Variational models for materials science: epitaxial growth and geometrically constrained walls

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by

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Contents

Introduction

This dissertation is based on the papers [28], [29], and [35]. The common thread among these works is the investigation of variational models arising from problems in materials science.
In this introduction, we provide a brief overview of materials science and present the main topics of this thesis, namely epitaxial growth and domain walls. For each topic, we review the relevant literature and models, and outline the focus of our study. The mathematical core of this dissertation relies on variational models, therefore we will carefully introduce them by motivating their relevance and show how they interact with materials science.
Chapters 1 and 2 are dedicated to our work on epitaxial growth, while Chapter 3 focuses on domain walls in three dimensions. Along the Chapters, we will present novel results that add to the state of art new perspectives and mathematical techniques that looks promising for future investigations on the subjects.

I Materials science and variational models

The connection between mathematics and industry has become predominant in many aspects over the last few decades. On the one hand, mathematical models are constantly tested by engineers; on the other hand, open questions arising from industry are actively investigated by mathematicians. In this context, materials science integrates many branches of science such as physics, chemistry, engineering, and, last but not least, mathematics. A full understanding of the micro- and macro-structural, as well as the physical properties of a material, chemical compound, or

alloy is highly relevant for scientific progress. Indeed, many applications in metallurgy, thermodynamics, optics, polymers, ceramics, biomaterials, and micro-components for electronics require a comprehensive knowledge from several disciplines, including mathematics. Within this framework, mathematics plays a crucial role both in practical applications and in the theoretical foundations, acting as a unifying element in the scientific process. In this sense, mathematics offers a wide range of tools to accurately describe the core principles of materials science. Moreover, the massive growth in computational power in recent decades has enabled mathematicians to perform simulations that are far more complex and accurate than ever before. This development enables a variety of numerical experiments which are, in general, sufficiently economically sustainable to attract investment from the industry. This advantage becomes even more significant when combined with the growing prevalence of artificial intelligence, where *trial-and-error* methods often result in optimization processes of particular interest for the industry.
A large part of the numerical analysis applied to materials science is rooted in variational models. This is because we typically seek equilibrium states, often from a kinetic point of view, which generally correspond to optimal conditions with respect to certain parameters or variables. For this reason, an entire branch of mathematics, namely the Calculus of Variations, has been extensively developed and now supports a large and diverse global community.

Generally speaking, a variational model usually incorporates a functional $F : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, often referred as an *energy,* where X is a Banach space, whose elements are usually called *configurations.* Applications suggest that is interesting the study of the local minimisers of F. In the simplest setting, in which $X = \mathbb{R}$, this corresponds to the study of the first derivative F', provided that F is regular enough. Despite X, in general, might consist of mathematical objects of different nature (functions, measures, sets, etc.), it is possible to find an analogue definition of local minimisers.
The first concern is the existence of such points, namely if the problem

$$\min_{x \in X} F(x) \tag{1}$$

has or not a solution. Here, the topology chosen on X and the properties

satisfied by F play a crucial role. Indeed, if F is coercive and lower semi-continuous (with respect the same topology) then problem (1) has a solution. Since those two properties are relevant for our analysis, we recall them.
We say that F is *lower semi-continuous* if for every $x \in X$ and for every sequence $(x_n)_n \subset X$ such that $x_n \to x$, as $n \to \infty$, we have

$$F(x) \leqslant \liminf_{n\to\infty} F(x_n).$$

Moreover, we say that F is *coercive* if, for every $t \in \mathbb{R}$, the sub-level sets

$$\overline{\{F(x) \leqslant t\}}$$

are compact. Since X is a Banach space and usually the weak topology is used, we have, equivalently, that F is coercive with respect to the weak topology on X if and only if

$$\lim_{||x||\to+\infty} F(x) = +\infty,$$

where $||\cdot||$ is a norm on X. In general, if we drop those assumptions on F we can have a lack of existence of points of minimum. One way to get around the lack of coercivity, it is to enlarge the space of configurations, by finding a Banach Y such that $X \subset Y$. This is usually the way in which *singular* configurations are introduced to the problem. In case the lower semi-continuity of F is missing, we look for a lower semi-continuous functional $\overline{F} : Y \to \mathbb{R}$, with $\overline{F} \leqslant F$, for which

$$\min_{y\in Y} \overline{F}(y) \tag{2}$$

has a solution. Moreover we would like to see minimum points of $\overline{F}$ as limit of minimising sequences for F, as explained in what follows. A *minimising sequence* $(x_n)_n \subset X$ for F is a sequence such that

$$F(x_n) \to \inf_{x\in X} F(x),$$

as $n \to \infty$. Consider a minimising sequence $(x_n)_n \subset X$ such that $x_n \to x \in Y$. If $\overline{F}(x) = \min_{y\in Y} \overline{F}(y)$, we would like that

$$F(x_n) \to \overline{F}(x),$$

as $n \to \infty$. If $\overline{F}$ is coercive, lower semi-continuous (namely (2) has a solution) and its minimum points are limit of minimising sequences for F, we say that $\overline{F}$ is the *relaxation* of F. As we can notice, the relaxation depends on the topology chosen. Usually, since we work with Banach spaces, we choose the weak topology. It is possible to prove that if the dual of X is separable and F is coercive, with respect to the weak topology, $\overline{F}$ has a sequential characterisation, and we have

$$\overline{F}(x) \coloneqq \inf \left\{ \liminf_{n\to\infty} F(x_n) : x_n \rightharpoonup x \right\}.$$

This idea has been generalised with the notion of Γ-convergence, introduced by De Giorgi (see [30]). In this case, we address the problem of understanding the behaviour of sequences of functionals and their minimising sequences. Indeed, it is usual to investigate the behaviour of a family of functionals depending on one (or more) parameter $(F_n)_{n\in\mathbb{N}}$, such that $F_n : X \to \overline{\mathbb{R}}$, where X is a Banach space. Usually, we can write an energy only for *regular* configurations and therefore we may have many information on F_n, but very little is known when we consider the limit of regular configurations, or if the sequence $(F_n)_{n\in\mathbb{N}}$ is converging to a functional in some sense. In general, if $F_n \to F$, as $n \to \infty$, for instance pointwise, there is no guarantee that

$$\inf_{x\in X} F_n(x_n) \to \min_{x\in X} F(x). \tag{3}$$

The Γ-convergence has been developed specifically to address this issue. With proper hypotheses on X and on the sequence $(F_n)_{n\in\mathbb{N}}$ it is possible to give a sequential characterisation of the Γ-convergence with respect to the weak (or weak*) topology. Indeed we have that $F_n \xrightarrow{\Gamma} F$ if and only if the two following inequalities hold:

(*i*) *Liminf inequality.* For every $x \in X$ and for every sequence $(x_n)_n \subset X$ converging to x in X we have

$$F(x) \leqslant \liminf_{n\to\infty} F_n(x_n);$$

(*ii*) *Limsup inequality.* For every $x \in X$ there is a sequence $(x_n)_n \subset X$ converging to x in X such that

$$\limsup_{n\to\infty} F_n(x_n) \leqslant F(x).$$

Again, in the proper setting, if $F_n \xrightarrow{\Gamma} F$ the we ensure the validity of (3). Note that the liminf and limsup inequalities hold for the relaxation, when $F_n = F$, for every $n \in \mathbb{N}$. In particular, form those two inequalities, we get that for every $x \in X$ there is a sequence $(x_n)_n \subset X$ such that

$$\lim_{n\to\infty} F_n(x_n) = F(x).$$

Such a sequence is called a *recovery sequence* for F at x.

This dissertation fits within this broader framework and focuses specifically on crystal growth and magnetism. Although the applications of these two phenomena differ, they share a common variational foundation. About crystal growth, we focus specifically on epitaxial growth. In Chapter (1), we model a thin film that grows on a substrate by using an energy defined on regular configurations. Then, prove a relaxation result in which the relaxed energy is describing vertical cracks into the thin film. In Chapter (2) we made an approximation of the relaxed energy, known as a phase-field formulation. Such an approximation allows numerical experiments on our study. Overall, the main difficulty is given by the approximation of the fractures inside the crystal, for which new techniques are developed.
Chapter 3 investigates the change of magnetisation on a domain with an extreme geometry and how the shape of the domain affects the behaviour of local minimisers of an energy. A crucial part of our analysis is given by a classification of where the transition happens (depending on several rescaled parameters) and the identification of the competitors that better adapts to the geometry of the domain. Moreover, those competitors, asymptotically, describe where the transition happens, by giving a quantitative estimate of the energy.

II Epitaxial growth

The first line of research, explored in Chapters 1 and 2, focuses on crystal growth. This topic is highly relevant in applications, particularly due to the increasing importance of rare-earth elements, which play a central role in the production of superconductors, magnets, alloys, catalysts, and

optical fibres.
The primary industrial goal of crystal growth is to produce materials with specific micro- or macro-structures. One of the most prominent examples is that of *thin films*, whose production and significance have grown exponentially due to the high demand from various sectors, such as electronic devices (e.g., memory or energy storage units), energy production (e.g., batteries or solar panels), and surface coatings (e.g., graphene, medical, and pharmacological applications).
In those Chapters, we investigate the case in which a crystalline material is deposited layer by layer onto a fixed crystalline substrate. If the atoms at the interface of the substrate align with the natural lattice positions of the deposited thin film, this process is known as *epitaxial* crystal growth.

Practitioners developed several techniques to grow crystals over a substrate. Vapour deposition techniques are among the most important and implemented: the substrate is immersed in a vapour, and mass transfer from the latter to the former is responsible for the growth of the crystal. In order for the crystal to grow, two conditions need to be satisfied: the vapour has to be saturated, and the substrate is kept at a significantly lower temperature than the vapour. The former ensures attachment of vapour atoms on the substrate, while the latter the quick thermalisation of deposited atoms. In particular, this implies that the entropic free energy is reduced after attachment.
In order to grow a crystal, attached atoms, called *adatoms*, need to have sufficient energy to move from the landing location to a position of equilibrium. This depends on the type of materials used in the vapour and for the substrate. Surface diffusion of adatoms is therefore the mechanism used by thin films to grow as a crystal.
The dynamics of the crystal growth process is extremely complicated, and it is influenced by many factors. In particular, the ratio between the tendency of the adatoms to stick to the substrate and their tendency to diffuse is a key factor influencing the growth behaviour.
Three modes of growth are defined based on this ratio: the *Frank-van der Merwe growth mode*, where diffusion is stronger and thus the crystal grows layer by layer, the *Volmer-Weber growth mode*, where diffusion is weaker, and therefore adatoms tend to form islands on the substrate, and an intermediate one, the *Stranski-Krastanov growth mode*, where the

first monolayers of the film behave like in a Frank-van der Merwe growth mode, while after a certain threshold, it starts forming islands. Here we consider the latter case.
In the epitaxial Stranski-Krastanov growth mode, it is observed that, after a few monolayers of material are deposited, the film accumulates too much elastic energy that it is no longer energetically convenient for atoms of the film to stick to the crystalline structure of the substrate. Thus, relaxation processes are employed in order to reduce the total energy of the system. The most important ones are surface corrugation and defect formation. These are known in the literature as stress driven rearrangement instabilities (see [45]). The former is responsible for non-flat surfaces as well as for the appearance of islands (agglomerates of atoms, also called *quantum dots*) on the surface. With the latter, instead, the film introduces singularities in its crystalline structure, such as cracks and dislocations. It is crucial to be able to control this complex process in such a way to reduce impurities as much as possible, or at least to be able to quantify them.
For a complete treatment of the subject we refer to [41] and the pioneering work [64] by Spencer and Tersoff.

Sharp models

In general, we refer to *sharp model* to identify a mathematical model that incorporates a variable or an expression that is a sharp interface (for instance the boundary of a set). Those models are known for being extremely precise when describing a phenomenon but with the downside of being hard to implement numerically. Sharp models on epitaxial growth have a vast scientific literature, as explained in what follows. The work by Fonseca, Fusco, Leoni and Morini [37], in which epitaxial growth is modelled in the two-dimensional case and the free profile of the thin film has the constraint to be a graph of a function. In this section we briefly explain the main ideas studied in [37]. This class of functions will allow us to see fracture as limiting objects of sequences of Lipschitz functions (see Figure 1). In mathematical terms, let Ω be the sub-graph of a Lipschitz function $h : (a, b) \to \mathbb{R}$, defined as

$$\Omega \coloneqq \{(x, y) \in \mathbb{R}^2 \,:\, x \in (a, b),\, y < h(x)\},$$

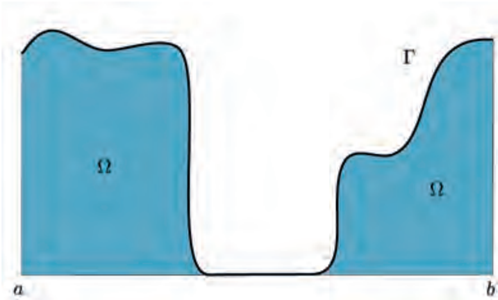

Figure 1: A regular configuration.

