is the desired pushout of the morphism $i_!(\text{id}_F \times u)$. This can be done in an analogous way as in the proof of [CM13b, Lemma 1.29]. $\square$

Proposition 4.4.27. *The morphisms in $\bar{J}$ are weak equivalences, where $\bar{J}$ denotes the saturation of J .*

Proof. Using the small object argument, we know that morphisms in $\bar{J}$ are retracts of sequential colimits of pushouts of morphisms in J . The class of weak equivalences in $\text{sOp}_{\mathbb{M}}^{\text{col}}$ is closed under retracts, and, using that pushouts of morphisms in J are monomorphisms on colours along with Proposition 4.4.25, we just have to prove that pushouts of morphisms in J are weak equivalences. Consider first pushouts along morphisms $(\text{id}_A, \phi): (A, \mathcal{O}) \rightarrow (A, \mathcal{P})$ in J_A . In this case, the pushout of (id_A, ϕ) along a morphism $(g, \psi): (A, \mathcal{O}) \rightarrow (B, \mathcal{Q})$ is the morphism (id_B, η) where η is obtained via the following pushout

$$\begin{array}{ccc} g_! \mathcal{O} & \xrightarrow{\psi'} & \mathcal{Q} \\ g_! \phi \downarrow & & \downarrow \eta \\ g_! \mathcal{P} & \longrightarrow & \mathcal{R} \end{array}$$

in the category $\text{sOp}_{\mathbb{M}}^B$ where $\psi': g_! \mathcal{O} \rightarrow \mathcal{Q}$ is obtained from ψ by adjunction. Since $g_!$ is a left Quillen functor and because ϕ is by definition a trivial cofibration in $\text{sOp}_{\mathbb{M}}^A$, it follows that η is a trivial cofibration in $\text{sOp}_{\mathbb{M}}^B$ and, in particular, (id_B, η) is a local weak equivalence in $\text{sOp}_{\mathbb{M}}^{\text{col}}$. The morphism (id_B, η) is also trivially essentially surjective, and it follows that it is a weak

equivalence in $\mathbf{sOp}_{\mathbb{M}}^{\text{col}}$. Consider now a morphism $(f, \phi): (A, \mathcal{O}) \rightarrow (B, \mathcal{P})$ that belongs to J_0 . There exists X in $\mathbb{M}_0$ and a simplicial category H satisfying the conditions of (A_2) such that (f, ϕ) is the image under $i_!$ of the associated morphism $\text{hom}(-, X) \rightarrow H \times \text{hom}(-, X)$ in $\text{Fun}(\mathbb{M}_0^{\text{op}}, \mathbf{sCat})$. Consider now a pushout square

$$\begin{array}{ccc} i_!(\text{hom}(-, X)) & \longrightarrow & (C, \mathcal{Q}) \\ \downarrow & & \downarrow (g, \eta) \\ i_!(H \times \text{hom}(-, X)) & \longrightarrow & (D, \mathcal{R}) \end{array} \quad (C)$$

in $\mathbf{sOp}_{\mathbb{M}}^{\text{col}}$ and let us show that (g, η) is a weak equivalence in $\mathbf{sOp}_{\mathbb{M}}^{\text{col}}$. As in the proof of [CM13b, Proposition A.25], the pushout square (C) can be decomposed as follows

$$\begin{array}{ccc} i_!(\text{hom}(-, X)) & \longrightarrow & (C, \mathcal{Q}) \\ \downarrow & & \downarrow (h, \nu) \\ i_!(K \times \text{hom}(-, X)) & \longrightarrow & (C, \mathcal{S}) \\ \downarrow & & \downarrow (k, \mu) \\ i_!(H \times \text{hom}(-, X)) & \longrightarrow & (D, \mathcal{R}) \end{array} \quad \begin{array}{c} \left. \begin{array}{c} \downarrow (h, \nu) \\ \downarrow (k, \mu) \end{array} \right\} (g, \eta) \end{array}$$

where K is $u^*(H)$ with $u: \{0\} \hookrightarrow \{0, 1\}$ and we have to show that both the morphisms $(h, \nu): (C, \mathcal{Q}) \rightarrow (C, \mathcal{S})$ and $(k, \mu): (C, \mathcal{S}) \rightarrow (D, \mathcal{R})$ are weak equivalences. The morphism $i_!(\text{hom}(-, X)) \rightarrow i_!(K \times \text{hom}(-, X))$ is constant on objects and it follows that $(h, \nu): (C, \mathcal{Q}) \rightarrow (C, \mathcal{S})$ is a weak equivalence if $i_!(\text{hom}(-, X)) \rightarrow i_!(K \times \text{hom}(-, X))$ is a trivial cofibration in $\mathbf{sOp}_{\mathbb{M}}^{\{0\}}$. Using [CM13b, Lemma 1.28], K is cofibrant in $\mathbf{sCat}^{\{0\}}$ and it follows that $i_!(\text{hom}(-, X)) \rightarrow i_!(K \times \text{hom}(-, X))$ is a trivial cofibration. We now have to prove that the morphism $(k, \mu): (C, \mathcal{S}) \rightarrow (D, \mathcal{R})$ is a weak equivalence. Using Proposition 4.4.10, we can compute the pushout levelwise in the category $\text{Fun}(\Delta^{\text{op}}, \mathbf{Op}_{\mathbb{M}}^{\text{col}})$. Levelwise, the simplicial functor $K \rightarrow H$ satisfies the conditions of Proposition 4.4.26 and we deduce that $(k, \mu): (C, \mathcal{S}) \rightarrow (D, \mathcal{R})$ is fully faithful (we even have isomorphisms of simplicial sets between the simplicial sets of operations). We now have to show that $(k, \mu): (C, \mathcal{S}) \rightarrow (D, \mathcal{R})$ is essentially surjective. The functor $\pi_0: \mathbf{sOp}_{\mathbb{M}}^{\text{col}} \rightarrow \mathbf{Op}_{\mathbb{M}}^{\text{col}}$ preserves colimits and it follows that the square

$$\begin{array}{ccc} i_!(\pi_0(K) \times F) & \longrightarrow & \pi_0(D, \mathcal{S}) \\ \downarrow & & \downarrow \pi_0(k, \mu) \\ i_!(\pi_0(H) \times F) & \longrightarrow & \pi_0(D, \mathcal{R}) \end{array}$$

is a pushout in $\mathbf{Op}_{\mathbb{M}}^{\text{col}}$, and, because $\pi_0(u): \pi_0(K) \rightarrow \pi_0(H)$ is an equivalence, it follows from Proposition 4.4.26 that $\pi_0(k, \mu)$ is essentially surjective. $\square$

Finally, we prove the main Theorem of this section regarding the existence of the model structure on simplicial coloured $\mathbb{M}$ -operads.

Theorem 4.4.28. *The category of simplicial coloured $\mathbb{M}$ -operads $\mathrm{sOp}_{\mathbb{M}}^{\mathrm{col}}$ is endowed with a cofibrantly generated model structure in which a morphism $(f, \phi): (A, \mathcal{O}) \rightarrow (B, \mathcal{P})$ is a weak equivalence if it is a fully faithful and essentially surjective morphism and in which a morphism is a fibration if it is an isofibration in the sense of Definition 4.4.11.*

Proof. We check the conditions of [Hov99, Theorem 2.1.19] where the sets of generating cofibrations and trivial cofibrations are respectively I and J . Note first that we have already proven that $\mathrm{sOp}_{\mathbb{M}}^{\mathrm{col}}$ is bicomplete in Proposition 4.4.9. The first condition is proven using both Proposition 4.4.23 and Proposition 4.4.24. The conditions (2) and (3) were proven in Proposition 4.4.21. The conditions (5), (4), and (6) follow from Proposition 4.4.16, Proposition 4.4.17, and Proposition 4.4.27. $\square$

Example 4.4.29. For every orbital pair $(\mathcal{O}, \mathcal{T})$, we obtain a model structure on the category $\mathrm{sOp}_{(\mathcal{O}, \mathcal{T})}^{\mathrm{col}}$. In particular, if we consider the final category $*$, we find back the existence of the model structure on $\mathrm{sOp}^{\mathrm{col}}$ as constructed in [CM13b, Theorem 1.14] and [Rob11, Theorem 6].

Example 4.4.30. If we consider the $\mathbf{F}$ -multicategory $\underline{\mathbb{N}}$ of Example 4.2.13, we obtain a model structure on the category of simplicial multicategories.

Chapter 5

A program to compare all models for equivariant operads

In this last chapter, we outline a program based on our notion of $\mathbb{M}$ -operads that aims at comparing all models for equivariant operads for a finite group G . In particular, the goal of this program would be to compare the graph model structure of Bonventre and Pereira on simplicial coloured G -operads [BP22c] with the ∞ -category of G - ∞ -operads of Nardin and Shah [NS22].