which represents the region occupied by the thin film, and let Γ be the sharp interface of the film, namely

$$\Gamma := \big\{ \big(x, h(x)\big) \ : \ x \in (a,b) \big\}.$$

Inside the bulk we consider a displacement variable, which describes the atomic internal structure of the film. The task of such a variable is to take into account all the possible rearrangements of the atoms inside the thin film. A different disposition of the atomic structure could lead to a different kind of crystal. Therefore we would like to be able to implement this feature to our model. The displacement is a function $v \in H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$, where the functional space is the Sobolev space

$$H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^2) := \{ v \in L^2_{\mathrm{loc}}(\Omega;\mathbb{R}^2) : \nabla v \in L^2_{\mathrm{loc}}(\Omega;\mathbb{R}^{2\times 2}) \},$$

where by ∇v we mean the weak gradient of v (see [13] for a complete description of Sobolev spaces). The functional considered in [37] is $F : X_{\mathrm{Lip}} \to \mathbb{R}$, defined as

$$F(v,\Omega) := \int_\Omega W\big(E(v) - E_0(y)\big)\ \mathrm{d}\mathbf{x} + \int_\Gamma \varphi_0(y)\ \mathrm{d}\mathcal{H}^1, \tag{4}$$

where $\mathbf{x} = (x,y) \in \mathbb{R}^2$. Here, $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure (in this case $n = 1$) and the admissible class of regular configurations for F is given by

$$X_{\mathrm{Lip}} := \{ (v,\Omega) \ : \ v \in H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^2),\ \Omega \text{ is the sub-graph of a}$$

$$\text{Lipschitz function } h : (a,b) \to \mathbb{R}\}. \tag{5}$$

Moreover, $\varphi_0 : \Gamma \to \mathbb{R}$ is given by

$$\varphi_0(y) := \begin{cases} \sigma_c & \text{if } y > 0, \\ \sigma_s & \text{if } y = 0, \end{cases}$$

for some constants $\sigma_c, \sigma_s > 0$. The surface energy taken into account is the length of the graph, up to the two constants σ_c and σ_s that, in the process of minimisation, tell us whether it is convenient to have the substrate exposed or not (wetting or non-wetting regimes). In particular if the constant relative to the substrate is σ_s and $\sigma_s < \sigma_c$, then energetically it is more convenient to leave the substrate exposed and favour the island formation (non-wetting regime). If $\sigma_s > \sigma_c$, covering the substrate with a thin crystal layer is energetically more favourable than leaving it exposed. We remark that this mechanism is at the base of the *island formation*. About this topic, we refer to the numerical simulation shown in [9], which is also the first mathematical paper on epitaxial growth.
The first term of (4) represents the stored elastic energy, which in the linearized setting is given by a quadratic form $W : \mathbb{R}^{2\times 2} \to \mathbb{R}$ defined by the elasticity tensor $\mathbb{C}$, namely

$$W(A) := \frac{1}{2} A \cdot \mathbb{C}[A], \tag{6}$$

for every $A \in \mathbb{R}^{2\times 2}$. The expression in (6) comes from a Taylor expansion around the starting configuration E_0 of the elastic energy W. First, $E_0 : \mathbb{R} \to \mathbb{R}^{2\times 2}$, represents the starting configuration of the substrate and the film, in which their lattices are perfectly aligned, and it is defined as

$$E_0(y) := \begin{cases} te_1 \otimes e_1 & \text{if } y \geqslant 0, \\ 0 & \text{if } y < 0, \end{cases}$$

where $t > 0$ is a constant depending on the lattice of the substrate, $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$ and, by definition,

$$e_1 \otimes e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In general, the strain tensor $E : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is defined as

$$E \coloneqq \frac{1}{2}(A^\top A - I). \tag{7}$$

An infinitesimal movement is a function $s : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $s(\mathbf{x}) = \mathbf{x} + v(\mathbf{x})$, where $v \in H^1_{\text{loc}}(\Omega; \mathbb{R}^2)$ is the displacement, for which

$$\nabla s(\mathbf{x}) = I + \nabla v(\mathbf{x}),$$

and $I \in \mathbb{R}^{2\times 2}$ is the identity matrix. Note that ∇s is a symmetric matrix as ∇v is symmetric as well (since $v \in H^1_{\text{loc}}(\Omega; \mathbb{R}^2)$). If we choose $A = I + \nabla v$ in (7), we obtain

$$E(v) = \frac{1}{2}\Big(\nabla v + \nabla^\top v + \nabla^\top v \cdot \nabla v\Big).$$

We assume that we are in the framework of small deformations. This hypothesis, that is rephrased as $||\nabla v||_{L^2} \ll 1$, allows us to simplify the above formula

$$E(v) = \frac{1}{2}\Big(\nabla v + \nabla^\top v + \nabla^\top v \cdot \nabla v\Big) \approx \frac{1}{2}\Big(\nabla v + \nabla^\top v\Big).$$

Therefore, by a Taylor expansion, we get

$$\begin{aligned} W(E(v)) = W(E_0(y)) &+ DW(E_0(y))[E(v) - E_0(y)] \\ &+ \frac{1}{2}D^2W(E_0(y))[E(v) - E_0(y), E(v) - E_0(y)] \\ &+ o(||E(v) - E_0(y)||^2). \end{aligned}$$

Assuming that $W(E_0(y)) = 0$ and that E_0 is an equilibrium point of the elastic energy, namely $DW(E_0(y)) = 0$, we are left with

$$W(E(v)) \approx \frac{1}{2}D^2W(E_0(y))[E(v) - E_0(y), E(v) - E_0(y)].$$

Thus, we define the elasticity tensor as the Hessian of W at E_0, that is

$$\mathbb{C} \coloneqq W(E_0(y))$$

and we obtain the linearised elastic energy (6).
In conclusion, it is convenient to our study to define the strain tensor E in (7) as a function of the displacement, that is $E : H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^2) \to \mathbb{R}^{2\times 2}$ which corresponds to the symmetric gradient

$$E(v) = \frac{1}{2}(\nabla v + \nabla^\top v).$$

The use of the symmetric gradient instead of the full gradient is given by the fact that its anti-symmetric part describes an infinitesimal rotation. In linear elasticity, it is assumed that the elastic energy W is invariant under rotations, namely

$$W(RA) = R(A),$$

for every anti-symmetric matrix $R \in \mathbb{R}^{2\times 2}$ and every $A \in \mathbb{R}^{2\times 2}$. Moreover, it is possible to prove (by using again a Taylor expansion) that

$$D^2W(E_0(y))[R,\cdot] = 0,$$

for every anti-symmetric matrix $R \in \mathbb{R}^{2\times 2}$.

Phase-field models

In order to better deal with the numerical implementations of an energy that presents a surface term like in (4), phase-field approximations of sharp energies have an extremely vast literature. In general, we refer to the term *phase-field* to indicate a mathematical model that implements variables which are only function regular enough to be used in numerics. A perfect example in this direction are Sobolev spaces.
In what follows we present some classical models as well as phase-field approximations closer to our topic. In this section, w plays the role of the phase-field variable, namely $w \in H^1(A)$ and $0 \leqslant w \leqslant 1$, where $A \subset \mathbb{R}^n$ and n depend on the problem we are referring to.
In the seminal paper by Ambrosio and Tortorelli [2], the authors introduce the functional named after them. This energy is defined on a bounded domain $\Omega \subset \mathbb{R}^n$, a given $g \in L^\infty(\Omega)$ and $v, w \in H^1(\Omega)$, which satisfy $(v, w) = (g, 1)$ on $\partial\Omega$. We define $AT_\varepsilon : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ as

$$AT_\varepsilon(v,w) := \int_\Omega (o(\varepsilon) + w^2)\,|\nabla v|^2 \,\mathrm{d}\mathbf{x}$$

$$+\frac{1}{2}\int_\Omega \Big(\varepsilon\,|\nabla w|^2 + \frac{1}{4\varepsilon}(w-1)^2\Big)\;\mathrm{d}\mathbf{x}. \qquad (8)$$

We remark that the definition of AT_ε does not coincide with the original one given by Ambrosio and Tortorelli. However, the above modern formulation of AT_ε is made in such a way that is easier to be dealt numerically.
AT_ε has been developed to be the phase-field formulation for the Mumford-Shah functional $MS : \mathrm{SBV}^2(\Omega) \to \mathbb{R}$, defined as

$$MS(v) = \int_\Omega |\nabla v|^2 \;\mathrm{d}\mathbf{x} + \mathcal{H}^{n-1}(\partial\Omega \cap \{v \neq g\}) + \mathcal{H}^{n-1}(\Gamma_v^j),$$

where Γ_v^j is the jump set of v, defined as

$$\Gamma_v^j \coloneqq \{(x,y) \in \mathbb{R}^2 \,:\, x \in (a,b),\, v^-(x) \leqslant y < v^+(x)\}.$$

The functional space SBV^2 is the space of special functions of bounded variations which are also in L^2 (we refer to [1] for more details on the topic). With respect to a suitable topology, we have that AT_ε Γ-converges to MS. For more details, we refer to [61] and to the related image segmentation problem (see [1]).

Closer to our topic, a phase-field model about epitaxial growth is present in the paper by Bonnetier and Chambolle [9]. The authors studied an approximation for the functional (4). Their model includes the two constants $\sigma_c, \sigma_s > 0$, already introduced above, that describe the wetting or non-wetting regimes. The energy to be approximated, in the same setting of (4), is $BC : X \to \mathbb{R}$, defined as

$$BC(v,\Omega) \coloneqq \int_\Omega W\big(E(v) - E_0(y)\big)\;\mathrm{d}\mathbf{x} + (\sigma_s \wedge \sigma_c)\mathcal{H}^1(\Gamma \cap \{y = 0\})$$
$$+\,\sigma_c\Big(\mathcal{H}^1(\Gamma \cap \{y > 0\}) + 2\sum_{\mathbf{x}\in\Gamma^c}(h^-(x) - h(x))\Big),$$

where $\sigma_s \wedge \sigma_c \coloneqq \min\{\sigma_s, \sigma_c\}$. If we consider the vertical strip

$$Q \coloneqq (a,b)\times\mathbb{R} \quad \text{and} \quad Q^+ = Q \cap \{y \geqslant 0\},$$

the phase-field approximation of BC is given by $BC_\varepsilon : H^1(Q)\times H^1(Q^+) \to \mathbb{R}$, defined as

$$BC_\varepsilon(v,w) := \int_{Q^+} (w + o(\varepsilon))W\big(E(v) - E_0(y)\big)\ \mathrm{d}\mathbf{x} \\ + 2\sigma_c\Big[\int_{Q^+}\Big(\frac{4\varepsilon}{\pi^2}|\nabla w|^2 + \frac{1}{\varepsilon}w(1-w)\Big)\ \mathrm{d}\mathbf{x}\Big]. \quad (9)$$

The main result contained in [9] is the Γ-convergence of BC_ε to BC.

Another phase-field approximation model, which also takes into account the adatom density, is proposed by Caroccia and Cristoferi in [17]. In this paper the authors prove a general Γ-convergence for functionals defined on the set of positive Radon measures $\mathcal{M}^+(\Omega)$. Such a result is immediately applied to a Modica-Mortola type of Theorem and we resume it in what follows. We define the phase-field energy $CC_\varepsilon : H^1(\Omega)\times L^1(\mathbb{R}^n) \to \mathbb{R}$ as

$$CC_\varepsilon(w,u) := \frac{1}{\sigma}\int_\Omega \Big(\varepsilon\,|\nabla w|^2 + \frac{1}{\varepsilon}P(w)\Big)\psi(u)\ \mathrm{d}\mathbf{x},$$

where $P : \mathbb{R} \to \mathbb{R}^+$ is a double-well potential,

$$\sigma := 2\int_0^1 \sqrt{P(t)}\ \mathrm{d}t$$

and $\psi : \mathbb{R} \to (0, +\infty]$ is a Borel function. The functional CC_ε is a variation of the classical Modica-Mortola functional (see [56]) and encodes the adatom variable $u \in L^1(\mathbb{R}^n)$. The authors prove that CC_ε Γ-converges to a functional $CC : \mathcal{O}\times\mathcal{M}(\Omega) \to \mathbb{R}$, given by

$$CC(E,\mu) := \int_{\partial^* E} \widetilde{\psi}(u)\ \mathrm{d}\mathcal{H}^{n-1} + \theta\mu^s(\Omega),$$

where $\mathcal{O}$ is the set of open and bounded subsets of $\mathbb{R}^n$, $\mu = u\mathcal{H}^{n-1}\llcorner\partial^* E + \mu^s$, where μ^s is the singular part of μ with respect to $\mathcal{H}^{n-1}\llcorner\partial^* E$ and $\partial^* E$ denotes the reduced boundary of E (see [54]). Moreover $\widetilde{\psi}$ is the convex and sub-additive envelope of ψ (see Definition 11) and $\theta > 0$ is the recession coefficient (see Definition 14).

We also mention the work by Conti, Focardi and Iurlano [25] (and later in [26]) in which the authors use an Ambrosio-Tortorelli phase-field formulation to model cohesive fractures. The work [36] by Forcardi proposes a phase-field formulation in the setting of Generalised Special functions with Bounded Variation ($GSBD$).

We notice that all the above phase-field models presented in this introduction display two phases. In BC_ε, CC_ε, those two phases are entirely encoded by the two wells of the potential. In this sense, it might seem that AT_ε has only one phase as the potential has one well at 1. However, the second phase of AT_ε is 0, since we expect that the phase-field variable w vanishes on the jump set of v. This difference is further remarked by the fact that in BC_ε, CC_ε the set Ω plays the role of a variable.

III Outline of the results on epitaxial crystal growth

In this section, we present the main results achieved in Chapters 1 and 2.

Review of previous works

From the mathematical point of view, several investigations have been carried out, focusing on different aspects of the crystal growth process. There are both discrete models, and continuum ones. Here we focus on these latter. In particular, the already mentioned work [9] by Bonnetier and Chambolle laid the foundations for rigorous mathematical investigations of stable equilibrium configurations of epitaxially strained elastic thin films in the linear elastic regime. We discussed the work [37], by Fonseca, Fusco, Leoni, and Morini proved a similar result by using an independent strategy, and also investigated the regularity of configurations locally minimizing the energy.
Questions about the stability of the flat profile were investigate by Fusco and Morini in [43] for the case of linear elasticity, and in [8] by Bonacini

in the non-linear regime. Moreover, in [7], Bonacini considered the same question for the case where surface energy is anisotropic, showing, surprisingly, that the flat interface is always stable. It was not until 2019, with the work [27] by Crismale and Friedrich that the three dimensional case was considered. Indeed, despite the existence of investigations for similar functionals in higher dimension (see the work [23] by Chambolle and Solci, and [12] by Braides, Chambolle, and Solci for the study of material void) were available, all of them considered elastic energies depending on the full gradient of the displacement. On the other hand, it is known that physically compatible models for elasticity must depend on the symmetrized gradient. The reason for such a time gap between the two and the three dimensional case was technical: it was not clear how to get compactness of a sequence of configurations with uniformly bounded energy. This required the introduction of a new functional space: GSBD, the space of Generalized Functions of Bounded Deformation, designed in the work [31] by Dal Maso in 2014 specifically designed to address this issue.