5.1 The coloured operadic Elmendorf conjecture

The goal of this section is to state a conjecture regarding the comparison between the model structure of Bonventre and Pereira [BP22c] on G -simplicial coloured operads and the model structure of Example 4.4.29 on simplicial coloured parametrized operads.

5.1.1 The Bonventre-Pereira model structure

We recall briefly the model structure on $(\mathrm{sOp}^{\mathrm{col}})^G$ defined in [BP22c]. The category $(\mathrm{sOp}^{\mathrm{col}})^G$ can be identified with the category of simplicial coloured BG -operads and is obtained as the Grothendieck construction of the functor $(\mathrm{Set}^G)^{\mathrm{op}} \rightarrow \mathrm{Cat}$ that sends a G -set A to sOp_{BG}^A , in the sense of Section 4.4. We now give an explicit description of the category Σ_{BG}^A , as defined in Construction 4.2.19.

Construction 5.1.1. Let A be a G -set of colours. Consider the set $A \times (A^{\times n})$ with the $(G \times \Sigma_n)$ -action on $A \times (A^{\times n})$ where Σ_n acts by permuting the components of $A^{\times n}$ and where G acts component-wise with the action of A . Choose a set

R_n of representatives of the orbits of $A \times A^{\times n}$. The category Σ_{BG}^A is the action groupoid of $\bigsqcup_{n \geq 0} (A \times A^{\times n})$ and, in particular, we have an equivalence

$$\Sigma_{BG}^A \simeq \bigsqcup_{n \geq 0} \bigsqcup_{X \in R_n} \text{BStab}(X). \quad (5.1)$$

For every X in R_n , we have an inclusion $\text{BStab}(X) \subseteq G \times \Sigma_n$, and we denote by $\mathcal{F}_X$ the family of subgroups Γ of $\text{BStab}(X)$ such that Γ is an admissible subgroup of $G \times \Sigma_n$. In particular, using the equivalence (5.1), we obtain a model structure on $\text{Coll}_{BG}^A(\text{sSet}) = \text{Fun}\left(\left(\Sigma_{BG}^A\right)^{\text{op}}, \text{sSet}\right)$ given by all the families $\mathcal{F}_X$ for all $n \geq 0$ and all X in R_n .

The following proposition recovers the results from [BP22b] in the simplicial case.

Proposition 5.1.2. *The model structure on $\text{Coll}_{BG}^A(\text{sSet})$ from Construction 5.1.1 lifts to a model structure on sOp_{BG}^A via the free-forgetful adjunction:*

$$F: \text{Coll}_{BG}^A(\text{sSet}) \rightleftarrows \text{sOp}_{BG}^A: U.$$

Proof. The proof is similar to Theorem 4.4.1. The monad is unchanged and it remains to check that Ex^∞ is still a fibrant replacement functor for this new model structure and that $\mathcal{O} \rightarrow \mathcal{O}^{\Delta^1} \rightarrow \mathcal{O} \times \mathcal{O}$ is still a functorial path object for fibrant operads. This follows from the fact that Ex^∞ preserves finite limits (hence fixed points for finite groups), from the fact that $(-)^{\Delta^1}$ preserves limits, and from the fact that products commute with limits. $\square$

Notation 5.1.3. For every subgroup H of G , we denote by $i_H: (\text{sOp}^{\text{col}})^G \rightarrow \text{sCat}$ the functor given by the composition

$$(\text{sOp}^{\text{col}})^G \xrightarrow{i^G} (\text{sCat})^G \xrightarrow{(-)^H} \text{sCat}$$

with i the functor that sends a simplicial coloured operad to its underlying simplicial category.

Definition 5.1.4. We say that a morphism of G -simplicial coloured operads $(f, \phi): (A, \mathcal{O}) \rightarrow (B, \mathcal{P})$ is:

1. *fully-faithful* if $\phi: \mathcal{O} \rightarrow f^*\mathcal{P}$ is a weak equivalence in sOp_{BG}^A ;
2. *essentially surjective* if for every subgroup H of G , $(i_H)^*(f, \phi)$ is essentially surjective in sCat ;
3. *a local fibration* if the morphism $\phi: \mathcal{O} \rightarrow f^*\mathcal{P}$ is a fibration in sOp_{BG}^A ;
4. *an isofibration* if it is a local fibration and if for every subgroup H of G , $(i_H)^*(f, \phi)$ is an isofibration in sCat .

We can now state the existence of the desired model structure.

Theorem 5.1.5 ([BP22c, Theorem A]). *There is a model structure on $(\text{sOp}^{\text{col}})^G$ for which the weak equivalences are the fully faithful and essentially surjective morphisms and for which the fibrations are the isofibrations in the sense of Definition 5.1.4.*

5.1.2 The operadic fixed points adjunction

Let $\mathcal{V}$ be a cartesian closed bicomplete category. The goal of this subsection is to construct a right adjoint to the functor $\phi^*: \text{Op}_{(\mathcal{O}_G, \mathcal{O}_T)}^{\text{col}}(\mathcal{V}) \rightarrow \text{Op}^{\text{col}}(\mathcal{V})^G$ from Proposition 4.2.37.

Definition 5.1.6. We say that a lax functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between monoidal categories is *half strong monoidal* if $F_{\otimes}: F(X) \otimes_{\mathcal{D}} F(Y) \rightarrow F(X \otimes_{\mathcal{C}} Y)$ is an isomorphism for all X and Y in $\mathcal{C}$.

Proposition 5.1.7. *For every multifunctor $\phi: \mathbb{M} \rightarrow \mathbb{N}$ between target right fibrant F -multicategories such that the square*

$$\begin{array}{ccc} \mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1 & \xrightarrow{\phi_1 \times_{\mathbb{F}\phi_0} \mathbb{F}\phi_1} & \mathbb{N}_1 \times_{\mathbb{F}\mathbb{N}_0} \mathbb{F}\mathbb{N}_1 \\ \mathcal{O}_{\mathbb{M}} \downarrow & & \downarrow \mathcal{O}_{\mathbb{N}} \\ \mathbb{M}_1 & \xrightarrow{\phi_1} & \mathbb{N}_1 \end{array} \quad (\text{A})$$

is cartesian, the functor $\phi^: \text{Coll}_{\mathbb{N}}(\mathcal{V}) \rightarrow \text{Coll}_{\mathbb{M}}(\mathcal{V})$ is half strong monoidal.*

Proof. We have to show that for every f in $\mathbb{M}_1$, the functor

$$\mathcal{O}_{\mathbb{M}/f}^{\text{op}} \rightarrow \mathcal{O}_{\mathbb{N}/\phi(f)}^{\text{op}}$$

is final. Consider the square

$$\begin{array}{ccc} (\mathcal{O}_{\mathbb{M}}^{\text{op}})^{-1}(f) & \longrightarrow & (\mathcal{O}_{\mathbb{N}}^{\text{op}})^{-1}(\phi(f)) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{M}/f}^{\text{op}} & \longrightarrow & \mathcal{O}_{\mathbb{N}/\phi(f)}^{\text{op}}. \end{array}$$

The vertical inclusions are final and, because the square (A) is a pullback square, the upper horizontal functor is an equivalence. It follows that the lower horizontal functor is also final. $\square$

Proposition 5.1.8. *For every multifunctor $\phi: \mathbb{M} \rightarrow \mathbb{N}$ satisfying the conditions of Proposition 5.1.7 and for every right fibration $F: \mathcal{C} \rightarrow \mathbb{N}_0$, the functor $F^*\phi: G^*\mathbb{M} \rightarrow F^*\mathbb{N}$, where G is $(\phi_0)^*F$, also satisfies the conditions of Proposition 5.1.7.*

Proof. The result follows directly from the fact that the square below is a pullback when the conditions of Proposition 5.1.7 are satisfied.

$$\begin{array}{ccc} (\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1) \times_{(\mathbb{N}_0 \times_{\mathbb{F}\mathbb{N}_0} \mathbb{F}^2\mathbb{N}_0)} (\mathcal{A} \times_{\mathbb{F}\mathcal{A}} \mathbb{F}^2\mathcal{A}) & \rightarrow & \mathbb{M}_1 \times_{(\mathbb{N}_0 \times_{\mathbb{F}\mathbb{N}_0})} (\mathcal{A} \times_{\mathbb{F}\mathcal{A}}) \\ \downarrow & & \downarrow \\ (\mathbb{N}_1 \times_{\mathbb{F}\mathbb{N}_0} \mathbb{F}\mathbb{N}_1) \times_{(\mathbb{N}_0 \times_{\mathbb{F}\mathbb{N}_0} \mathbb{F}^2\mathbb{N}_0)} (\mathcal{A} \times_{\mathbb{F}\mathcal{A}} \mathbb{F}^2\mathcal{A}) & \rightarrow & \mathbb{N}_1 \times_{(\mathbb{N}_0 \times_{\mathbb{F}\mathbb{N}_0})} (\mathcal{A} \times_{\mathbb{F}\mathcal{A}}). \end{array}$$

$\square$

Proposition 5.1.9. *For every $\mathcal{O}_G$ -set of colours A , the functor $(\phi^A)^*$ is strong monoidal.*

Proof. We prove first that $(\phi^A)^*$ is half strong monoidal for every A . Using Proposition 5.1.8 along with Proposition 5.1.7, it is enough to show that the square

$$\begin{array}{ccc} \Sigma_{BG} \times_{\mathbf{FBG}} \mathbf{F}\Sigma_{BG} & \longrightarrow & \Sigma_{(\mathcal{O}_G, \mathcal{O}_T)} \times_{\mathbf{F}\mathcal{O}_G} \mathbf{F}\Sigma_{(\mathcal{O}_G, \mathcal{O}_T)} \\ \downarrow & & \downarrow \\ \Sigma_{BG} & \longrightarrow & \Sigma_{(\mathcal{O}_G, \mathcal{O}_T)} \end{array} \quad (\text{B})$$

is a pullback. The functors $BG \rightarrow \mathcal{O}_G$ and $\Sigma_{BG} \rightarrow \Sigma_{(\mathcal{O}_G, \mathcal{O}_T)}$ are fully faithful and it follows, using that $\mathbf{F}$ preserves fully faithful functors, that the upper functor of the square (B) is fully faithful. In particular, it is enough to show that the square (B) is a pullback on objects. Consider an object n of Σ_{BG} , seen as the object $(G/e, \dots, G/e) \rightarrow G/e$ (with n occurrences of G/e) of $\Sigma_{(\mathcal{O}_G, \mathcal{O}_T)}$. Given an object $(f, (f_1, \dots, f_n))$ of $\Sigma_{(\mathcal{O}_G, \mathcal{O}_T)} \times_{\mathbf{F}\mathcal{O}_G} \mathbf{F}\Sigma_{(\mathcal{O}_G, \mathcal{O}_T)}$ such that $f \circ (f_1, \dots, f_n) = n$, we have to show that $(f, (f_1, \dots, f_n))$ belongs to $\Sigma_{BG} \times_{\mathbf{FBG}} \mathbf{F}\Sigma_{BG}$. This is a consequence of the fact that f is a retract of every f_i and that retracts in $\mathcal{O}_G$ are isomorphisms. We now have to show that, for every $\mathcal{O}_G$ -set of colours A , the functor $(\phi^A)^*$ preserves the unit. The unit of the category $\text{Coll}_{\Sigma_{BG}}^A(\mathcal{V})$ is the functor $1_{\text{Coll}_{\Sigma_{BG}}^A(\mathcal{V})}: (\Sigma_{BG}^A)^{\text{op}} \rightarrow \mathcal{V}$ defined by

$$1_{\text{Coll}_{\Sigma_{BG}}^A(\mathcal{V})}(n, (a_0, a_1, \dots, a_n)) = \begin{cases} 1_{\mathcal{V}} & \text{if } n = 1 \text{ and } a_0 = a_1 \\ \emptyset & \text{else.} \end{cases}$$

A similar description can be given for the unit $1_{\text{Coll}_{(\mathcal{O}_G, \mathcal{O}_T)}(\mathcal{V})}$ in the category $\text{Coll}_{(\mathcal{O}_G, \mathcal{O}_T)}(\mathcal{V})$ where $1_{\text{Coll}_{(\mathcal{O}_G, \mathcal{O}_T)}(\mathcal{V})}(f, (a_0, a_1, \dots, a_n))$ is given by

$$\begin{cases} 1_{\mathcal{V}} & \text{if } f \text{ belongs in } \mathcal{O}_G \text{ and is an isomorphism and } f^*(a_0) = a_1; \\ \emptyset & \text{else.} \end{cases}$$

It is now clear that the restriction of $1_{\text{Coll}_{(\mathcal{O}_G, \mathcal{O}_T)}(\mathcal{V})}$ along the inclusion $\Sigma_{BG}^A \hookrightarrow (\Sigma_{(\mathcal{O}_G, \mathcal{O}_T)}^A)^{\text{op}}$ is $1_{\text{Coll}_{\Sigma_{BG}}^A(\mathcal{V})}$. $\square$

Proposition 5.1.10. *The functor $\phi^*: \text{Op}_{(\mathcal{O}_G, \mathcal{O}_T)}^{\text{col}}(\mathcal{V}) \rightarrow \text{Op}^{\text{col}}(\mathcal{V})^G$ admits a right adjoint that we denote by ϕ_* .*

Proof. The existence of the right adjoint follows from Proposition 5.1.9 and Proposition 4.3.25. $\square$

We will now provide a more explicit description of the right adjoint ϕ_* . We first give an explicit description of $\Sigma_{(\mathcal{O}_G, \mathcal{O}_T)}^A$.

Proposition 5.1.11. *Let A be a G -set of colours. If one also denotes by A the presheaf $\mathcal{O}_G^{\text{op}} \rightarrow \text{Set}$ of fixed points of A , there is an equivalence*

$$\Sigma_{(\mathcal{O}_G, \mathcal{O}_T)}^A \simeq \bigsqcup_{n \geq 0} \bigsqcup_{X \in R_n} \mathcal{O}_{\text{Stab}(X)}^{\mathcal{F}_X} \quad (5.2)$$

and the functor $\Sigma_{\text{BG}}^A \hookrightarrow \Sigma_{(\mathcal{O}_G, \mathcal{O}_T)}^A$ corresponds to the canonical inclusions of groups into their orbit categories through the equivalences (5.1) and (5.2).

Proof. We only sketch the construction of the equivalence at the level of objects. Consider an object $(f, (a_1, \dots, a_n), a)$ of $\Sigma_{(\mathcal{O}_G, \mathcal{O}_T)}^A$. The map f is of the form $(f_1, \dots, f_n): G/H_1 \sqcup \dots \sqcup G/H_n \rightarrow G/H$ where every f_i can be represented by an element g_i of G such that $g_i^{-1}H_i g_i \subseteq H$ and such that $H/(g_i^{-1}H_i g_i)$ is admissible. Moreover, every a_i belongs to A^{H_i} . The object $(f, (a_1, \dots, a_n), a)$ is isomorphic in $\Sigma_{(\mathcal{O}_G, \mathcal{O}_T)}^A$ to $(f', (g_1^{-1}a_1, \dots, g_n^{-1}a_n), a)$ with

$$f': G/(g_1^{-1}H_1 g_1) \sqcup \dots \sqcup G/(g_n^{-1}H_n g_n) \rightarrow G/H$$

the map given component-wise by the inclusions $g_i^{-1}H_i g_i \subseteq H$. In particular, we can, without loss of generality, assume that every g_i is the neutral element of G . Consider the H -set $T := H/H_1 \sqcup \dots \sqcup H/H_n$, which is admissible. For every $i = 1 \dots, n$, choose a set $\{h_1^i, \dots, h_{|H/H_i|}^i\}$ of coset representatives of H/H_i . Every element of T is of the form (i, g_j^i) and this provides an isomorphism $T \simeq \{1, \dots, |T|\}$. We thus obtain an admissible graph subgroup $\Gamma_T \subseteq G \times \Sigma_{|T|}$. Consider now the element $X = (b, (b_1, \dots, b_{|T|}))$ of $A \times A^{|T|}$ where $b = a$ and where $b_k = g_j^i a_i$ if k corresponds to (i, g_j^i) in T . Using that a_i is a H_i -fixed point, Γ_T is a subgroup of the stabilizer of X and we obtain that $\text{Stab}(X)/\Gamma_T$ belongs to $\mathcal{O}_{\text{Stab}(X)}^{\mathcal{F}_X}$. □

Lemma 5.1.12. *For every inclusion of groups $K \subseteq H$, the following square*

$$\begin{array}{ccc} \text{sSet}^H & \xrightarrow{\text{Res}_K^H} & \text{sSet}^K \\ \downarrow & & \downarrow \\ \text{Fun}(\mathcal{O}_H^{\text{op}}, \text{sSet}) & \longrightarrow & \text{Fun}(\mathcal{O}_K^{\text{op}}, \text{sSet}) \end{array}$$

commutes, where the vertical functors are the fixed points functors and where the lower horizontal functor is given by restricting along the inclusion $\mathcal{O}_K \subseteq \mathcal{O}_H$.

Proof. This boils down to the fact that, for every subgroup L of K and every X in sSet , we have $X^L = (\text{Res}_K^H X)^L$. □

Proposition 5.1.13. *For every map $f: A \rightarrow B$ of G -sets, the following square*

$$\begin{array}{ccc} \mathrm{sOp}_{BG}^B & \xrightarrow{f^*} & \mathrm{sOp}_{BG}^A \\ (\phi^B)_* \downarrow & & \downarrow (\phi^A)_* \\ \mathrm{sOp}_{(\mathcal{O}_G, \mathcal{O}_T)}^B & \xrightarrow{f^*} & \mathrm{sOp}_{(\mathcal{O}_G, \mathcal{O}_T)}^A \end{array}$$

commutes.