We now describe some of the main results achieved in [37]. Since the functional F in (4) is neither lower semi-continuous nor coercive with respect to the natural topology given by the model, we have that the problem

$$\min_{X_{\mathrm{Lip}}} F$$

might not have a solution. Therefore, according to the general theory introduced in Section I, we look for the relaxation of F, also by enlarging the admissible configuration space X_{Lip}. This process allows us to identify vertical fractures inside the thin film as limits of regular configurations. Physically, this is justified by the fact that when excessive stress accumulates in a crystal, it becomes energetically favourable to form a crack and thereby reduce the internal structural tension.
First, since sharp energies like in (4) are hard to implement numerically, the authors define a regularised energy, depending on a parameter δ, as

$$F_\delta(v,\Omega) := \int_\Omega W\big(E(v) - E_\delta(y)\big)\, \mathrm{d}\mathbf{x} + \int_\Gamma \varphi_\delta(y)\, \mathrm{d}\mathcal{H}^1,$$

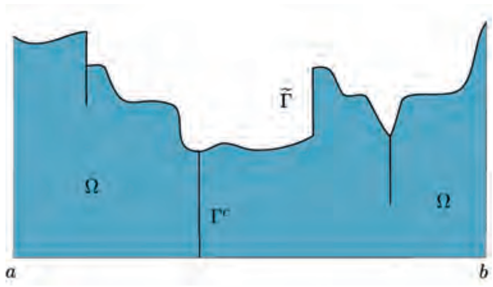

Figure 2: A possible limiting configuration.

for a suitable regularised surface energy density φ_δ and starting configuration E_δ (we refer to [37] for a precise formulation). The first result achieved is a mass constrained relaxation theorem. First, the class of admissible configuration is enlarged and functions of bounded variation are used to describe stress-free configurations (see Figure 2). We say that $h \in L^1(a,b)$ has *bounded variation* in (a,b) if there exists a Radon measure μ such that

$$\int_a^b h\varphi' \,\mathrm{d}x = -\int_{(a,b)} \varphi \,\mathrm{d}\mu,$$

for all $\varphi \in \mathcal{C}^1_c(a,b)$. In this case, we write $h \in \mathrm{BV}(a,b)$, and we denote the measure μ by $\mathrm{D}h$.
Consider the class

$$X \coloneqq \{(v,\Omega) \;:\; v \in H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^2),\ \Omega \text{ is the sub-graph of a lower-semi continuous function } h \in \mathrm{BV}(a,b)\}$$

and define $\overline{F}_\delta : X \to \mathbb{R}$ as

$$\overline{F}_\delta(v,\Omega) \coloneqq \int_\Omega W\big(E(v) - E_\delta(y)\big)\,\mathrm{d}\mathbf{x} + \int_{\widetilde{\Gamma}} \varphi_\delta(y) \;\; \mathrm{d}\mathcal{H}^1 + 2\int_{\Gamma^c} \varphi_\delta(y) \;\; \mathrm{d}\mathcal{H}^1. \tag{10}$$

Here, Γ^c is the cut part of h, given by

$$\Gamma^c := \{(x,y) \in \mathbb{R}^2 \,|\, x \in (a,b),\, h(x) \leqslant y < h^-(x)\},$$

where

$$h^-(x) = \liminf_{y\to x} h(y), \qquad h^+(x) = \limsup_{y\to x} h(y).$$

(see Definition 6) and $\widetilde{\Gamma} = \Gamma \setminus \Gamma^c$ (see Definition 6).
In [37, Theorem 2.8] the authors proved that $\overline{F}_\delta$ is the relaxation of F_δ with respect to a suitable topology made in such a way that fractures and jumps of the thin films are seen as limiting configurations of sequences in X_{Lip}. Moreover, since the relaxed energy is lower semi-continuous and coercive, we have that the problem

$$\min_X \overline{F}_\delta$$

has as solution.
We remark that in the energy in (10), the contribution given by the cut part of h is counted twice. Although it might seem that creating a fracture is unfavourable energetically, it actually can decrease the bulk energy. The interplay between the bulk and surface terms makes the study of the Γ-convergence of $\overline{F}_\delta$ particularly interesting from a mathematical standpoint, as well as the regularity of local minimisers of the Γ-limit (see [37, Theorem 2.9 and Section 3]).
The work contained in [28] continues this line of research by studying a generalisation that includes the presence of adatoms in the energy (4) and its relaxation (see Chapter 1 for a full description).

Novelty of the contribution

What all of the above continuum models are neglecting is the role of adatoms in the creation of equilibrium stable interfaces. The importance of considering their effect was made clear by Specer and Tersoff in [64], where the authors highlighted that considering the effect of adatoms, and in particular of surface segregation of several species of deposited material, affect the equilibrium configurations predicted by the model,

and hopefully provide a more accurate description of those observed in experiments.
This was made even clearer in the seminal paper [42] by Fried and Gurtin. The manuscript unified several ad hoc investigations that focused on specific aspects on crystal growth or used specific assumptions to derive the model. In particular, it was noted that considering adatoms will, on the one hand, add a new variable to the problem, while, on the other hand, will make the evolution equations parabolic. Note that this is a huge mathematical advantage, since in [38] and in [39], the authors had to add an extra term to the energy (that nevertheless has some physical interpretation) to regularize the non-parabolic evolution equations obtained from the model that does not take into consideration adatoms.
Following this direction of investigation, in [18], Cristoferi, together with Caroccia and Dietrich, started the study of a variational characterization of the evolution equations derived by Fried and Gurtin. In that paper, the authors considered a variational model describing the equilibrium shape of a crystal, where the elastic energy is neglected, and without the constraint of growing as the graph of a function. From the energy for regular configurations, a natural topology was identified, and a representation formula for the relaxed energy was obtained. The result highlighted the interplay between oscillations of crystal surfaces and changes in adatom density in order to lower the total energy. The result obtained in that paper was different from previous investigations by Bouchitté (see [10]), Bouchitté and Buttazzo (see [11]), and Buttazzo and Freddi (see [15]), due to the choice of the topology.
In a subsequent paper (see [17]), a phase field model was considered in a more general setting, to pave the way towards the analysis of the convergence of the gradient flows.

In Chapter 1, we describe the main results achieved in [28]. We introduce a new variable on the free surface of the film Γ that describes the adatom density, seen as a Radon measure $\mu = u\mathcal{H}^1 \llcorner \Gamma$. The functional taken into account is of the form $\mathcal{H} : \mathcal{A}_r \to \mathbb{R}$ and is defined as

$$\mathcal{H}(\Omega, v, \mu) := \int_\Omega W\big(E(v) - E_0(y)\big)\, \mathrm{d}\mathbf{x} + \int_\Gamma \psi(u)\, \mathrm{d}\mathcal{H}^1, \tag{11}$$

where $\psi : \mathbb{R} \to (0, \infty)$ is a Borel function with $\inf_{s\in\mathbb{R}} \psi(s) > 0$ and $\mathcal{A}_r$

is a suitable admissible class of regular of configurations (see Definition 16) that describe the thin film via Lipschitz functions. The surface term in (11) penalises not only the length of the graph, but also an elevate adatom density. For a precise formulation and additional details, we refer to Section 2.1.
We address the problem of finding the relaxed functional of (11). In order to observe vertical fractures inside the film, one way to see them is by using functions of bounded variation and a suitable topology in the relaxation. The topology is chosen in such a way that fractures are the limit of regular configurations (see Definition 20). One can expect a similar result on the interface as in [18], in which the surface energy has the form

$$\int_\Gamma \widetilde{\psi}(u)\ \mathrm{d}\mathcal{H}^1,$$

where $\widetilde{\psi}$ is the convex sub-additive envelope of ψ, defined as

$$\widetilde{\psi}(s) := \sup\{f(s) \mid f : [0,+\infty) \to \mathbb{R} \textit{ is convex, sub-additive } \textit{and } f \leqslant \psi\}.$$

However, we prove that on vertical fractures we have a different behaviour. Indeed, we have that the interface energy is relaxed in to a term of the form

$$\int_\Gamma \psi^c(u)\ \mathrm{d}\mathcal{H}^1.$$

Here, ψ^c (see Definition 12) is defined as

$$\psi^c(s) := \min\{\widetilde{\psi}(r) + \widetilde{\psi}(t) \,|\, s = r + t\}.$$

Heuristically, since we are approximating a fracture with regular functions, we expect that a cut is the limit of the contributions given by functions on the left and on the right of it. In that sense ψ^c detects the best possible way to do so.
We prove that the functional $\mathcal{G} : \mathcal{A} \to \mathbb{R}$ is the relaxation of $\mathcal{H}$, and is defined as

$$\mathcal{G}(\Omega, v, \mu) := \int_\Omega W\big(E(v) - E_0(y)\big)\ \mathrm{d}\mathbf{x} + \int_{\widetilde{\Gamma}} \widetilde{\psi}(u)\ \mathrm{d}\mathcal{H}^1$$

$$+\int_{\Gamma^c}\psi^c(u)\,\mathrm{d}\mathcal{H}^1+\theta\mu^s(\mathbb{R}^2). \tag{12}$$

Here, $\mathcal{A}$ is the class of admissible configurations (see Definition 15) which describe the thin film with the use of functions of bounded variation. Moreover, $\theta > 0$ is the recession coefficient (see Definition 14) and μ^s is the singular part of μ with respect to $\mathcal{H}^1 \llcorner \Gamma$. Both proofs of the liminf and limsup inequalities are challenging and technical. New techniques needed to be developed in order to deal with vertical cracks and the surface term with ψ^c. In particular, the liminf inequality relies on an argument based on geometric measure theory. The limsup inequality is obtained by an approximation argument and by a careful adaptation (and generalisation) of the *wriggling process* introduced in [18] and [17]. Briefly, given a Lipschitz function $f : (a,b) \to [0,+\infty)$, and $r > 1$, we can prove that there exists a sequence of Lipschitz functions $f_n : (a,b) \to [0,+\infty)$ with $\mathcal{H}^1 \llcorner \Gamma_{f_n} \overset{*}{\rightharpoonup} r\mathcal{H}^1 \llcorner \Gamma_f$ as $n \to \infty$, such that

$$\mathcal{H}^1(\Gamma_{f_n}) = r\mathcal{H}^1(\Gamma_f),$$

with fixed boundary conditions $f_n(a) = f(a)$, $f_n(b) = f(b)$, for each $n \in \mathbb{N}$, and satisfying other technical properties (such as the preservation of the mass constraints). What the above inequality is pinpointing is a quantitative lack of lower semi-continuity of the perimeter (see Proposition 3).

In Chapter 2, we explain the main results obtained in [35], where we develop a phase-field model for the sharp functional defined in (12). Our goal is to approximate (12) using only variables that are functions not defined on a sharp interface, that is, in our case, not defined on a one-dimensional domain in $\mathbb{R}^2$. Consider the two vertical strips

$$Q \coloneqq (a,b)\times\mathbb{R} \quad \text{and} \quad Q^+ \coloneqq Q\cap\{y>0\}.$$

We aim to approximate the set variable Ω in (12) with a phase-field sequence $(w_\varepsilon)_\varepsilon \subset H^1(Q^+)$ such that $0 \leqslant w_\varepsilon \leqslant 1$ and $w_\varepsilon \to \chi_\Omega$ in $L^1(Q^+)$, as $\varepsilon \to 0$. In the same way, the Radon measure $\mu = u\mathcal{H}^1 \llcorner \Gamma_h$ is seen as the limit (in the sense of the weak* convergence) of a sequence $(u_\varepsilon)_\varepsilon \subset L^1(Q)$.

To this end, we introduce the functional $\mathcal{G}_\varepsilon : \mathcal{A}_p \to \mathbb{R}$ defined as

$$\mathcal{G}_\varepsilon(w, v, u) := \int_{Q^+} (w(\mathbf{x}) + o(\varepsilon)) W\big(E(v) - E_0(y)\big) \, \mathrm{d}\mathbf{x} + \frac{1}{\sigma} \int_{Q^+} \Big[\varepsilon |\nabla w|^2 + \frac{1}{\varepsilon} P(w)\Big] \psi(u) \, \mathrm{d}\mathbf{x}, \tag{13}$$

where $\mathcal{A}_p$ is the admissible set of phase-field configurations (see Definition 22), which consists of functions in H^1 or L^1, P is a double-well potential and

$$\sigma := 2 \int_0^1 \sqrt{P(t)} \, \mathrm{d}t.$$

The main result of this work is proving that $\mathcal{G}_\varepsilon$ Γ-converges to $\mathcal{G}$, as $\varepsilon \to 0$, with respect to a suitable topology (see Definition 25, Theorems 14 and 16).
We remark that the role of $o(\varepsilon)$ in (13) and also in the functionals presented in this section is to ensure the compactness of minimising sequences.
The proof makes use of the strategies introduced in [9] and [17]. Moreover we show that the new techniques introduced in [28] proved to be solid enough to be adapted in a phase-field formulation.

IV Geometrically constrained walls

The second line of research, discussed in Chapter 3, is on *Magnetic domain walls*, which are regions in which the magnetisation of a material changes from one value to another one. The most familiar setting is that of a magnet. In a simple geometry, such as a cylinder, we can observe two distinct magnetizations at the bases. For clarity, we may assume that the two magnetization vectors are $(0, 0, \pm 1)$. It is intuitive to expect that the magnetization changes uniformly along the body of the magnet, and we can imagine the transition to be linear due to the shape of the domain. However, in the presence of extreme geometries, such as that of a dumbbell-shaped domain (see Figure 3.1), the magnetic wall is more

likely to be found in or around the neck; in this and similar geometry-driven situations, one usually speaks of *geometrically constrained walls*, to stress the fact that the domain shape plays a pivotal role in the localisation of the transition region of the magnetisation, for instance, when prescribing it in the bulky parts of the dumbbell.
The model that allows us to describe geometrically constrained walls features a sufficiently smooth potential which is minimal at the imposed values of the magnetisation in the bulk parts of the dumbbell, and a gradient term penalising transitions; the two are competing as soon as the values of the magnetisation in the bulks are not the same. To better explain, we consider a double well potential $W\colon \mathbb{R} \to [0, +\infty)$ of class $\mathcal{C}^2$ such that $W^{-1}(0) = \{\alpha, \beta\}$ for some $\alpha < \beta$ and

$$\lim_{|t|\to+\infty} W(t) = +\infty.$$

In this setting α and β represents the two state of the magnetisation preferred by the potential. We notice that we are assuming that the magnetisation $u : \mathbb{R}^3 \to \mathbb{R}$ is a scalar function. The general case, in which the magnetisation is a field $u : \mathbb{R}^3 \to \mathbb{R}^3$ is extremely hard to handle as it leads to a vectorial variational problem, which is still an open problem. We now describe the geometries that are of interest in the applications. We consider an infinitesimally small neck (see Figure 3), whose size is determined by three parameters $\varepsilon, \delta, \eta > 0$:

$$N_\varepsilon := \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 : |x| \leqslant \varepsilon, |y| < \delta, |z| < \eta\}, \tag{14}$$

with the understanding that all three of them vanish when $\varepsilon \to 0$, that is $\delta = \delta(\varepsilon) \to 0$ and $\eta = \eta(\varepsilon) \to 0$, as $\varepsilon \to 0$. We study a mathemati-

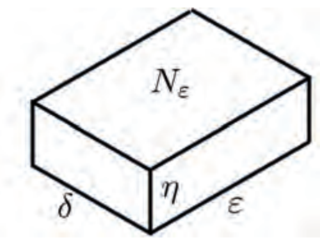

Figure 3: The neck N_ε

cal model to characterise magnetic domain walls in a three-dimensional dumbbell-shaped domain (see Figure 3.1). The two bulks are modelled by two bounded, connected, open sets $\Omega^\ell, \Omega^{\mathrm{r}} \subset \mathbb{R}^3$ with Lipschitz boundary such that

(*i*) the origin $(0,0,0)$ belongs to $\partial\Omega^\ell \cap \partial\Omega^{\mathrm{r}}$;
(*ii*) $\Omega^\ell \subset \{x < 0\}$ and $\Omega^{\mathrm{r}} \subset \{x > 0\}$;
(*iii*) there exists $r_0 > 0$ such that $(\partial\Omega^\ell) \cap B_{r_0}(0,0,0)$ and $(\partial\Omega^{\mathrm{r}}) \cap B_{r_0}(0,0,0)$ are contained in the plane $\{x = 0\}$, i.e., the bulks are flat and vertical near the origin, where the conjunction with the neck will be located.