Proof. It is enough to show the result at the level of collections, as all the functors of the square are obtained from lax monoidal functors. Using the equivalence of Proposition 5.1.11, the result follows from Lemma 5.1.12 applied, for all $n \geq 0$ and all X in $A \times A^{\times n}$, to the inclusion

$$\mathrm{Stab}(X) \subseteq \mathrm{Stab}((f \times f^{\times n})(X)).$$

□

Remark 5.1.14. Using Proposition 5.1.13, we can give an explicit description of the functor $\phi_*: (\mathrm{sOp}^{\mathrm{col}})^G \rightarrow \mathrm{sOp}_{(\mathcal{O}_G, \mathcal{O}_T)}^{\mathrm{col}}$. For every G -simplicial coloured operad $(A, \mathcal{O})$, its image under ϕ_* is the parametrized operad $(A, (\phi^A)_* \mathcal{O})$ where $(\phi^A)_*$ is the right adjoint of the following adjunction

$$(\phi^A)^*: \mathrm{sOp}_{(\mathcal{O}_G, \mathcal{O}_T)}^A \rightleftarrows \mathrm{sOp}_{BG}^A: (\phi^A)_*$$

where we identify A with its presheaf of fixed points. For every morphism $(f, \psi): (A, \mathcal{O}) \rightarrow (B, \mathcal{P})$ of G -simplicial coloured operads, its image under ϕ_* is the morphism $(f, (\phi^A)_* \psi)$.

5.1.3 Cellular extension of $\mathbb{M}$ -operads

One important ingredient in operadic theory is the notion of *cellular extension*. This was first considered in [BM03] for classical operads and it has since been extended to other situations, such as in [BP21] for *genuine equivariant operads*. In this subsection, we explain how such considerations could be generalised to $\mathbb{M}$ -operads.

In Subsection 4.3.6, we proved, under some assumptions on $\mathbb{M}$, that $\mathbb{M}$ -operads are algebras over a monad $\mathbf{T}_{\mathbb{M}}$. We will now give a more explicit description of the monad $\mathbf{T}_{\mathbb{M}}$ using the *category of $\mathbb{M}$ -trees*.

Definition 5.1.15. The *category of $\mathbb{M}$ -trees of height at most n* , denoted by $\mathcal{T}_{\mathbb{M}}(n)$, and the *root functor* $r(n): \mathcal{T}_{\mathbb{M}}(n) \rightarrow \mathbb{M}_0$ are defined by induction on n as follows:

1. the category $\mathcal{T}_{\mathbb{M}}(0)$ is $\mathbb{M}_0$ and the root functor $r(0): \mathbb{M}_0 \rightarrow \mathbb{M}_0$ is the identity;

- the category $\mathcal{T}_{\mathbb{M}}(n+1)$ is the pullback $(\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathcal{T}_{\mathbb{M}}(n)) \sqcup \mathbb{M}_0$ and the root functor $r(n+1): \mathcal{T}_{\mathbb{M}}(n+1) \rightarrow \mathbb{M}_0$ is the composition

$$(\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathcal{T}_{\mathbb{M}}(n)) \sqcup \mathbb{M}_0 \xrightarrow{p_1 \sqcup \text{id}_{\mathbb{M}_0}} \mathbb{M}_1 \sqcup \mathbb{M}_0 \xrightarrow{(t, \text{id}_{\mathbb{M}_0})} \mathbb{M}_0.$$

We now define a monomorphism $i_n: \mathcal{T}_{\mathbb{M}}(n) \hookrightarrow \mathcal{T}_{\mathbb{M}}(n+1)$ by induction as follows:

- the functor $i_0: \mathbb{M}_0 \hookrightarrow \mathbb{M}_1 \sqcup \mathbb{M}_0$ is the canonical inclusion;
- the functor $i_{n+1}: \mathcal{T}_{\mathbb{M}}(n+1) \rightarrow \mathcal{T}_{\mathbb{M}}(n+2)$ is the functor

$$(\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathcal{T}_{\mathbb{M}}(n)) \sqcup \mathbb{M}_0 \longrightarrow (\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathcal{T}_{\mathbb{M}}(n+1)) \sqcup \mathbb{M}_0$$

$$\text{given by } (\text{id}_{\mathbb{M}_1} \times_{\text{id}_{\mathbb{F}\mathbb{M}_0}} i_n) \sqcup \text{id}_{\mathbb{M}_0}.$$

The category of $\mathbb{M}$ -trees $\mathcal{T}_{\mathbb{M}}$ is the category $\text{colim}_n \mathcal{T}_{\mathbb{M}}(n)$. We define $\mathbb{M}$ -vertex functors

$$v_{\mathbb{M}}(n): \mathcal{T}_{\mathbb{M}}(n) \rightarrow \mathbb{F}\mathbb{M}_1$$

by induction as follows:

- the functor $v_{\mathbb{M}}(0): \mathbb{M}_0 \rightarrow \mathbb{F}\mathbb{M}_1$ is the functor constant on the initial object of $\mathbb{F}\mathbb{M}_1$;
- the functor $v_{\mathbb{M}}(n+1): \mathcal{T}_{\mathbb{M}}(n+1) \rightarrow \mathbb{F}\mathbb{M}_1$ is the composition

$$(\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathcal{T}_{\mathbb{M}}(n)) \sqcup \mathbb{M}_0 \longrightarrow \mathbb{M}_1 \times \mathbb{F}^2 \mathbb{M}_1 \longrightarrow \mathbb{M}_1 \times \mathbb{F}\mathbb{M}_1 \longrightarrow \mathbb{F}\mathbb{M}_1.$$

The functors $v_{\mathbb{M}}(n)$ induce an $\mathbb{M}$ -vertex functor $v_{\mathbb{M}}: \mathcal{T}_{\mathbb{M}} \rightarrow \mathbb{F}\mathbb{M}_1$. Finally, we define $\mathbb{M}$ -leaf functors

$$\odot_{\mathcal{T}_{\mathbb{M}}}(n): \mathcal{T}_{\mathbb{M}}(n) \rightarrow \mathbb{M}_1$$

by induction as follows:

- the functor $\odot_{\mathcal{T}_{\mathbb{M}}}(0)$ is the identity functor $1_{\mathbb{M}}: \mathbb{M}_0 \rightarrow \mathbb{M}_1$;
- the functor $\odot_{\mathcal{T}_{\mathbb{M}}}(n+1)$ is the composition

$$(\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathcal{T}_{\mathbb{M}}(n)) \sqcup \mathbb{M}_0 \longrightarrow (\mathbb{M}_1 \times_{\mathbb{F}\mathbb{M}_0} \mathbb{F}\mathbb{M}_1) \sqcup \mathbb{M}_0 \longrightarrow \mathbb{M}_1$$

where the first functor is $\text{id}_{\mathbb{M}_1} \times_{\text{id}_{\mathbb{F}\mathbb{M}_0}} \mathbb{F} \odot_{\mathcal{T}_{\mathbb{M}}}(n)$ and where the second functor is $(\odot_{\mathbb{M}}, 1_{\mathbb{M}})$. The functors $\odot_{\mathcal{T}_{\mathbb{M}}}(n)$ induce an $\mathbb{M}$ -leaf functor $\odot_{\mathcal{T}_{\mathbb{M}}}: \mathcal{T}_{\mathbb{M}} \rightarrow \mathbb{M}_1$.

Remark 5.1.16. A $\mathbb{M}$ -tree can be seen as a planar rooted tree where every vertex is decorated with an object of $\mathbb{M}$ and where every edge is decorated with an object of $\mathbb{M}_0$ in such a way that the inputs of every vertex are decorated with the source of the multimorphism decorating this vertex.

We now conjecture that the monad $\mathbf{T}_{\mathbb{M}}$ admits the following description involving the category of $\mathbb{M}$ -trees, which would generalise the classical formula for free operads.

Conjecture 5.1.17. *Let S be an $\mathbb{M}$ -collection in $\mathcal{V}$. The $\mathbb{M}$ -collection $\mathbf{T}_{\mathbb{M}}(S)$ is the left Kan extension*

$$\begin{array}{ccc} \mathcal{T}_{\mathbb{M}}^{\text{op}} & \xrightarrow{v_{\mathbb{M}}^{\text{op}}} & \mathbf{FIM}_1^{\text{op}} \xrightarrow{\tilde{S}} \mathcal{V}. \\ \downarrow \odot_{\mathcal{T}_{\mathbb{M}}} & \swarrow & \uparrow \\ \mathbb{M}_1^{\text{op}} & \xrightarrow{\quad \mathbf{T}_{\mathbb{M}}(S) \quad} & \end{array}$$

We now consider the following generalisation of the notion of cellular extension of operads. We also generalise the notion of Σ -cofibrant operads.