We let $\varepsilon > 0$ and define the neck region as in (14), so that the dumbbell-shaped domain Ω_ε is defined as

$$\Omega_\varepsilon := \Omega^\ell_\varepsilon \cup N_\varepsilon \cup \Omega^{\mathrm{r}}_\varepsilon,$$

where $\Omega^\ell_\varepsilon = \Omega^\ell - (\varepsilon, 0, 0)$ and $\Omega^{\mathrm{r}}_\varepsilon = \Omega^{\mathrm{r}} + (\varepsilon, 0, 0)$, We notice that Ω_ε is a bounded, connected, open set with Lipschitz boundary. This geometry makes the x direction the preferred one, whereas the y- and z-direction can be interchanged upon a change of coordinates; this motivates the fact that we will use, throughout the work, the subscript ε alone as an indication of the smallness of the neck.
We study an energy of the form

$$\mathcal{F}(u, \Omega_\varepsilon) := \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} + \int_{\Omega_\varepsilon} W(u(\mathbf{x}))\, \mathrm{d}\mathbf{x}\,,$$

which resembles a Modica-Mortola type of functional, in which also the domain of integration varies with ε.

V Outline of the results on geometrically constrained walls

In this section we describe the main results achieved in Chapter 3.

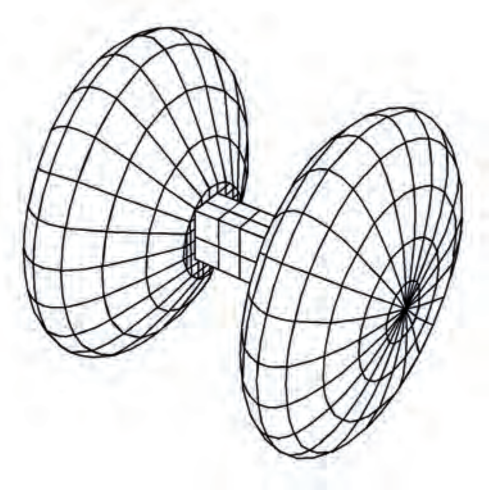

Figure 4: A pictorial representation of a typical domain of interest.

Review of previous works

We discuss two recent contributions in detail, since the results we obtain are related to them. In the work by Kohn and Slastikov [52], the problem is studied in the full three-dimensional setting, with the assumption that the geometry be axisymmetric: the dumbbell Ω_ϵ is a rotation body around the x axis, so that the shape parameters of the neck are essentially its length ε and its radius δ. By taking advantage of a scale-invariant Poincaré inequality for Sobolev functions and by reducing the problem to a one-dimensional variational one, the authors proved the existence of three possible regimes, according to the value of the limit $\lim_{\varepsilon\to 0} \delta/\varepsilon = \lambda \in [0, +\infty]$ and singled out a *thin neck* regime ($\lambda = 0$), a *normal neck* regime ($\lambda \in (0, +\infty)$), and a *thick neck* regime ($\lambda = +\infty$). In the first case the transition happens entirely inside the neck and is an affine function of the x-coordinate, in the second case the transition happens across the neck, partially inside and partially outside, depending on the value of λ, whereas in the third case the transition happens entirely outside the neck. These behaviours are found by studying the energy of particular competitors (essentially, an affine transition inside the neck and a harmonic transition in a spherical shell just outside the neck) and then rescaling the minimiser in the vicinity of the neck.

In the works by Morini and Slastikov [59, 60] the same problem was addressed in the case of a magnetic thin film, that is when the domain has the shape of a dumbbell, but it is two-dimensional, that is, in Bruno's setting in the limit as $h \to 0$. Mathematically speaking, the endeavour is more difficult on two accounts: the scale-invariant Poincaré inequality is not available in dimension two, and the problem loses its variational character. Methods that are typical from the study of PDE's were employed to construct suitable barriers to estimate the solutions. Moreover, due to the slow decay of the logarithm (the fundamental solution to Laplace's equation in two dimensions), sub-regimes became available in addition to the thin, normal, and thick neck regimes already analysed by Kohn and Slastikov: the sub-critical, critical, and supercritical thin neck regimes were found according to the value of the limit $\lim_{\varepsilon\to 0}(\delta|\ln\delta|)/\varepsilon \in [0,+\infty]$, displaying a richer variety. In the case of sub-regimes, the rescaling of the minimisers to study their asymptotic behaviour is not trivial; nonetheless, the authors managed to characterise the profiles as the unique solutions to certain PDE's where the boundary conditions track the expected asymptotic behaviour. Both in Kohn and Slastikov's and in Morini and Slastikov's papers the technique involves two steps: the first is to estimate the energy of minimisers to understand if the wall is located all inside, all outside, or across the neck; the second is to rescale the whole domain Ω_ε to an appropriate Ω_∞ in a way that either a variational problem or a PDE can be studied in Ω_∞, which brings to the attention that the boundary $\partial\Omega_\infty$ must be a set in which boundary conditions can be prescribed.

Novelty of the contribution

We are interested in understanding the asymptotic behaviour, as $\varepsilon \to 0$, of stable critical points (see Definition 1) of the energy

$$\mathcal{F}(u,\Omega_\varepsilon) \coloneqq \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} + \int_{\Omega_\varepsilon} W(u(\mathbf{x}))\,\mathrm{d}\mathbf{x}\,, \tag{15}$$

defined for $u \in H^1(\Omega_\varepsilon)$, where $\mathrm{d}\mathbf{x} = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ and

$$\begin{cases} W\colon \mathbb{R} \to [0,+\infty) \text{ is a function of class } \mathcal{C}^2 \text{ such that} \\ W^{-1}(0) = \{\alpha, \beta\} \text{ for some } \alpha < \beta \\ \text{and } \lim\limits_{|t|\to+\infty} W(t) = +\infty. \end{cases}$$

In (15), the function u represents a suitable quantity related to the magnetisation field defined on Ω_ε and the potential W favours the values $u(\mathbf{x}) = \alpha$ and $u(\mathbf{x}) = \beta$ for the magnetisation, corresponding to those to be imposed in the bulks. Here, the competition between the potential and the gradient terms is significantly influenced by the geometry of Ω_ε. The energy (15) was considered in [14] as a simplified model for studying the magnetisation inside a thin dumbbell domain under the assumption that the magnetic field is of the form

$$\mathbf{m}(\mathbf{x}) = (0, \cos(u(x)), \sin(u(x)))\,.$$

Despite this simplifying assumption, the mathematical analysis is rich enough to exhibit non-trivial behaviours of the magnetisation.
We now give the relevant definitions of critical points and isolated local minimiser for the functional $\mathcal{F}(\cdot, \Omega_\varepsilon)$ introduced in (15).

Definition 1. For $\varepsilon > 0$, let $u_\varepsilon \in H^1(\Omega_\varepsilon)$ be a critical point of $F(\cdot, \Omega_\varepsilon)$. We say that the family $(u_\varepsilon)_\varepsilon$ is an *admissible family of nearly locally constant critical points* if

(a) there exists $\bar{\varepsilon} > 0$ such that

$$\sup\left\{\|u_\varepsilon\|_\infty : \varepsilon \in (0, \bar{\varepsilon}]\right\} =: \overline{M} < +\infty;$$

(b) $\|u_\varepsilon - \alpha\|_{L^1(\Omega_\varepsilon^\ell)} \to 0$ and $\|u_\varepsilon - \beta\|_{L^1(\Omega_\varepsilon^{\mathrm{r}})} \to 0$ with $\alpha < \beta$, as $\varepsilon \to 0$.

Moreover, we say that $(u_\varepsilon)_\varepsilon \subset H^1(\Omega_\varepsilon)$ is an *admissible family of local minimisers* if it satisfies, additionally,

(c) there exist $\varepsilon_0 > 0$ and $\theta_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$ we have $\mathcal{F}(v, \Omega_\varepsilon) \geqslant \mathcal{F}(u_\varepsilon, \Omega_\varepsilon)$ for all $v \in H^1(\Omega_\varepsilon)$ such that $0 < \|v - u_\varepsilon\|_{L^1(\Omega_\varepsilon)} \leqslant \theta_0$.

Unlike [52], we do not assume axial symmetry of the domain and this results in a richer variety of regimes. In particular, we find that some of these regimes admit sub-regimes, as was discovered for magnetic thin films in [59, 60]. We discuss all the possible cases in the next section.

First of all, we note that, given the privileged role of the parameter ε, it is trivial to see that the roles of δ and η can be interchanged upon switching the coordinate axes y and z. The regimes investigated in [52] correspond to the cases where $\delta \sim \eta$. Therefore, here we limit ourselves to consider the other following regimes:

(*i*) *Super thin*: $\varepsilon \gg \delta \gg \eta$;

(*ii*) *Flat thin*: $\varepsilon \approx \delta \gg \eta$;

(*iii*) *Window thick*: $\delta \gg \eta \gg \varepsilon$;

(*iv*) *Narrow thick*: $\delta \gg \varepsilon \approx \eta$;

(*v*) *Letter-box*: $\delta \gg \varepsilon \gg \eta$.

In the Letter-box regime, in particular, we have the presence of sub-regimes, depending on the value of the following limit

$$\ell := \lim_{\varepsilon \to 0} \frac{\eta\, |\ln \eta|}{\varepsilon}. \tag{16}$$

Then we have the additional sub-regimes

(v') *Sub-critical Letter-box*: $\delta \gg \varepsilon \gg \eta$ and $\ell = 0$;

(v'') *Critical Letter-box*: $\delta \gg \varepsilon \gg \eta$ and $\ell \in (0, +\infty)$;

(v''') *Super-critical Letter box*: $\delta \gg \varepsilon \gg \eta$ and $\ell = +\infty$.

Depending on the regime, the transition will happen: completely inside, completely outside, or in both regions. To understand this, we reason as follows. For $\varepsilon > 0$ and $d < \min\{|\alpha|\,|\Omega^\ell|^{1/2}, |\beta|\,|\Omega^{\mathrm{r}}|^{1/2}\}$, we consider

$$B_\varepsilon := \{u \in H^1(\Omega_\varepsilon) : ||u - u_{0,\varepsilon}||_{L^2(\Omega_\varepsilon)} \leqslant d,\ \|u\|_{L^\infty(\Omega_\varepsilon)} < \infty\},$$

with $u_{0,\varepsilon}\colon \Omega_\varepsilon \to \mathbb{R}$ defined as

$$u_{0,\varepsilon}(\mathbf{x}) := \begin{cases} \alpha & \text{if } \mathbf{x} \in \Omega_\varepsilon^\ell\,, \\ \dfrac{\alpha+\beta}{2} & \text{if } \mathbf{x} \in N_\varepsilon\,, \\ \beta & \text{if } \mathbf{x} \in \Omega_\varepsilon^{\mathrm{r}}\,. \end{cases}$$

First of all, we expect the main part of the energy to be the Dirichlet integral. Then, we consider two harmonic functions that play the role of competitors for the minimisation problem

$$\min\{\mathcal{F}(v,\Omega_\varepsilon)\ : v \in B_\varepsilon\};$$

one where the transition from α to β happens inside the neck, and the other one where it happens only outside (and the competitor is constant inside the neck). We then compare their energies (whose computations will be carried out in Section 3.2) to get a guess of where the transition will occur.
The main achievements of Chapter 3 can be resumed as follows.

(A1) For all of the above-mentioned regimes, we identify where the transition happens. More precisely, we find sequences $(\varrho_\varepsilon)_\varepsilon$, with $\varrho_\varepsilon \to +\infty$, as $\varepsilon \to 0$, such that

$$\lim_{\varepsilon\to 0} \varrho_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon,\Omega_\varepsilon) =: \kappa \in (0,+\infty),$$
$$\lim_{\varepsilon\to 0} \varrho_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon,N_\varepsilon) =: \kappa_N \in [0,\kappa].$$

Their interpretation is the following: κ is the asymptotic energy in the whole domain, and κ_N that in the neck. Therefore, if $\kappa_N = \kappa$, we say that the transition happens entirely inside the neck, if $\kappa_N = 0$, the transition happens entirely outside the neck, while if $\kappa_N \in (0,\kappa)$, the transition happens both inside and outside the neck. In particular, we rigorously justify the expectations derived from the above heuristics.

(A2) We identify a proper rescaling of the independent variables that allows us to prove that such rescaled profile converges to a solution of a Dirichlet energy in a limiting domain with suitable boundary conditions.

VI Overview of the results

This section aims to review the results achieved and the technical difficulties encountered in this dissertation.
Starting with the study of epitaxial growth, in the first paper [28], the generalization of the wriggling process for the recovery sequence proved to be extremely challenging. Not only did the graph constraint on the free surface of the film introduce an additional layer of complexity, but its interaction with the two mass constraints also had to be handled with great care. Indeed, since the wriggling process relies on oscillations in the graph of a function, the mass constraints had to be adjusted at each step accordingly. This aspect became particularly delicate when approximating a function of bounded variation with a sequence of Lipschitz functions.
The phase-field model for epitaxial growth developed in [35] provides an investigation that demonstrates the techniques introduced in the previous paper are also suitable for describing a different type of convergence of functionals.
Both papers opened new lines of investigation, thanks to the adaptability of the proposed techniques to similar settings. Current directions of research include the case of non-linear elasticity, anisotropic surface energy, and the three-dimensional setting. It would be of mathematical interest to model fractures that are not vertical, in order to explore whether our study can be generalized in that direction.

The study on magnetic domain walls in [29] was motivated by the imbalance between the number of publications in physics and mathematics. Despite the extensive literature in the former field, mathematical investigations were lacking. The aim of our study was to clarify and classify the various regimes arising in the dumbbell geometry. The presence of three independently varying parameters significantly increased the number of possible rescalings. The choice of an appropriate rescaling was dictated not only by physical considerations, but also by the geometry of the competitors found in the bulk regions. In fact, due to the mismatch between all the parameters, we were able to identify an elliptical competitor that yielded a lower energy order. This insight was crucial, as it led us to

discover the existence of sub-regimes in the three-dimensional case—a behaviour previously thought to be typical only of the two-dimensional setting.
Future investigations will proceed in several directions. The most promising one concerns a dimension reduction problem. Now that we have a full characterization of both the two- and three-dimensional cases, we aim to develop a theory that connects them. More specifically, if the dumbbell shape is considered as a thin film, we are interested in studying the asymptotic behaviour of the various regimes as the film's thickness tends to zero. Our hypothesis is that the variational problems arising in the three-dimensional setting converge to the solutions of the PDE's corresponding to the two-dimensional case.

Chapter 1

A sharp model for epitaxial growth with adatoms

This chapter follows the study presented in [28]. We investigate epitaxial growth by considering the case in which a crystalline material is deposited on a substrate. In addition, its profile can be described by a function, and both the elastic energy of the film and the surface energy of adatoms are taken into account. The goal is to obtain a representation formula for the relaxed energy in the natural topology of the problem. In order to develop the main ideas needed for such an investigation, we focus on the two dimensional case. The main contribution of [28] is to show how the mechanism identified in [18], where oscillations of the profile interact with adatom concentration, plays a role in the case where the geometry of the configuration is constrained to be a graph. This might seem as an easier case than that treated in [18], where the profile of the crystal was free to grow in any direction. Nevertheless, the graph constraint poses several challenges that have to be tackled with the utmost care, in order to be properly overcome. Indeed, we prove that the relaxed energy differs from that of [18] exactly on vertical cracks of the deposited layer. In particular, we introduce a strategy to deal with oscillations and adatom concentration on vertical cracks, whose robustness will be tested in Chapter 2, where we investigate a phase-field approximation of the model.

1.1 The model

In this section we introduce the model that we will study. We consider the two dimensional case. This corresponds to three dimensional configurations that are constant in one direction. We work within the continuum theory of epitaxial growth. The main assumptions of the model are the following:

(i) The profile of the configurations of the thin film can be described as the graph of a function;

(ii) We neglect surface stress;

(iii) The exchange of atoms between the substrate and the deposited film is negligible;

(iv) The atoms of the substrate do not change position.

The free energy of a configuration is the sum of a bulk energy and a surface energy. The former is the elastic energy due to rearrangement of the atoms of the deposited film from a stress free configuration (atoms sitting in their natural lattice position) to another disposition. The latter, instead, stems from the net work needed to create an interface with a specific density of adatoms. We first prescribe the energy of regular configurations, and will then obtain that of more irregular configurations by relaxing the former.

We model the substrate as the set $\{(x, y) \in \mathbb{R}^2 \,:\, y \leqslant 0\}$. We consider a portion of the deposited film in a region $(a, b) \times \{y \geqslant 0\}$. To describe the free profile of the film, let $h : (a, b) \to \mathbb{R}$ be a non-negative Lipschitz function. Consider its graph

$$\Gamma_h \coloneqq \big\{ \big(x, h(x)\big) \,:\, x \in (a, b) \big\}, \tag{17}$$

and its subgraph (see Figure 1.2 on the left)

$$\Omega_h \coloneqq \{(x, y) \in \mathbb{R}^2 \,:\, x \in (a, b),\, y < h(x)\}. \tag{18}$$

The set $\Omega_h \cap \{y \geqslant 0\}$ represents the deposited film.
We first introduce the surface energy. The adatom density will be described by a positive function $u \in L^1(\mathcal{H}^1 \llcorner \Gamma_h)$. The surface energy corresponding to such an adatom density distribution will be

$$\int_{\Gamma_h} \psi\big(u(\mathbf{x})\big)\, \mathrm{d}\mathcal{H}^1(\mathbf{x}),$$

where $\mathbf{x} = (x, y) \in \mathbb{R}^2$, and $\psi : [0, +\infty) \to (0, +\infty)$ is a Borel function such that

$$\inf_{s \geqslant 0} \psi(s) > 0. \tag{19}$$

Note that such a requirement has the physical interpretation that no matter what the adatom density is, there is always an amount of energy needed to construct a profile.

We now discuss the elastic energy. For each macroscopic configuration Ω_h, there are several arrangements of atoms inside the thin film that produce that same profile. To each of these arrangements there is an elastic energy associated to: this energy will depend on the displacement between the actual position of each atom and its position in the natural crystal lattice. This displacement will be described by a function $v : \Omega_h \to \mathbb{R}^2$, and we assume it to be of class $H^1(\Omega_h; \mathbb{R}^2)$. The natural crystal configuration of the crystalline substrate and that of the deposited film are represented by a function $E_0 : \mathbb{R} \to \mathbb{R}^{2\times 2}$, defined as

$$E_0(y) \coloneqq \begin{cases} t e_1 \otimes e_1 & \text{if } y \geqslant 0, \\ 0 & \text{if } y < 0. \end{cases}$$

Here, $t > 0$ is a constant depending on the lattice of the substrate, and $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. The crystalline structure of the film and the substrate might be slightly different, but we assume their difference to be very small, namely $|t| \ll 1$. This assumption allows us to work in the framework of linearized elasticity. In particular, the relevant object needed to compute the elastic energy is the symmetric gradient of the displacement

$$E(v) \coloneqq \frac{1}{2}(\nabla v + \nabla^\top v),$$

where $\nabla^\top v$ is the transpose of the matrix ∇v. Note that $E(v)$ is zero if ∇v is zero for any anti-symmetric matrix (for instance, a rotation matrix).

Finally, we assume that the substrate and the film share similar elastic properties, so they are described by the same positive definite elasticity tensor $\mathbb{C}$. The elastic energy density will be given by a function $W : \mathbb{R}^{2\times 2} \to \mathbb{R}$ defined as

$$W(A) := \frac{1}{2} A \cdot \mathbb{C}[A] = \frac{1}{2} \sum_{i,j,m,n=1}^{2} c_{ijnm} a_{ij} a_{nm},$$

for a 2×2 matrix $A = (a_{ij})_{i,j=1}^2$. In addition, we ask that W is a positive quadratic form, namely

$$W(A) > 0, \tag{20}$$

for all symmetric matrices $A \neq 0$.
The elastic energy will then be

$$\int_{\Omega_h} W\big(E(v(\mathbf{x})) - E_0(y)\big)\, \mathrm{d}\mathbf{x}.$$

Therefore, the energy of a regular configuration that we consider is given by

$$\mathcal{E}(\Omega_h, v, u) := \int_{\Omega_h} W\big(E(v(\mathbf{x})) - E_0(y)\big)\, \mathrm{d}\mathbf{x} + \int_{\Gamma_h} \psi\big(u(\mathbf{x})\big)\, \mathrm{d}\mathcal{H}^1(\mathbf{x}), \tag{21}$$

where $h : (a, b) \to \mathbb{R}$ is a non-negative Lipschitz function, $u \in L^1(\mathcal{H}^1 \llcorner \Gamma_h)$, and $v \in H^1(\Omega_h; \mathbb{R}^2)$. In the following, we will refer to such triples as regular admissible configurations, and we will denote it by the class $\mathcal{A}_r$ (see Definition 16).

1.2 Main result

In order to study the relaxation of the energy $\mathcal{E}$, we need to first discuss what topology to use. This will determine the types of limiting configurations to expect, and how these effect the value of the effective energy.

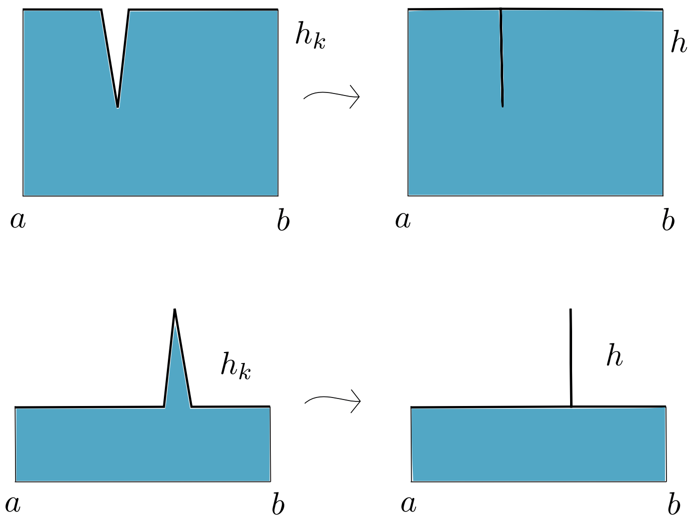

Figure 1.1: Two ways that a sequence of graphs can *close up*: on the top by giving rise to a crack inside Ω_h, while on the bottom to a crack outside Ω_h. We want a topology that *sees* the crack in the former case, but not in the latter.

Here we justify the definition of the topology we use, that will be stated precisely in Definition 20.

We first consider the notion of convergence for the profiles of the film. This will be the same used in [37]. Here we give the heuristics for such a choice. There are several mechanisms that a film can use to release elastic energy. Our model allows for three of these: rearrangement of atoms inside the film, corrugation of the surface, and creation of cracks. The topology on the profile will be concerned only with the last two. We are interested in how a crack forms. There are two mechanisms: as a fracture inside the film, or when the free profile becomes vertical, like it is depicted in Figure 1.1 on the top. We choose to model situations where only the latter is allowed. Note that this forces cracks to be vertical segments touching the free profile. What we want to avoid are configurations where cracks happen outside of the film (Figure 1.1 on the bottom). Thus, we need to differentiate the two situations. The right way to do it is by considering the Hausdorff convergence of the complement of the subgraphs (the so called Hausdorff-complement topology). We note that, in the latter case, the sets $\mathbb{R}^2 \setminus \Omega_{h_k}$ will converge to the limiting configuration $\mathbb{R}^2 \setminus \Omega_h$ where there is no vertical cut (see Figure 1.1 on the bottom). This topology also accommodates for corrugation of the profile.

We now consider the convergence of the displacements. Since the energy has quadratic growth in the symmetric gradient of the displacement, the natural topology will be the weak H^1 topology. In particular, in order to take care of the fact that the displacements are defined in different domains (the subgraphs of the profiles), we take advantage of the fact that the complement of these latter are converging in the Hausdorff sense. Thus, local convergence in the final domain will do the job.

Finally, we discuss the topology for the adatom density. In [18] the idea was to see the adatom density as a Radon measure μ concentrated on the graph describing the profile. Namely, for each $u \in L^1(\Gamma_h)$, we consider

$$\mu := u\mathcal{H}^1 \llcorner \Gamma_h.$$

This identification allows not only to consider concentration of measures, but it turns out to be the right way to model adatoms in order to exploit the interplay between oscillations of the profile and change in adatom density. Thus, for the adatom density, the weak* convergence of measures will be used.

The question we now have to address is what are the possible limiting objects that we need to consider. This is a discussion of compactness of sequences $(\Omega_{h_k}, v_k, \mu_k)_k$ with uniformly bounded energy, namely such that

$$\sup_{k\in\mathbb{N}} \mathcal{E}(\Omega_{h_k}, v_k, \mu_k) < +\infty,$$

We start by investigating the convergence of graphs, and the others will follow. Thanks to the lower bound (19) on the energy density ψ, the energy $\mathcal{F}$ is lower bounded by the length of the graph of h_k. Indeed, there exists $c > 0$ such that

$$\sup_{k\in\mathbb{N}} c\mathcal{H}^1(\Gamma_{h_k}) \leqslant \sup_{k\in\mathbb{N}} \int_{\Gamma_{h_k}} \psi(u_k)\ \mathrm{d}\mathcal{H}^1 < +\infty$$

which in turn is a lower bound on the total variation of h_k:

$$\sup_{k\in\mathbb{N}} \mathcal{H}^1(\Gamma_{h_k}) = \sup_{k\in\mathbb{N}} \int_a^b \sqrt{1+|h_k'|^2}\ \mathrm{d}x \geqslant \int_a^b |h_k'|\ \mathrm{d}x.$$

Thus, if a mass constraint on the area of Ω_k, or a Dirichlet boundary condition at a and b are imposed, we get that the limiting configuration will be the subgraph of a function $h : (a,b) \to [0,+\infty)$ of bounded variation. In particular, since we are in the one dimensional case, such a function will have countably many jumps and countably many cuts.

Now, we consider the convergence of the displacement. Due to the choice of the topology, the limiting displacement will be a function $v \in H^1(\Omega_h;\mathbb{R}^2)$. Note that one of the technical advantages of working in dimension two is that we can avoid having to rely on functions of bounded deformation, and use instead Sobolev functions and the free profile to describe cracks.

Finally, let us discuss the adatom densities. Each of them is seen as the Radon measure $u_k\mathcal{H}^1 \llcorner \Gamma_{h_k}$. By imposing a mass constraint on the total amount of adatoms, we have that their total variation is bounded, and thus they converge (up to a subsequence), to a Radon measure μ. Noting that each μ_k is supported on the graph Γ_{h_k}, and these latter also converge in the Hausdorff sense to the graph of the limiting profile h, the limiting measure μ will be supported on Γ_h.

Therefore, the class $\mathcal{A}$ of limiting admissible configurations we will need to consider is given by the triples (Ω_h, v, μ), where $h \in \mathrm{BV}(a,b)$, $v \in H^1(\Omega_h;\mathbb{R}^2)$, and μ is a Radon measure supported on Γ_h. Moreover, we denote by Γ_h^c the cuts of h, and by $\widetilde{\Gamma}_h$ the rest of the extended graph of h, namely regular part and jumps (see Figure 1.2 on the right, and Definition 6 for the precise definition).