Definition 5.1.18. If $\mathcal{O}$ is a simplicial $\mathbb{M}$ -operad and if $u: U(\mathcal{O}) \rightarrow S$ is a cofibration of $\mathbb{M}$ -collections in $\mathbf{sSet}$, the *cellular extension* $\mathcal{O} \rightarrow \mathcal{O}[u]$ is the pushout of $F(u): FU(\mathcal{O}) \rightarrow F(S)$ along the counit $FU(\mathcal{O}) \rightarrow \mathcal{O}$. We say that an $\mathbb{M}$ -operad $\mathcal{O}$ is *$\mathbb{M}$ -cofibrant* if the underlying morphism of $\mathbb{M}$ -collections $I \rightarrow \mathcal{O}$ given by the identities of $\mathcal{O}$ is a cofibration of $\mathbb{M}$ -collections.

One could try to modify the description of free $\mathbb{M}$ -operads that we conjectured in Conjecture 5.1.19 in order to obtain a possible description of cellular extensions. Such a description would be based on a notion of *marked $\mathbb{M}$ -tree* and would involve iterated pushout-products indexed on the vertices of $\mathbb{M}$ -trees. Based on this potential description of cellular extensions, we make the following conjecture.

Conjecture 5.1.19. *For every cofibration $u: U(\mathcal{O}) \rightarrow S$ of $\mathbb{M}$ -collections with $\mathcal{O}$ an $\mathbb{M}$ -cofibrant simplicial $\mathbb{M}$ -operad, the cellular extension $\mathcal{O} \rightarrow \mathcal{O}[u]$ is an $\mathbb{M}$ -cofibration.*

One important motivation for the resolution of Conjecture 5.1.19 is the following proposition, which is a direct consequence of it.

Proposition 5.1.20. *Assuming that Conjecture 5.1.19 is verified, for every target right fibrant $\mathbf{F}$ -multicategory $\mathbb{M}$ and for every simplicial $\mathbb{M}$ -operad $\mathcal{O}$, if $\mathcal{O}$ is cofibrant in $\mathbf{sOp}_{\mathbb{M}}$, then $\mathcal{O}$ is $\mathbb{M}$ -cofibrant.*

Remark 5.1.21. The Conjecture 5.1.17 should be relatively easy to solve, and the author does not expect other assumptions on $\mathbb{M}$ to be needed. However, it is not clear whether the assumption of target right fibrancy on $\mathbb{M}$ would actually suffice to prove Conjecture 5.1.19. We expect however that this result should be true on a sufficient class of $\mathbf{F}$ -multicategories, which would contain the examples of interest to the study of parametrized and equivariant operads.

5.1.4 The coloured operadic Elmendorf conjecture

In this subsection, we show, assuming that the Conjecture 5.1.19 is true, that the adjunction of Proposition 5.1.10 is a Quillen equivalence.

Proposition 5.1.22. *Let $\mathbb{M}$ be a target right fibrant $\mathbf{F}$ -multicategory. For every cofibrant simplicial coloured $\mathbb{M}$ -operad $(A, \mathcal{O})$, $\mathcal{O}$ is cofibrant in $\mathrm{sOp}_{\mathbb{M}}^A$.*

Proof. Denote by $\emptyset$ the initial object of $\mathrm{sOp}_{\mathbb{M}}^{\mathrm{col}}$ and denote by $\emptyset_A$ the initial object of $\mathrm{sOp}_{\mathbb{M}}^A$. Let $(A, \mathcal{O})$ be a cofibrant simplicial coloured $\mathbb{M}$ -operad. Consider a commutative square

$$\begin{array}{ccc} \emptyset_A & \longrightarrow & \mathcal{P} \\ \downarrow & & \downarrow f \\ \mathcal{O} & \longrightarrow & \mathcal{Q} \end{array}$$

in $\mathrm{sOp}_{\mathbb{M}}^A$ such that f is a trivial fibration. Consider the following diagram in $\mathrm{sOp}_{\mathbb{M}}^{\mathrm{col}}$

$$\begin{array}{ccc} \emptyset & \longrightarrow & (A, \mathcal{P}) \\ \downarrow & & \downarrow \mathrm{id}_{(A, \mathcal{P})} \\ (A, \emptyset_A) & \longrightarrow & (A, \mathcal{P}) \\ \downarrow & & \downarrow f \\ (A, \mathcal{O}) & \longrightarrow & (A, \mathcal{Q}). \end{array}$$

The outer square admits a lift because $(A, \mathcal{O})$ is cofibrant in $\mathrm{sOp}_{\mathbb{M}}^{\mathrm{col}}$ and because f is also a trivial fibration in $\mathrm{sOp}_{\mathbb{M}}^{\mathrm{col}}$. Note that because the lower triangle provided by this lift commutes, we obtain that the lift belongs to $\mathrm{sOp}_{\mathbb{M}}^A$ and thus provides a lift for the first square. $\square$

Proposition 5.1.23. *For every cofibrant simplicial coloured $\mathbb{M}$ -operad $\mathcal{O}$, the $\mathbb{M}$ -set of colours of $\mathcal{O}$ is a retract of a coproduct of representables.*

Proof. Consider the class C in $\mathrm{Psh}(\mathbb{M}_0)$ of maps $A \hookrightarrow A \sqcup B$ with B a coproduct of representables. We show first that cellular maps of simplicial coloured $\mathbb{M}$ -operad belong to C , at the level of colours. Using that the functor $\mathrm{sOp}_{\mathbb{M}}^{\mathrm{col}} \rightarrow \mathrm{Psh}(\mathbb{M}_0)$ that sends an $\mathbb{M}$ -operad $(A, \mathcal{O})$ to its $\mathbb{M}$ -set of colours A preserves colimits and that the class C is closed under pushouts and sequential colimits, it is enough to show that generating cofibrations in $\mathrm{sOp}_{\mathbb{M}}^{\mathrm{col}}$ belong to C , at the level of colours. This follows directly from the explicit description that we gave of these generating cofibrations. It follows that the $\mathbb{M}$ -set of colours of cellular objects in $\mathrm{sOp}_{\mathbb{M}}^{\mathrm{col}}$ are coproduct of representables and the result follows from the fact that cofibrant objects are retract of cellular objects. $\square$

Theorem 5.1.24. *If Conjecture 5.1.19 is true, then the adjunction*

$$\phi^*: \mathrm{sOp}_{(\mathcal{O}_G, \mathcal{O}_T)}^{\mathrm{col}} \rightleftarrows (\mathrm{sOp}^{\mathrm{col}})^G: \phi_*$$

of Proposition 5.1.10 is a Quillen equivalence where $(\mathrm{sOp}^{\mathrm{col}})^G$ is endowed with the graph model structure of [BP22c] for the G -transfer system T and where $\mathrm{sOp}_{(\mathcal{O}_G, \mathcal{O}_T)}^{\mathrm{col}}$ is endowed with the model structure of Example 4.4.29.

Proof. We prove first that (ϕ^*, ϕ_*) is a Quillen adjunction by showing that ϕ_* is a right Quillen functor. To do this, we prove that ϕ_* preserves fully-faithful and essentially surjective morphisms, the same arguments prove that ϕ_* preserves isofibrations. For fully faithful morphisms, the result follows directly from the fact that $((\phi^A)^*, (\phi^A)_*)$ is a Quillen adjunction, using the description that we gave of ϕ_* in Remark 5.1.14. For essentially surjective morphisms, the result follows from the fact that the triangle below commutes for every subgroup H of G .

$$\begin{array}{ccc} (\mathrm{sOp}^{\mathrm{col}})^G & \xrightarrow{\phi_*} & \mathrm{sOp}_{(\mathcal{O}_G, \mathcal{O}_T)}^{\mathrm{col}} \\ & \searrow i_H \quad \swarrow i_{G/H} & \\ & \mathrm{sCat} & \end{array}$$

We now prove that (ϕ^*, ϕ_*) is a Quillen equivalence. The adjunction (ϕ^*, ϕ_*) is reflexive and it is thus enough to prove that, for every cofibrant simplicial coloured $(\mathcal{O}_G, \mathcal{O}_T)$ -operad $(A, \mathcal{O})$, the unit $(A, \mathcal{O}) \rightarrow \phi_* \phi^*(A, \mathcal{O})$ is a weak equivalence. Using Proposition 5.1.23, the $\mathcal{O}_G$ -set A is obtained as the fixed points of a G -set B and it follows that the unit $(A, \mathcal{O}) \rightarrow \phi_* \phi^*(A, \mathcal{O})$ is the unit of the adjunction

$$(\phi^B)^*: \mathrm{sOp}_{(\mathcal{O}_G, \mathcal{O}_T)}^B \rightleftarrows \mathrm{sOp}_{BG}^B: (\phi^B)_*$$

Using Conjecture 5.1.19, the previous adjunction is a Quillen equivalence if and only if the adjunction

$$(\phi^B)^*: \mathrm{Coll}_{(\mathcal{O}_G, \mathcal{O}_T)}^A(\mathrm{sSet}) \rightleftarrows \mathrm{Coll}_{BG}^B(\mathrm{sSet}): (\phi^B)_*$$

is a Quillen equivalence. This result now follows from Proposition 5.1.11 and Elmendorf's Theorem. $\square$

5.2 Complete $\mathbb{M}$ -Segal spaces

In this final section, we define a notion of *complete $\mathbb{M}$ -Segal spaces* that should produce the same homotopy theory as simplicial coloured $\mathbb{M}$ -operads. We define first, for every $\mathbb{F}$ -multicategory $\mathbb{M}$, an algebraic pattern $\Delta_{\mathbb{M}}^{\mathrm{op}}$ in the sense of [CH21, Definition 2.1], which will encode our $\mathbb{M}$ -Segal spaces.

Definition 5.2.1. Let $\mathbb{D}$ be a double category. The category $\Delta_{\mathbb{D}}$ is the (cocartesian) Grothendieck construction of the nerve $N(\mathbb{D}): \Delta^{\mathrm{op}} \rightarrow \mathrm{Cat}$ of $\mathbb{D}$. If $\mathbb{M}$ is an $\mathbb{F}$ -multicategory, then we define $\Delta_{\mathbb{M}}$ as the category $\Delta_{\mathbb{F}\mathbb{M}}$. We have a cocartesian fibration $\pi_{\mathbb{D}}: \Delta_{\mathbb{D}}^{\mathrm{op}} \rightarrow \Delta^{\mathrm{op}}$ and it follows that the algebraic pattern structure of Δ^{op} , denoted by $\Delta^{\mathrm{op}, \mathfrak{h}}$ in [CH21, Example 3.3], induces the structure of an algebraic pattern on $\Delta_{\mathbb{D}}^{\mathrm{op}}$. Explicitly, the structure of an algebraic pattern on $\Delta_{\mathbb{D}}^{\mathrm{op}}$ is defined as follows:

1. a morphism in $\Delta_{\mathbb{D}}^{\mathrm{op}}$ is *inert* if the underlying morphism in Δ^{op} is inert;
2. a morphism in $\Delta_{\mathbb{D}}^{\mathrm{op}}$ is *active* if it is $\pi_{\mathbb{D}}$ -cocartesian and if the underlying morphism in Δ^{op} is active;

3. an object $([n], f)$ in $\Delta_{\mathbb{D}}^{\text{op}}$ is elementary if $n = 1$ or $n = 0$.

For every $\mathbf{F}$ -multicategory $\mathbb{M}$, the category $\Delta_{\mathbb{M}}^{\text{op}}$ admits the structure of an algebraic pattern whose inert and active morphisms are the ones of $\Delta_{\mathbf{F}\mathbb{M}}^{\text{op}}$ and where the elementary objects are the objects $([1], f)$ where f belongs to $\mathbb{M}_1$ and $([0], x)$ where x belongs to $\mathbb{M}_0$.

Remark 5.2.2. Note that, in Definition 4.2.16, we were considering the cartesian Grothendieck construction, whereas in the previous definition, we consider the cocartesian one.

Definition 5.2.3. An $\mathbb{M}$ -Segal space is a $\Delta_{\mathbb{M}}^{\text{op}}$ -Segal object in $\mathcal{S}$, in the sense of [CH21, Definition 2.7]. We denote by $\text{Seg}_{\mathbb{M}}(\mathcal{S})$ the ∞ -category of $\mathbb{M}$ -Segal spaces.

We define a notion of completeness for $\mathbb{M}$ -Segal spaces. Every object X of $\mathbb{M}$ provides a section $i_X: \Delta \rightarrow \Delta_{\mathbb{M}}$ of $\pi_{\mathbb{M}}$. Moreover, the functor i_X is a Segal morphism in the sense of [CH21, Definition 4.2] and it induces a functor $i_X^*: \text{Seg}_{\mathbb{M}}(\mathcal{S}) \rightarrow \text{Seg}(\mathcal{S})$.

Definition 5.2.4. An $\mathbb{M}$ -Segal space $\mathcal{O}$ is *complete* if for every X in $\mathbb{M}_0$, the Segal space $i_X^*(\mathcal{O})$ is complete. We denote by $\text{CSeg}_{\mathbb{M}}(\mathcal{S})$ the full subcategory of $\text{Seg}_{\mathbb{M}}(\mathcal{S})$ spanned by complete $\mathbb{M}$ -Segal spaces.

Example 5.2.5. For the $\mathbf{F}$ -multicategory Σ_* of Example 4.2.12, we obtain a slight modification of the algebraic pattern Δ_{Fin} defined by Barwick in [Bar18]. More precisely, the category Δ_{Fin} is a wide subcategory of our Δ_{Σ_*} and this inclusion satisfies the conditions of [Bar23, Corollary 2.64]. This implies that our Σ_* -Segal spaces correspond to the Fin-Segal spaces of Barwick.

Example 5.2.6. For every orbital pair $(\mathcal{O}, \mathcal{T})$, we obtain a notion of *complete* $(\mathcal{O}, \mathcal{T})$ -Segal spaces using the $\mathbf{F}$ -multicategory $\Sigma_{(\mathcal{O}, \mathcal{T})}$ of Definition 4.2.30. We denote by $\text{CSeg}_{(\mathcal{O}, \mathcal{T})}$ the associated ∞ -category.

In the same way simplicial categories can be compared with complete Segal spaces, we make the following conjecture.

Conjecture 5.2.7. The ∞ -category obtained from $\text{sOp}_{\mathbb{M}}^{\text{col}}$ by inverting fully faithful essentially surjective morphisms (defined in Definition 4.4.11) is equivalent to the ∞ -category $\text{CSeg}_{\mathbb{M}}(\mathcal{S})$ of complete $\mathbb{M}$ -Segal spaces.

Definition 5.2.8. For every orbital pair $(\mathcal{O}, \mathcal{T})$, we denote by $\infty\text{-Op}_{(\mathcal{O}, \mathcal{T})}$ the ∞ -category of fibrous patterns over $\text{Span}_{\mathcal{T}}(\mathbf{F}\mathcal{O})$ in the sense of [Ste25b]. The objects of $\infty\text{-Op}_{(\mathcal{O}, \mathcal{T})}$ will be referred to as $(\mathcal{O}, \mathcal{T})$ - ∞ -operads. Note that if $\mathcal{O}$ is an atomic orbital category, then it follows from [Ste25b, Theorem A.1] that the ∞ -category $\infty\text{-Op}_{(\mathcal{O}, \mathcal{O})}$ is equivalent to the ∞ -category of $\mathcal{O}$ - ∞ -operads of Nardin and Shah [NS22].

A direct consequence of Conjecture 5.2.7 would be that the ∞ -category associated with $\text{sOp}_{(\mathcal{O}, \mathcal{T})}^{\text{col}}$ can be compared with complete $(\mathcal{O}, \mathcal{T})$ -Segal spaces. We also conjecture that Barwick's work [Bar18] should generalise to the algebraic pattern $\Delta_{\Sigma_{(\mathcal{O}, \mathcal{T})}}^{\text{op}}$.

Conjecture 5.2.9. *For every orbital pair $(\mathcal{O}, \mathcal{T})$, the ∞ -category $\text{CSeg}_{(\mathcal{O}, \mathcal{T})}(\mathcal{S})$ is equivalent to the ∞ -category of $(\mathcal{O}, \mathcal{T})$ - ∞ -operads.*

Finally, if Conjecture 5.1.19, Conjecture 5.2.7 and Conjecture 5.2.9 are true, one would obtain, for every G -transfer system $\mathcal{T}$, the following chain of equivalences

$$\left(\text{sOp}^{\text{col}} \right)^G \xrightarrow{\simeq} \text{sOp}_{(\mathcal{O}_G, \mathcal{O}_T)}^{\text{col}} \xrightarrow{\simeq} \text{CSeg}_{(\mathcal{O}_G, \mathcal{O}_T)} \xrightarrow{\simeq} \infty\text{-Op}_{(\mathcal{O}_G, \mathcal{O}_T)}.$$

In particular, this would provide an equivalence between the ∞ -category of G -simplicial coloured operads with the Bonventre Pereira model structure associated with the complete G -transfer system and the ∞ -category of G - ∞ -operads of Nardin and Shah.