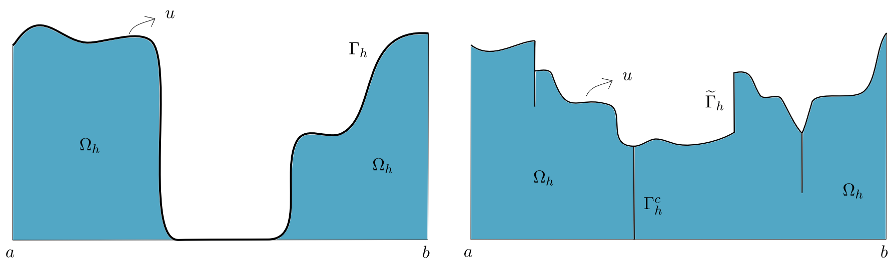

Figure 1.2: A regular configuration on the left, and a possible limiting configuration on the right: cracks and jumps can appear.

Thus, in light of the above discussion, given a sequence $(\Omega_{h_k}, v_k, \mu_k)_k \subset \mathcal{A}_r$, we will say that $(\Omega_{h_k}, v_k, \mu_k) \to (\Omega_h, v, \mu) \in \mathcal{A}$ if

(i) $\mathbb{R}^2 \setminus \Omega_{h_k} \overset{H}{\to} \mathbb{R}^2 \setminus \Omega_h$ in the Hausdorff convergence of sets;

(ii) $v_k \rightharpoonup v$ weakly in $H^1_{\mathrm{loc}}(\Omega_h; \mathbb{R}^2)$;

(iii) $\mu_k \overset{*}{\rightharpoonup} \mu$ weakly* in the sense of measures;

as $k \to \infty$.

The two main results of this Chapter provide representations of the relaxation of the functional $\mathcal{E}$ when a mass constraint is in force, and when it is not.

Theorem 1. Let $(\Omega_h, v, \mu) \in \mathcal{A}$, and write $\mu = u\mathcal{H}^1 \llcorner \Gamma_h + \mu^s \llcorner \Gamma_h$, where μ^s is the singular part of μ with respect to $\mathcal{H}^1 \llcorner \Gamma_h$. Then, the relaxation of the functional $\mathcal{E}$ defined in (21), with respect to the above topology, is given by

$$\mathcal{G}(\Omega_h, v, \mu) = \int_{\Omega_h} W\big(E(v) - E_0(y)\big)\, \mathrm{d}\mathbf{x} \\ + \int_{\widetilde{\Gamma}_h} \widetilde{\psi}(u)\, \mathrm{d}\mathcal{H}^1 + \int_{\Gamma_h^c} \psi^c(u)\, \mathrm{d}\mathcal{H}^1 + \theta\mu^s(\Gamma_h),$$

where $\widetilde{\psi}$ is the convex sub-additive envelope of ψ (see Definition 11), the function ψ^c is defined as

$$\psi^c(s) \coloneqq \min\{\widetilde{\psi}(r) + \widetilde{\psi}(t) \,:\, s = r + t\},$$

for all $s \in [0, +\infty)$, and

$$\theta := \lim_{t \to +\infty} \frac{\widetilde{\psi}(t)}{t} = \lim_{t \to +\infty} \frac{\psi^c(t)}{t},$$

is the common recession coefficient of $\widetilde{\psi}$ and of ψ^c.

Theorem 2. Fix $M, m > 0$. Denote by $\mathcal{A}_r(m, M)$ the triples $(\Omega_h, v, \mu) \in \mathcal{A}_r$ such that

$$\int_{\Gamma_h} u \, \mathrm{d}\mathcal{H}^1 = m, \qquad \mathcal{L}^2 \left(\Omega_h \cap \{y \geqslant 0\}\right) = M,$$

and by $\mathcal{A}(m, M)$ the triples $(\Omega_h, v, \mu) \in \mathcal{A}$ such that

$$\mu(\Gamma_h) = m, \qquad \mathcal{L}^2 \left(\Omega_h \cap \{y \geqslant 0\}\right) = M.$$

Define $\mathcal{H}^{m,M} : \mathcal{A} \to [0, +\infty]$ as

$$\mathcal{H}^{m,M}(\Omega_h, v, \mu) := \begin{cases} \mathcal{E}(\Omega_h, v, \mu) & (\Omega_h, v, \mu) \in \mathcal{A}_r(m, M), \\ +\infty & \text{else.} \end{cases}$$

Then, the relaxation of $\mathcal{H}$ in the above topology is given by $\mathcal{F}^{m,M} : \mathcal{A} \to [0, +\infty]$ as

$$\mathcal{F}^{m,M}(\Omega_h, v, \mu) := \begin{cases} \mathcal{G}(\Omega_h, v, \mu) & \text{if } (\Omega_h, v, \mu) \in \mathcal{A}(m, M), \\ +\infty & \text{else,} \end{cases}$$

where $\mathcal{G}(\Omega_h, v, \mu)$ denotes the right-hand side of the representation formula of Theorem 1. Namely, the mass constraint is maintained by the relaxation procedure.

Remark 1. In general, it is not possible to say more on the singular part of the measure.

1.3 Strategy of the proof

Now, we would like to comment on the strategy to prove the main results. First of all, in Theorem 9 we will prove the liminf inequality for the case of no mass constraint, and in Theorem 10 the limsup inequality for the case with the mass constraint. These theorems will give both Theorem 1, and Theorem 2.

Similarly for the functional considered in [37], the bulk and the surface terms of the energy do not interact in the relaxation process. Since the former is quite standard, we will comment on how to deal with the latter. In this lies the novelty of the contributions of [28]. Our strategy relies on ideas inspired by results obtained in [18]. The main difference with the case treated in that paper is the graph constraint. This reflects on the fact that oscillations of the thin film profile must be in the vertical direction in order to preserve such a constraint, and that cracks can be created only in a specific way. The former term only gives technical challenges, while the latter is responsible for the different energy densities $\widetilde{\psi}$ and ψ^c. Despite this, note that the recession coefficients for the singular part of the measure in the two parts of the extended graph (the cuts, and the rest of the graph) agree.

Let us discuss the strategy for the liminf inequality for the surface terms. We avoid mentioning the fine details and focus instead on the main ideas. Let $(h_k)_{k\in\mathbb{N}}$ be a sequence of Lipschitz functions such that $\mathbb{R}^2 \setminus \Omega_{h_k}$ converge to $\mathbb{R}^2 \setminus \Omega_h$, for some function h of bounded variation. This implies that Ω_{h_k} converges to Ω_h in L^1 (see Lemma 4). Let $(u_k)_{k\in\mathbb{N}}$ the be adatom densities defined on each Γ_{h_k}, and let $\mu = u\mathcal{H}^1 \llcorner \Gamma_h + \mu^s$ be the limiting measure. We need to prove that

$$\liminf_{k\to\infty} \int_{\Gamma_{h_k}} \psi(u_k)\, \mathrm{d}\mathcal{H}^1 \geqslant \int_{\widetilde{\Gamma}_h} \widetilde{\psi}(u)\, \mathrm{d}\mathcal{H}^1 + \int_{\Gamma_h^c} \psi^c(u)\, \mathrm{d}\mathcal{H}^1 + \theta\mu^s(\Gamma_h). \tag{22}$$

The idea is to separate the contribution that the energy on the left-hand side has on a neighborhood of each cut of h, and on the other part of the graph of h. Despite there might be a countable number of cuts, it is just a technicality to show that we can reduce to finitely many of them (see the beginning of the proof of Theorem 9). Thus, let us assume that the final configuration described by h has finitely many cuts. Since the energy is local, for the sake of simplicity, we will consider the case where

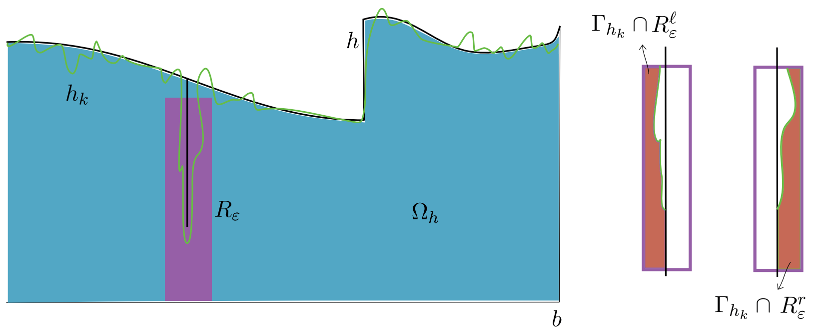

Figure 1.3: In order to get the liminf inequality, we separate the effects on a neighborhood R_ε of the cut, and outside of it.

there is one single cut. In case the measure μ has a Dirac delta at the point P (see Figure 1.3), we want to count its contribution to the energy as part of the energy of the regular part of $\widetilde{\Gamma}_h$. For this reason, we take $\varepsilon > 0$ and consider a rectangle R_ε around the cut as in Figure 1.3.

Now, we claim that

$$\liminf_{k\to\infty} \int_{\Gamma_{h_k}\setminus R_\varepsilon} \psi(u_k)\, \mathrm{d}\mathcal{H}^1 \geqslant \int_{\widetilde{\Gamma}_h\setminus R_\varepsilon} \widetilde{\psi}(u)\, \mathrm{d}\mathcal{H}^1 + \theta\mu^s(\widetilde{\Gamma}_h \setminus R_\varepsilon), \qquad (23)$$

and that

$$\liminf_{k\to\infty} \int_{\Gamma_{h_k}\cap R_\varepsilon} \psi(u_k)\, \mathrm{d}\mathcal{H}^1 \geqslant \int_{\Gamma_h^c\setminus R_\varepsilon} \psi^c(u)\, \mathrm{d}\mathcal{H}^1 + \theta\mu^s(\Gamma_h^c \cap R_\varepsilon). \qquad (24)$$

Given (23) and (24), we obtain the desired liminf inequality (22) by sending ε to zero.

To obtain both (23) and (24), we rely on (a localized version of) the lower semicontinuity result proved in [18, Theorem 5] (see Theorem 8). In the first case, the idea is to view the graph of each h_k, and the regular and the jump part of the extended graph of h as ($\mathcal{H}^1$-equivalent to) the reduced boundaries of the corresponding subgraph.

For (24), we instead have to consider the contributions of the surface energy from both sides of the crack. Therefore, we reason as follows: the rectangle R_ε in Figure 1.3 is split by the vertical line passing through the crack in two parts, one on the left and one on the right. Call them R_ε^ℓ, and R_ε^r, respectively. Then, we consider the sets $\Omega_{h_k} \cap R_\varepsilon^\ell$ and $\Omega_{h_k} \cap R_\varepsilon^r$. Since $\mathbb{R}^2 \setminus \Omega_{h_k} \overset{H}{\to} \mathbb{R}^2 \setminus \Omega_h$, they converge in L^1 to R_ε^ℓ, and R_ε^r, respectively. Moreover, it holds

$$u_k \mathcal{H}^1 \llcorner (\Gamma_{h_k} \cap R_\varepsilon^\ell) \overset{*}{\rightharpoonup} \mu^\ell = u^\ell \llcorner (\Gamma_h^c \cap R_\varepsilon) + \mu_s^\ell,$$

$$u_k \mathcal{H}^1 \llcorner (\Gamma_{h_k} \cap R_\varepsilon^r) \overset{*}{\rightharpoonup} \mu^r = u^r \llcorner (\Gamma_h^c \cap R_\varepsilon) + \mu_s^r.$$

Thus, thanks to the lower semicontinuity result (see Theorem 8), we get that

$$\liminf_{k\to\infty} \int_{\Gamma_{h_k} \cap R_\varepsilon^\ell} \psi(u_k)\, \mathrm{d}\mathcal{H}^1 \geqslant \int_{\Gamma_h^c \setminus R_\varepsilon} \widetilde{\psi}(u^\ell)\, \mathrm{d}\mathcal{H}^1 + \theta \mu_s^\ell(\Gamma_h^c \cap R_\varepsilon)$$

and

$$\liminf_{k\to\infty} \int_{\Gamma_{h_k} \cap R_\varepsilon^r} \psi(u_k)\, \mathrm{d}\mathcal{H}^1 \geqslant \int_{\Gamma_h^c \setminus R_\varepsilon} \widetilde{\psi}(u^r)\, \mathrm{d}\mathcal{H}^1 + \theta \mu_s^r(\Gamma_h^c \cap R_\varepsilon).$$

We then show that $u^\ell + u^r = u$, and $\mu_s^\ell + \mu_s^r = \mu_s$. Thus, by definition of ψ^c, we obtain

$$\psi^c(u) \leqslant \widetilde{\psi}(u^r) + \widetilde{\psi}(u^\ell).$$

This gives (24), and, in turn, the desired liminf inequality for the surface energy.