Summary

In this dissertation, we compare two approaches for equivariant algebra in homotopy theory. A central principle of homotopy theory is the Grothendieck hypothesis, which states that topological spaces should be interpreted, up to homotopy, as ∞ -groupoids. This links two viewpoints on classical homotopy theory namely, topology and higher category theory. In equivariant homotopy theory, a similar story can be told with Elmendorf's theorem which provides a description of equivariant spaces in terms of G - ∞ -groupoids. In particular, equivariant homotopy theory can be studied through the lens of equivariant higher category theory.

A natural question now arises:

“Given a mathematical notion with some topological and equivariant flavor, can it be described, up to homotopy, within the framework of equivariant higher categories ?”

In this thesis, we study this question in the case of algebraic objects. We provide a full answer for commutative algebras and we develop ideas and constructions toward a complete answer to more general algebraic objects.

Samenvatting

In dit proefschrift vergelijken we twee manieren om equivariante algebra in homotopietheorie te doen. Een centraal principe van de homotopietheorie is de hypothese van Grothendieck, die stelt dat topologische ruimten, tot op homotopie, geïnterpreteerd moeten worden als ∞ -groepoïden. Dit verbindt twee perspectieven van de klassieke homotopietheorie, namelijk topologie en hogere categorietheorie. Een soortgelijk verhaal geldt in de homotopietheorie met de stelling van Elmendorf, die een beschrijving geeft van equivariante ruimten in termen van G - ∞ -groepoïden. In het bijzonder kan equivariante homotopietheorie worden bestudeerd vanuit het perspectief van equivariante hogere categorietheorie.

Een natuurlijke vraag is dan:

“Gegeven een wiskundig begrip met een zekere topologische en equivariante structuur, kan dit, tot op homotopie, worden beschreven binnen het raamwerk van equivariante hogere categorieën?”

In dit proefschrift bestuderen wij deze vraag in het geval van algebraïsche objecten. Wij geven een volledig antwoord in het geval van commutatieve algebra's en ontwikkelen ideeën en constructies die leiden tot een volledig antwoord voor meer algemene algebraïsche objecten.

Acknowledgments

First of all, I would like to thank my supervisor, Magdalena Kędziorek, for giving me the chance to do a PhD in mathematics. You gave me the opportunity to explore my own ideas and to work on my own projects from the very beginning of my PhD, and I could not be more grateful for that. I am also thankful for all your advice, in particular on things I am not comfortable with, such as writing introductions. Finally, you have been the best moral support during these last four years, and I would not have survived this without you.

I am also grateful to Ben Moonen for being my promotor and for your help with administrative matters.

I would also like to thank Angélica Osorno, Julie Bergner, Ieke Moerdijk, Mike Hill, and Erik Koelink for being part of my manuscript committee.

Finally, I would like to thank the other people who helped me mathematically during these last four years, including Seffen Sagave, Maxime Wybouw, Bastiaan Cnossen, Julius Groenjes, Giorgi Tigilauri, Niall Taggart, Miguel Barrero, and Natalie Stewart.

Research data management

This thesis research has been carried out under the institute research data management policy of the Institute for Mathematics, Astrophysics and Particle Physics, Radboud University. No data has been produced or analysed in this project.

About the author

Grégoire Marc was born on the 5th of September 1998 in Étampes, France. After finishing high school in Saint-Girons, he started in 2016 the bachelor parcours spécial at Université Paul-Sabatier in Toulouse. In 2019, he completed his first year of a Master's degree in mathematics in Marseille. In 2020, he received the Lebesgue Master Scholarship and completed his second year of the Master's program at Université Rennes 1. He wrote his master's thesis, entitled "Problèmes de modules formels", under the supervision of Sinan Yalin in Angers. Since April 2022, he has been a PhD candidate at the mathematics department of Radboud University, under the supervision of Magdalena Kędziołek.

Bibliography

- [AR94] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994. MR 1294136
- [Bar17] C. Barwick, *Spectral Mackey functors and equivariant algebraic K-theory (I)*, Adv. Math. **304** (2017), 646–727. MR 3558219
- [Bar18] ———, *From operator categories to higher operads*, Geom. Topol. **22** (2018), no. 4, 1893–1959. MR 3784514
- [Bar23] S. Barkan, *Arity approximation of ∞ -operads*, arXiv:2207.07200, 2023.
- [BBR21] S. Balchin, D. Barnes, and C. Roitzheim, *N_∞ -operads and associahedra*, Pacific J. Math. **315** (2021), no. 2, 285–304. MR 4366744
- [BDG⁺16] C. Barwick, E. Dotto, S. Glasman, D. Nardin, and J. Shah, *Parametrized higher category theory and higher algebra: Exposé I – elements of parametrized higher category theory*, arXiv:1608.03657, 2016.
- [Ben85] J. Benabou, *Fibered categories and the foundations of naive category theory*, J. Symbolic Logic **50** (1985), no. 1, 10–37. MR 780520
- [Ber07a] J. E. Bergner, *A model category structure on the category of simplicial categories*, Trans. Amer. Math. Soc. **359** (2007), no. 5, 2043–2058. MR 2276611
- [Ber07b] ———, *Three models for the homotopy theory of homotopy theories*, Topology **46** (2007), no. 4, 397–436. MR 2321038
- [BH15] A. J. Blumberg and M. A. Hill, *Operadic multiplications in equivariant spectra, norms, and transfers*, Adv. Math. **285** (2015), 658–708. MR 3406512
- [BH18] ———, *Incomplete Tambara functors*, Algebr. Geom. Topol. **18** (2018), no. 2, 723–766. MR 3773736
- [BH21] T. Bachmann and M. Hovey, *Norms in motivic homotopy theory*, Astérisque (2021), no. 425, ix+207. MR 4288071

- [BHS25] S. Barkan, R. Haugseng, and J. Steinebrunner, *Envelopes for algebraic patterns*, *Algebr. Geom. Topol.* **25** (2025), no. 9, 5319–5388. MR 5005649
- [BM03] C. Berger and I. Moerdijk, *Axiomatic homotopy theory for operads*, *Comment. Math. Helv.* **78** (2003), no. 4, 805–831. MR 2016697
- [BM07] ———, *Resolution of coloured operads and rectification of homotopy algebras*, *Categories in algebra, geometry and mathematical physics*, *Contemp. Math.*, vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 31–58. MR 2342815
- [BMO25] S. Balchin, E. MacBrough, and K. Ormsby, *The combinatorics of N_∞ operads for C_{qp^n} and D_{p^n}* , *Glasg. Math. J.* **67** (2025), no. 1, 50–66. MR 4857506
- [BMZ23] N. Behr, P. Melliès, and N. Zeilberger, *Convolution Products on Double Categories and Categorification of Rule Algebras*, 8th International Conference on Formal Structures for Computation and Deduction (FSCD 2023) (Dagstuhl, Germany) (Marco Gaboardi and Femke van Raamsdonk, eds.), *Leibniz International Proceedings in Informatics (LIPIcs)*, vol. 260, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2023, pp. 17:1–17:20.
- [Bon19] P. Bonventre, *The genuine operadic nerve*, *Theory Appl. Categ.* **34** (2019), 736–780. MR 4011810
- [BP20] P. Bonventre and L. A. Pereira, *Equivariant dendroidal Segal spaces and G - ∞ -operads*, *Algebr. Geom. Topol.* **20** (2020), no. 6, 2687–2778. MR 4185927
- [BP21] ———, *Genuine equivariant operads*, *Adv. Math.* **381** (2021), Paper No. 107502, 133. MR 4205708
- [BP22a] ———, *Equivariant dendroidal sets and simplicial operads*, *J. Topol.* **15** (2022), no. 2, 745–805. MR 4441603
- [BP22b] ———, *Homotopy theory of equivariant operads with fixed colors*, *Tunis. J. Math.* **4** (2022), no. 1, 87–158. MR 4401789
- [BP22c] ———, *On the homotopy theory of equivariant colored operads*, *Algebr. Geom. Topol.* **22** (2022), no. 6, 2631–2688. MR 4520675
- [BV73] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, *Lecture Notes in Mathematics*, vol. Vol. 347, Springer-Verlag, Berlin-New York, 1973. MR 420609
- [CH21] H. Chu and R. Haugseng, *Homotopy-coherent algebra via Segal conditions*, *Adv. Math.* **385** (2021), Paper No. 107733, 95. MR 4256131
- [CHH18] H. Chu, R. Haugseng, and G. Heuts, *Two models for the homotopy theory of ∞ -operads*, *J. Topol.* **11** (2018), no. 4, 857–873. MR 3847208

- [CLL25a] B. Cnossen, T. Lenz, and S. Linskens, *Parametrized (higher) semiadditivity and the universality of spans*, arXiv:2403.07676, 2025.
- [CLL25b] ———, *Parametrized stability and the universal property of global spectra*, J. Topol. **18** (2025), no. 4, Paper No. e70044, 119. MR 4990070
- [CLL25c] ———, *Partial parametrized presentability and the universal property of equivariant spectra*, Trans. Amer. Math. Soc. **378** (2025), no. 10, 6913–6974. MR 4956364
- [CM11] D. Cisinski and I. Moerdijk, *Dendroidal sets as models for homotopy operads*, J. Topol. **4** (2011), no. 2, 257–299. MR 2805991
- [CM13a] ———, *Dendroidal Segal spaces and ∞ -operads*, J. Topol. **6** (2013), no. 3, 675–704. MR 3100887
- [CM13b] ———, *Dendroidal sets and simplicial operads*, J. Topol. **6** (2013), no. 3, 705–756. MR 3100888
- [Dre71] A. Dress, *Notes on the theory of representations of finite groups, Pt. 1: the Burnside ring of a finite group and some AGN-applications*, Bielefeld University, 1971.
- [Ehr63] C. Ehresmann, *Catégories doubles et catégories structurées*, C. R. Acad. Sci. Paris **256** (1963), 1198–1201. MR 152561
- [Elm83] A. D. Elmendorf, *Systems of fixed point sets*, Trans. Amer. Math. Soc. **277** (1983), no. 1, 275–284. MR 690052
- [GGN15] D. Gepner, M. Groth, and T. Nikolaus, *Universality of multiplicative infinite loop space machines*, Algebr. Geom. Topol. **15** (2015), no. 6, 3107–3153. MR 3450758
- [GM24] B. J. Guillou and J. P. May, *Models of G -spectra as presheaves of spectra*, Algebr. Geom. Topol. **24** (2024), no. 3, 1225–1275. MR 4767872
- [GP99] M. Grandis and R. Pare, *Limits in double categories*, Cahiers Topologie Géom. Différentielle Catég. **40** (1999), no. 3, 162–220. MR 1716779
- [GR03] A. Grothendieck and M. Raynaud, *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris), vol. 3, Société Mathématique de France, Paris, 2003. MR 2017446
- [Gro83] A. Grothendieck, *Pursuing stacks*, 1983, Manuscript, available at <https://arxiv.org/abs/math/0306104>.
- [Gui10] B. J. Guillou, *Strictification of categories weakly enriched in symmetric monoidal categories*, Theory Appl. Categ. **24** (2010), No. 20, 564–579. MR 2770075

- [Hau22] R. Haugseng, *∞ -operads via symmetric sequences*, Math. Z. **301** (2022), no. 1, 115–171. MR 4405646
- [HHR16] M. A. Hill, M. J. Hopkins, and D. C. Ravenel, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. (2) **184** (2016), no. 1, 1–262. MR 3505179
- [Hil22] M. A. Hill, *On the algebras over equivariant little disks*, J. Pure Appl. Algebra **226** (2022), no. 10, Paper No. 107052, 21. MR 4389764
- [Hil24] K. Hilman, *Parametrised presentability over orbital categories*, Appl. Categ. Structures **32** (2024), no. 3, Paper No. 15, 53. MR 4756154
- [Hov99] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR 1650134
- [HP15] Y. Harpaz and M. Prasma, *The Grothendieck construction for model categories*, Adv. Math. **281** (2015), 1306–1363. MR 3366868
- [Ill73] S. Illman, *Equivariant algebraic topology*, Ann. Inst. Fourier (Grenoble) **23** (1973), no. 2, 87–91. MR 358761
- [Joy08] A. Joyal, *Notes on quasi-categories*, 2008, available at <https://www.math.uchicago.edu/~may/IMA/Joyal.pdf>.
- [JT07] A. Joyal and M. Tierney, *Quasi-categories vs Segal spaces*, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 277–326. MR 2342834
- [Kel74] G. M. Kelly, *Doctrinal adjunction*, Category Seminar (Proc. Sem., Sydney, 1972/1973), Lecture Notes in Math., vol. 420, Springer, Berlin-New York, 1974, pp. 257–280. MR 360749
- [Kel80] ———, *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on*, Bull. Austral. Math. Soc. **22** (1980), no. 1, 1–83. MR 589937
- [Lei04] T. Leinster, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series, vol. 298, Cambridge University Press, Cambridge, 2004. MR 2094071
- [Len25] T. Lenz, *Global homotopy theory via spectral Mackey functors*, arXiv:2202.07272, 2025.
- [Lew88] L. G. Lewis, Jr., *The $RO(G)$ -graded equivariant ordinary cohomology of complex projective spaces with linear $\mathbb{Z}/p$ actions*, Algebraic topology and transformation groups (Göttingen, 1987), Lecture Notes in Math., vol. 1361, Springer, Berlin, 1988, pp. 53–122. MR 979507
- [Lur09] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659

- [Lur17] ———, *Higher algebra*, 2017, available at <https://www.math.ias.edu/~lurie/papers/HA.pdf>.
- [Mar22a] L. Martini, *Cocartesian fibrations and straightening internal to an ∞ -topos*, arXiv:2204.00295, 2022.
- [Mar22b] ———, *Yoneda’s lemma for internal higher categories*, arXiv:2103.17141, 2022.
- [Mar24] G. Marc, *A higher mackey functor description of algebras over an N_∞ -operad*, arXiv:2402.12447, 2024.
- [Mar25] ———, *Homotopy theory of simplicial parametrized operads*, arXiv:2510.06932, 2025.
- [Mat71] T. Matumoto, *Equivariant K-theory and Fredholm operators*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **18** (1971), 109–125. MR 290354
- [May72] J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics, vol. Vol. 271, Springer-Verlag, Berlin-New York, 1972. MR 420610
- [MW24] L. Martini and S. Wolf, *Colimits and cocompletions in internal higher category theory*, High. Struct. **8** (2024), no. 1, 97–192. MR 4752519
- [Nar16] D. Nardin, *Parametrized higher category theory and higher algebra: Exposé IV – stability with respect to an orbital ∞ -category*, arXiv:1608.07704, 2016.
- [Nar17] ———, *Stability and distributivity over orbital ∞ -categories*, Ph.D. thesis, Massachusetts Institute of Technology, 2017.
- [NS22] D. Nardin and J. Shah, *Parametrized and equivariant higher algebra*, arXiv:2203.00072, 2022.
- [Per18] L. A. Pereira, *Equivariant dendroidal sets*, Algebr. Geom. Topol. **18** (2018), no. 4, 2179–2244. MR 3797065
- [Pis22] C. Pisani, *Operads as double functors*, arXiv:2208.07028, 2022.
- [PS18a] D. Pavlov and J. Scholbach, *Admissibility and rectification of colored symmetric operads*, J. Topol. **11** (2018), no. 3, 559–601. MR 3830876
- [PS18b] ———, *Homotopy theory of symmetric powers*, Homology Homotopy Appl. **20** (2018), no. 1, 359–397. MR 3785971
- [Rez01] C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007. MR 1804411
- [Rob11] M. Robertson, *The homotopy theory of simplicially enriched multicategories*, arXiv:1111.4146, 2011.
- [Rub18] J. Rubin, *Equivariant categorical coherence theory*, Ph.D. thesis, The University of Chicago, 2018.

- [Rub19] ———, *Characterizations of equivariant Steiner and linear isometries operads*, arXiv:1903.08723, 2019.
- [Rub20] ———, *Normed symmetric monoidal categories*, arXiv:1708.04777, 2020.
- [Rub21] ———, *Combinatorial N_∞ operads*, *Algebr. Geom. Topol.* **21** (2021), no. 7, 3513–3568. MR 4357612
- [Rui24] J. Ruit, *Formal category theory in ∞ -equipments II: Lax functors, monoidality and fibrations*, arXiv:2408.15190, 2024.
- [Rui25] ———, *Formal category theory in ∞ -equipments I*, arXiv:2308.03583, 2025.
- [Sha17] J. Shah, *Parametrized higher category theory*, ProQuest LLC, Ann Arbor, MI, 2017, Thesis (Ph.D.)—Massachusetts Institute of Technology. MR 3781930
- [Sta63] J. D. Stasheff, *Homotopy associativity of H-spaces. I, II*, *Trans. Amer. Math. Soc.* **108** (1963), 293–312, **108** (1963), 275–292; *ibid.* MR 158400
- [Ste16] M. Stephan, *On equivariant homotopy theory for model categories*, *Homology, Homotopy and Applications* **18** (2016), 183–208.
- [Ste25a] N. Stewart, *Eckmann-Hilton arguments in equivariant higher algebra*, arXiv:2508.05556, 2025.
- [Ste25b] ———, *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*, arXiv:2501.02129, 2025.
- [Ste25c] ———, *On tensor products with equivariant commutative operads*, arXiv:2504.02143, 2025.
- [Ste25d] ———, *Orbital categories and weak indexing systems*, arXiv:2409.01377, 2025.
- [Web93] P. Webb, *Two classifications of simple Mackey functors with applications to group cohomology and the decomposition of classifying spaces*, *J. Pure Appl. Algebra* **88** (1993), no. 1-3, 265–304. MR 1233331



Radboud Universiteit