We now discuss the strategy for the limsup inequality for the surface energy. This is more involved, and requires several steps. The idea is to reduce to the situation where the limiting profile h is Lipschitz, and the adatom measure μ is a piecewise constant density (more precisely, it is possible to find a square grid where the density has the same value on each of the parts of the graph inside each of these squares). In such a case, in Proposition 13 we construct a sequence $(\Omega_{h_k}, v_k, \mu_k)_k$ that satisfies the mass constraints such that

$$\limsup_{k\to\infty} \int_{\widetilde{\Gamma}_{h_k}} \psi(u_k)\, \mathrm{d}\mathcal{H}^1 \leqslant \int_{\widetilde{\Gamma}_h} \widetilde{\psi}(u)\, \mathrm{d}\mathcal{H}^1. \tag{25}$$

Without loss of generality (see Lemma 5), we can assume ψ to be convex. Then, ψ and $\widetilde{\psi}$ agree on $[0, s_0)$, for some $s_0 \in (0, +\infty]$. In particular, if $s_0 < +\infty$ the function $\widetilde{\psi}$ is linear on $(s_0, +\infty)$ (see Lemma 5)). Thus, in squares where $u \leqslant s_0$, we define h_k as h and u_k as u. We just have to care about those squares Q where $u > s_0$. The energy in such a square is $\widetilde{\psi}(u)\mathcal{H}^1(\Gamma_h \cap Q)$. The idea is to write

$$\begin{aligned}\widetilde{\psi}(u)\mathcal{H}^1(\Gamma_h \cap Q) &= \widetilde{\psi}(rs_0)\mathcal{H}^1(\Gamma_h \cap Q)\\ &= r\widetilde{\psi}(s_0)\mathcal{H}^1(\Gamma_h \cap Q)\\ &= \psi(s_0)\left[r\mathcal{H}^1(\Gamma_h \cap Q)\right],\end{aligned}$$

for some $r > 1$, where in the last step we used the fact that $\psi(s_0) = \widetilde{\psi}(s_0)$. Then, we want to obtain the quantity $r\mathcal{H}^1(\Gamma_h \cap Q)$ as the length of an oscillating profile h_k in Q, and define u_k as s_0. This ensures the validity of (25). Such a construction is done in Proposition 3, where we prove an extension of the so called *wriggling lemma* (see [18, Lemma 4]). Namely, given a Lipschitz function $f : (a, b) \to [0, +\infty)$, and a number $r > 1$, there exists a sequence of Lipschitz graphs $f_n : (a, b) \to [0, +\infty)$ with $\mathcal{H}^1 \llcorner \Gamma_{f_n} \stackrel{*}{\rightharpoonup} r\mathcal{H}^1 \llcorner \Gamma_f$ as $n \to \infty$, such that

$$\mathcal{H}^1(\Gamma_{f_n}) = r\mathcal{H}^1(\Gamma_f),$$

and $f_n(a) = f(a)$, $f_n(b) = f(b)$, for each $n \in \mathbb{N}$, and satisfying other technical properties (see Proposition 3 for the precise statement). What the above inequality is using is a quantitative lack of lower semi-continuity of the perimeter. The difference with the result in [18, Lemma 4] is that only vertical oscillations are allowed. Moreover, we also fill in details that were not fully explained in that paper. Note that in our case, there is an additional technical difficulty to be faced: ensuring that both mass constraints are satisfied by each $(\Omega_{h_k}, v_k, \mu_k)$ will be achieved by carefully modifying both the profile and the density. Note that modifications of the graphs have to be done in such a way that the profile is always non-negative.

In order to reduce from a general profile $(\Omega_h, v, \mu) \in \mathcal{A}(m, M)$ to the above case, we argue as follows. First of all, by using averages, we prove that it suffices to consider the situation where the adatom measure μ is a piecewise constant function (see Proposition 11). Then, we need to approximate a general profile $h \in \mathrm{BV}(a,b)$ with a sequence of Lipschitz profiles $(h_k)_{k\in\mathbb{N}}$, and corresponding piecewise constant adatom densities $(u_k)_{k\in\mathbb{N}}$, in such a way that

$$\lim_{k\to\infty} \int_{\widetilde{\Gamma}_{h_k}} \widetilde{\psi}(u_k)\, \mathrm{d}\mathcal{H}^1 + \int_{\Gamma^c_{h_k}} \psi^c(u_k)\, \mathrm{d}\mathcal{H}^1 = \int_{\widetilde{\Gamma}_h} \widetilde{\psi}(u)\, \mathrm{d}\mathcal{H}^1 + \int_{\Gamma^c_h} \psi^c(u)\, \mathrm{d}\mathcal{H}^1. \tag{26}$$

This is done in Proposition 12. In order to obtain the approximation of the profiles, we employ an idea by Bonnettier and Chambolle in Section 5.2 of [9], later adapted to the case of graphs in [37, Lemma 2.7]: to use the Moreau-Yosida transform to define a Lipschitz approximation of h to the left and to the right of each cut (again, we are reducing to the case of finitely many of them). To also approximate the cracks, we use a linear interpolation. As for defining the adatom density on the graph of h_k, we exploit the fact that $\mathbb{R}^2 \setminus \Omega_{h_k} \xrightarrow{H} \mathbb{R}^2 \setminus \Omega_h$ implies that the graphs $(h_k)_{k\in\mathbb{N}}$ are converging in the Hausdorff topology to h. In particular, for k large enough, the graphs of the h_k's will be inside the same squares where the graph of h is. This allows to define u_k on the part of the graph of h_k inside a square, as the value that u has inside that square. Then, the convergence of the energy required in (26) is ensured since the length of the graph of h_k inside each cube converges to the length of h inside the same cube.

1.4 Preliminaries

We here introduce the main definitions and basic results that will be used throughout the Chapters 1 and 2.

1.4.1 Γ-convergence

We introduce the definition of convergence of functionals used. We refer to [30] for a complete description and to the proofs of the following results.

We start with the topological definition of Γ-limit. Let X be a topological space and denote $\mathcal{O}(x)$, for $x \in X$ the set of all open neighbourhoods of x. As a notation, we write $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

Definition 2. The Γ-*lower limit* and the Γ-*upper limit* of a sequence of functionals $F_\varepsilon : X \to \overline{\mathbb{R}}$ are defined as

$$(\Gamma - \liminf_{\varepsilon\to 0} F_\varepsilon)(x) := \sup_{U\in\mathcal{O}(x)} \liminf_{\varepsilon\to 0} \inf_{y\in U} F_h(y),$$
$$(\Gamma - \limsup_{\varepsilon\to 0} F_\varepsilon)(x) := \sup_{U\in\mathcal{O}(x)} \limsup_{\varepsilon\to 0} \inf_{y\in U} F_\varepsilon(y).$$

If there exists $F : X \to \overline{\mathbb{R}}$ such that

$$F = \Gamma - \liminf_{\varepsilon\to 0} F_\varepsilon = \Gamma - \limsup_{\varepsilon\to 0} F_\varepsilon,$$

we write

$$F = \Gamma - \lim_{\varepsilon\to 0} F_h$$

and we say that $F_\varepsilon \xrightarrow{\Gamma} F$ in X as $\varepsilon \to 0$.

From Definition 2 is clear that the notion of Γ-convergence depends on the topology taken into account. By making additional assumptions on the topological space X, we can provide a sequential characterization of Γ-convergence. In case X satisfies the first axiom of countability, namely the neighbourhood system of every point has a countable base, we can state the following theorem, whose proof can be found in [30, Proposition 8.1].

Theorem 3. Let X be a topological space which satisfies the first axiom of countability. We have that $F_\varepsilon \xrightarrow{\Gamma} F$ in X, as $\varepsilon \to 0$, if and only if

(*i*) For every $x \in X$ and for every sequence $(x_\varepsilon)_\varepsilon \subset X$ converging to x in X we have

$$F(x) \leqslant \liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon);$$

(*ii*) For every $x \in X$ there is a sequence $(x_\varepsilon)_\varepsilon \subset X$ converging to x in X such that

$$\limsup_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) \leqslant F(x).$$

In this Chapter, we are working in the framework of Banach spaces endowed with the weak or the weak* topology. Since the weak topology is not a metrisable, we a result that ensures that such a topology is locally metrisable. To this end, we use the following proposition whose proof is contained in [33, Proposition 2.6].

Proposition 1. There exists a distance d on the space of positive measures with the following property. Let $(\mu_\varepsilon)_\varepsilon$ be a sequence of positive measures on $\mathbb{R}^n$ such that $\sup_{\varepsilon>0} \mu_\varepsilon(\mathbb{R}^n) < \infty$. Then, $\mu_\varepsilon \xrightarrow{*} \mu$ for some positive measure μ on $\mathbb{R}^n$ if and only if

$$\lim_{\varepsilon \to 0} d(\mu_\varepsilon, \mu) = 0.$$

Definition 3. We say that a sequence of functionals $(F_n)_n$ is equi-coercive if there exists a lower semi-continuous coercive functional $\rho : X \to \mathbb{R}$ such that $F_n \geqslant \rho$ for all $n \in \mathbb{N}$.

We now state a theorem that ensure the convergence of minimising sequences.

Theorem 4. Let $(F_n)_n$ be a sequence of is equi-coercive functions. Then, if $F_n \xrightarrow{\Gamma} F$, we have that

$$\exists \min_X F = \lim_{n\to\infty} \inf_X F_n.$$

Moreover, if $(x_n)_n$ is minimising sequence for F, then every of its cluster point is a minimum for F.

1.4.2 Function of (pointwise) bounded variation in one dimension

A comprehensive treatment of this topic can be found in the book [53] by Leoni.

Definition 4. Let $h : (a,b) \to \mathbb{R}$. We say that h is a function of *pointwise bounded variation* in (a,b) if $\mathrm{Var}(h) < +\infty$, where

$$\mathrm{Var}(h) \coloneqq \sup \left\{ \sum_{i=1}^{k} |h(x_i) - h(x_{i-1})| \right\},$$

where the supremum is taken over all finite partitions of (a,b). In this case, we write $h \in \mathrm{BVP}(a,b)$.

The main properties of functions of pointwise bounded variations that will be used in the paper are collected in the following result (see [53, Theorem 2.17, Theorem 2.36]).

Theorem 5. *Let $h \in BPV(a,b)$. Then, the limits*

$$h(x^-) \coloneqq \lim_{y \to x^-} h(y), \qquad h(x^+) \coloneqq \lim_{y \to x^+} h(y),$$

exist for all $x \in (a,b)$. In particular, if we define the functions

$$h^-(x) \coloneqq \min\{h(x^+), h(x^-)\}, \qquad h^+(x) \coloneqq \max\{h(x^+), h(x^-)\},$$

we have that there are at most countably many points $x \in (a,b)$ for which $h^-(x)$, $h^+(x)$ and $h(x)$ do not agree. Finally, h admits a lower semi-continuous representative.

We now connect functions of pointwise bounded variation with those of bounded variation.

Definition 5. Let $u \in L^1(a,b)$. We say that u has *bounded variation* in (a,b) if there exists a Radon measure μ such that

$$\int_a^b u\varphi' \,\mathrm{d}x = -\int_{(a,b)} \varphi \,\mathrm{d}\mu,$$

for all $\varphi \in \mathcal{C}^1_c(a,b)$. In this case, we write $u \in \mathrm{BV}(a,b)$, and we denote the measure μ by $\mathrm{D}u$.

The relation between functions of pointwise bounded variation and functions of bounded variation is given by the following result (see [53, Theorem 7.3]).

Theorem 6. Let $u \in BV(a,b)$. Then, there exists a right-continuous function $h \in BVP(a,b)$ with $u(x) = h(x)$ for a.e. $x \in (a,b)$ such that $\mathrm{Var}(h) = |Du|(a,b)$.

Finally, we recall that the subgraph of a function of bounded variation is a set of finite perimeter (see [44, Theorem 14.6]), and that its reduced boundary coincides with the non cut part of the extended graph (see [34, Theorem 4.5.9 (3)].

Remark 2. In Theorem 5 we introduced the functions $h^{\pm}$. Note that

$$h^-(x) = \liminf_{y \to x} h(y), \qquad h^+(x) = \limsup_{y \to x} h(y).$$

In particular, if $x \in (a,b)$ is a point of continuity for h, then $h^-(x) = h^+(x) = h(x)$.

Definition 6. Let $h \in \mathrm{BV}(a,b)$. We call

$$\Gamma_h \coloneqq \big\{(x,y) \in \mathbb{R}^2 \,:\, x \in (a,b),\, h(x) \leqslant y \leqslant h^+(x)\big\}$$

the *extended graph* of h. Moreover, we define:

(*i*) The *jump part* of Γ_h as

$$\Gamma_h^j \coloneqq \{(x,y) \in \mathbb{R}^2 \,:\, x \in (a,b),\, h^-(x) \leqslant y < h^+(x)\};$$

(*ii*) The *cut part* of Γ_h as

$$\Gamma_h^c \coloneqq \{(x,y) \in \mathbb{R}^2 \,:\, x \in (a,b),\, h(x) \leqslant y < h^-(x)\};$$

(*iii*) The *regular part* of Γ_h as

$$\Gamma_h^r \coloneqq \Gamma_h \setminus (\Gamma_h^j \cup \Gamma_h^c).$$

Moreover, we introduce the notation $\widetilde{\Gamma}_h \coloneqq \Gamma_h^r \cup \Gamma_h^j$.

Remark 3. Note that

$$\Gamma_h = \widetilde{\Gamma}_h \cup \Gamma_h^c = \Gamma_h^r \cup \Gamma_h^j \cup \Gamma_h^c,$$

holds for every $h \in \mathrm{BV}(a,b)$.

When there is no room for confusion, we will drop the suffix h in the notation above, as well as for the notation of the subgraph Ω_h. We denote by Δ the symmetric difference between sets.

Lemma 1. Let $h \in \mathrm{BV}(a,b)$. Then, the subgraph Ω has finite perimeter in $(a,b) \times \mathbb{R}$, and

$$\mathcal{H}^1(\widetilde{\Gamma} \triangle \partial^*\Omega) = 0,$$

where $\partial^*\Omega$ is the reduced boundary of Ω.

An important result that will be used several times is the following.

Lemma 2. Let $h \in \mathrm{BV}(a,b)$ be lower semi-continuous, and let $\varepsilon > 0$. Define

$$P(\varepsilon) := \left\{ x \in (a,b) \, : \, \exists\, y \in \Gamma \text{ s.t. } h(x) \leqslant y \leqslant h^-(x) - \varepsilon \right\}.$$

Then, $P(\varepsilon)$ is a finite set.

Proof. By [1, Corollary 3.33], it holds that

$$|\mathrm{D}h|(a,b) = \|h'\|_{L^1(a,b)} + \sum_{x \in S} [h^+(x) - h(x)] + |\mathrm{D}^c|(a,b),$$

where S denotes the set of points $x \in (a,b)$ such that $h^+(x) > h(x)$, and $\mathrm{D}^c h$ is the Cantor part of the measure $\mathrm{D}h$. we recall that from Theorem 5 we have that J_h is at most countable. Therefore, we obtain that

$$\sum_{x \in S} [h^-(x) - h(x)] < +\infty.$$

Notice that the set $P(\varepsilon)$ corresponds to points in S where the quantity $h^-(x) - h(x)$ is at least ε. From the convergence of the series above, we get the desired result. $\square$

In particular, we will need to work with a specific class of piecewise constant functions, that we introduce here.

Definition 7. Let $h \in \mathrm{BV}(a,b)$, and $\delta > 0$. We say that a finite family $(Q^j)_{j=1}^N$ of open and pairwise disjoint rectangles is *δ-admissible covering* for Γ, if

(i) The side lengths of each Q^j is less than δ;

(ii) It holds
$$\Gamma \subset \bigcup_{i=1}^{N} \overline{Q^j};$$

(iii) $\mathcal{H}^1(\Gamma \cap \partial Q^j) = 0$ for all $j = 1, \dots, N$.

A simple result that will be use repeatedly without mentioning it is the following (see (a) of 8).

Lemma 3. Let $h \in \mathrm{BV}(a,b)$, and $\delta > 0$. Then, there exists a δ-admissible covering for Γ.

Definition 8. Let $h \in \mathrm{BV}(a,b)$, and $\delta > 0$. Let $\Gamma \coloneqq \mathrm{graph}(h)$. A function $u \in L^1(\Gamma)$ is called *δ-grid constant* if there exists a δ-admissible covering for Γ, such that $u_{|Q^j \cap \Gamma} = u^j \in \mathbb{R}$, for every $j = 1, \dots, N$. Moreover, we say that $u \in L^1(\Gamma)$ is *grid constant* if there exists $\delta > 0$ such that it is δ-grid constant.

1.4.3 Hausdorff convergence

We now introduce the Hausdorff metric.

Definition 9. Let $E, F \subset \mathbb{R}^N$. We define
$$\mathrm{d}_H(E,F) \coloneqq \inf\{r > 0 \,:\, E \subset F_r,\, F \subset E_r\},$$
where, for $A \subset \mathbb{R}^N$ and $r > 0$, we set $A_r \coloneqq \{x + y : x \in A, y \in B_r(0)\}$. Moreover, we say that a sequence of sets $(E_k)_k$ with $E_k \subset \mathbb{R}^N$ *Hausdorff converges* to a set $E \subset \mathbb{R}^N$, and we write $E_k \xrightarrow{H} E$, if $\mathrm{d}_H(E_k, E) \to 0$ as $k \to \infty$.

In order for the Hausdorff distance to actually be a distance, we need to work with compact sets. This will also give compactness of the metric space. This latter fact is known as Blaschke Theorem (see [1, Theorem 6.1]).

Theorem 7 (Blaschke Theorem)**.** The family of compact sets of $\mathbb{R}^N$ endowed with the Hausdorff distance is a compact metric space.

The convergence of subgraphs in the Hausdorff-complement topology we use implies their L^1 convergence, as it was shown in [37, Lemma 2.5].

Lemma 4. Let $(h_k)_k \subset \mathrm{BV}(a,b)$ be a sequence of lower semi-continuous functions such that

$$\sup_{k\in\mathbb{N}} |Dh_k|(a,b) < +\infty, \qquad \mathbb{R}^2 \setminus \Omega_{h_k} \xrightarrow{H} \mathbb{R}^2 \setminus A,$$

for some open set $A \subset \mathbb{R}^2$. Then, there exists $h \in \mathrm{BV}(a,b)$ such that $A = \Omega_h$, $h_k \to h$ in L^1. Moreover, $\Omega_{h_k} \to \Omega_h$ in L^1.

We now relate the Hausdorff metric with the notion of Kuratowski convergence (see [1, Theorem 6.1]).

Proposition 2. Let $(E_k)_k$, with $E_k \subset \mathbb{R}^2$, and let $E \subset \mathbb{R}^2$. Then, $E_k \xrightarrow{H} E$ if and only if the followings hold:

(i) Any cluster point of a sequence $(x_k)_k$, with $x_k \in E_k$, belongs to E;

(ii) For any $x \in E$, there exists $(x_k)_k$, with $x_k \in E_k$, such that $x_k \to x$.

These equivalent properties are those defining the so called Kuratowski convergence.

1.4.4 Convex sub-additive envelope

Here we introduce all the notation and recall the result that are needed to treat the surface term.

Definition 10. A function $\psi : [0,+\infty) \to \mathbb{R}$ is said to be *sub-additive* if

$$\psi(s+t) \leqslant \psi(s) + \psi(t),$$

for any $s, t \geqslant 0$.

Definition 11. Let $\psi : [0, +\infty) \to \mathbb{R}$. The *convex sub-additive envelope* of ψ is the function $\widetilde{\psi} : [0, +\infty) \to \mathbb{R}$ defined as

$$\widetilde{\psi}(s) := \sup\{f(s) : f : [0, +\infty) \to \mathbb{R} \textit{ is convex, sub-additive} \\ \textit{and } f \leqslant \psi\}.$$

for all $s \in [0, +\infty)$.

Remark 4. Note that $\widetilde{\psi}$ is the greatest convex and sub-additive function that is no greater than ψ.

Definition 12. Let $\psi : [0, +\infty) \to \mathbb{R}$. We define the function $\psi^c : [0, +\infty) \to \mathbb{R}$ as

$$\psi^c(s) := \min\{\widetilde{\psi}(r) + \widetilde{\psi}(t) \, : \, s = r + t\},$$

for all $s \in [0, +\infty)$.

Remark 5. It is easy to see that the function ψ^c is well defined. Indeed, fix $s \geqslant 0$. Since ψ is defined only for non-negative real numbers, by compactness there exist $a, b \geqslant 0$ with $s = a + b$ such that

$$\psi^c(s) = \widetilde{\psi}(a) + \widetilde{\psi}(b).$$

Moreover, note that $\psi^c(0) = 2\widetilde{\psi}(0)$. This is consistent with the result obtained in [37], where they consider the case $\psi \equiv 1$. We will prove in Lemma 6 that ψ^c is convex and sub-additive.

We now recall two results on the surface energy. The first is a combination of [18, Lemma A.11] and [17, Lemma 2.2].

Definition 13. Let $\psi : \mathbb{R} \to \mathbb{R}$. We define its *convex envelope* $\psi^{\mathrm{cvx}} : \mathbb{R} \to \mathbb{R}$ as

$$\psi^{\mathrm{cvx}}(x) := \sup\{\rho(x) : \rho \text{ is convex and } \rho \leqslant \psi\},$$

for all $x \in \mathbb{R}$.

Lemma 5. Let $\psi : [0,+\infty) \to (0,+\infty)$. Then

$$\widetilde{\psi} = \widetilde{\psi^{\mathrm{cvx}}}.$$

Namely, in order to compute the convex sub-additive envelope of ψ, we can assume, without loss of generality, that ψ is convex.

Moreover, assume ψ to be convex. Then, there exists $s_0 \in (0,+\infty]$ such that

$$\widetilde{\psi}(s) = \begin{cases} \psi(s) & s \leqslant s_0, \\ \theta s & s > s_0, \end{cases}$$

for some $\theta > 0$.

Remark 6. Note that, if ψ is differentiable at s_0, then $\theta = \psi'(s_0)$. In particular, if $s_0 < +\infty$, it holds that $\widetilde{\psi}$ is linear in $[s_0,+\infty)$.

Definition 14. Let $\psi : [0,+\infty) \to \mathbb{R}$. We define the *recession coefficients* of $\widetilde{\psi}$ and ψ^c as

$$\widetilde{\theta} := \lim_{s\to+\infty} \frac{\widetilde{\psi}(s)}{s} \quad \text{and} \quad \theta^c := \lim_{s\to+\infty} \frac{\psi^c(s)}{s},$$

respectively, where $\widetilde{\psi}$ is as in Definition 11 and ψ^c as in Definition 12.

In Lemma 7 we will prove that $\widetilde{\theta} = \theta^c$. The common value will be denoted by θ.

Lemma 6. Let $\psi : [0,+\infty) \to \mathbb{R}$. Then, the function ψ^c (see Definition 12) is convex and sub-additive.

Proof. Step 1. We prove that ψ^c is sub-additive. Fix $z \geqslant 0$. Then, by definition of $\psi^c(z)$, there exist $x, y \geqslant 0$ with $z = x + y$ such that

$$\psi^c(z) = \widetilde{\psi}(x) + \widetilde{\psi}(y).$$

Thus,

$$\psi^c(z) = \widetilde{\psi}(x) + \widetilde{\psi}(y) \geqslant \widetilde{\psi}(x+y) = \widetilde{\psi}(z),$$

where last inequality follows from the sub-additivity of $\widetilde{\psi}$. Moreover,

$$\psi^c(z+w) \leqslant \widetilde{\psi}(z) + \widetilde{\psi}(w) \leqslant \psi^c(z) + \psi^c(w),$$

for every $z, w \geqslant 0$.

Step 2. We prove that ψ^c is convex. Let $z, w \geqslant 0$ and $\lambda \in [0,1]$. By definition of $\psi^c(z)$, and of $\psi^c(w)$, there exist $z_1, z_2, w_1, w_2 \geqslant 0$ with $z = z_1 + z_2$ and $w = w_1 + w_2$ such that

$$\psi^c(z) = \widetilde{\psi}(z_1) + \widetilde{\psi}(z_2), \qquad \psi^c(w) = \widetilde{\psi}(w_1) + \widetilde{\psi}(w_2).$$

Note that

$$\begin{aligned} \lambda z + (1-\lambda)w &= \lambda(z_1 + z_2) + (1-\lambda)(w_1 + w_2) \\ &= \big(\lambda z_1 + (1-\lambda)w_1\big) + \big(\lambda z_2 + (1-\lambda)w_2\big). \end{aligned}$$

Thus, we get that

$$\begin{aligned} \psi^c\big(\lambda z + (1-\lambda)z\big) &\leqslant \widetilde{\psi}\big(\lambda z_1 + (1-\lambda)w_1\big) + \widetilde{\psi}\big(\lambda z_2 + (1-\lambda)w_2\big) \\ &\leqslant \lambda\widetilde{\psi}(z_1) + (1-\lambda)\widetilde{\psi}(w_1) + \lambda\widetilde{\psi}(z_2) + (1-\lambda)\widetilde{\psi}(w_2) \\ &= \lambda\psi^c(z) + (1-\lambda)\widetilde{\psi}(w), \end{aligned}$$

where, in the second step, we used the convexity of $\widetilde{\psi}$. $\square$

We now prove that the recession coefficients of $\widetilde{\psi}$ and of ψ^c, defined in Definition 14, coincide.

Lemma 7. Let $\psi : [0, +\infty) \to \mathbb{R}$. Then, $\widetilde{\theta} = \theta^c$.

Proof. We first prove that $\theta^c \leqslant \widetilde{\theta}$. Indeed, since $\psi^c(s) \leqslant 2\widetilde{\psi}(s/2)$, for all $s \geqslant 0$, we have that

$$\theta^c = \lim_{s\to+\infty} \frac{\psi^c(s)}{s} \leqslant \lim_{s\to+\infty} \frac{2}{s}\widetilde{\psi}\Big(\frac{s}{2}\Big) = \widetilde{\theta}.$$

We now prove that $\theta^c \geqslant \widetilde{\theta}$. Fix $z \geqslant 0$, and let $x, y \geqslant 0$ with $z = x + y$ such that

$$\psi^c(z) = \widetilde{\psi}(x) + \widetilde{\psi}(y).$$

Then, we get

$$\psi^c(z) = \widetilde{\psi}(x) + \widetilde{\psi}(y) \geqslant \widetilde{\psi}(z),$$

where last inequality follows from the sub-additivity of $\widetilde{\psi}$. Therefore,

$$\theta^c = \lim_{s\to+\infty} \frac{\psi^c(s)}{s} \geqslant \lim_{s\to+\infty} \frac{\widetilde{\psi}(s)}{s} = \widetilde{\theta}.$$

This concludes the proof. □

1.5 Technical results

In this section we collect the main technical results that will be needed in the proof of the integral representation of the relaxation and, implicitly, in the phase-field approximation contained in Chapter 2.

The following result proved in [18, Theorem 3] gives a lower bound for the surface energy.

Theorem 8. Let $E \subset \mathbb{R}^N$ be a set of finite perimeter and μ be a Radon measure supported on ∂E. Let $A \subset \mathbb{R}^N$ be an open set with $\mu(\partial A) = 0$. Let $(E_k)_{k\in\mathbb{N}} \subset \mathbb{R}^N$ be a sequence of sets of finite perimeter, and let $(u_k)_{k\in\mathbb{N}}$, with $u_k \in L^1(\partial E_k)$ be such that

(*i*) $E_k \cap A \to E \cap A$ in $L^1(\mathbb{R}^N)$;

(*ii*) $u_k \mathcal{H}^1 \llcorner (\partial^* E_k \cap A) \overset{*}{\rightharpoonup} \mu \llcorner A$.

Then,

$$\liminf_{k\to\infty} \int_{\partial^* E_k \cap A} \psi(u_k)\, \mathrm{d}\mathcal{H}^1 \geqslant \int_{\partial^* E \cap A} \widetilde{\psi}(u)\, \mathrm{d}\mathcal{H}^1 + \theta \mu^s(A),$$

where $\widetilde{\psi}$ is as in Definition 11.

We now prove a result that will be needed in the limsup inequality.

Lemma 8. Let $r > 0$, and let $\{z_j\}_{j\in\mathbb{N}}$ be an enumeration of $\mathbb{Z}^2$. Define

$$Q^j := r\left(z_j + (0,1)^2\right).$$

Let $h \in \mathrm{BV}(a,b)$, and let $(h_k)_k$ be a sequence of Lipschitz functions such that $\mathbb{R}^2 \setminus \Omega_{h_k} \xrightarrow{H} \mathbb{R}^2 \setminus \Omega_h$, as $k \to \infty$. Then, there exists $v \in \mathbb{R}^2$, and $\overline{k} \in \mathbb{N}$ such that the grid defined as

$$\widetilde{Q}^j := v + Q^j$$

satisfies:

(a) The intersection between the graph of h and the boundary of the new grid is finite, namely

$$\mathcal{H}^0\big(\Gamma \cap (\bigcup_{j\in\mathbb{N}} \partial\widetilde{Q}^j)\big) < +\infty.$$

(b) We have that

$$\mathcal{H}^1(\Gamma_k \cap \widetilde{Q}^j) \neq 0 \qquad \text{if and only if} \qquad \mathcal{H}^1(\Gamma \cap \widetilde{Q}^j) \neq 0,$$

for every $k \geqslant \bar{k}$.

Proof. We first prove (a). We first consider a horizontal translation. Since $h \in \mathrm{BV}(a,b)$, it has at most a countable number of jumps and cuts. Therefore, there is $v_1 \in \mathbb{R}$ such that

$$\mathcal{H}^0\big((\Gamma^j \cup \Gamma^c) \cap \big[\bigcup_{j\in\mathbb{N}} \partial\big((v_1,0) + Q^j\big)\big]\big) < +\infty.$$

Now we need to find a suitable vertical translation. Using the coarea formula (see [1, Theorem 3.40]), we infer that

$$\mathrm{Per}\big(\{x \in (a,b) : h(x) > t\}\big) < +\infty,$$

for almost every $t \in \mathbb{R}$, where Per denotes the perimeter. Since we are using the lower semi-continuous representative of h, the sup-level set

$\{x \in (a,b) : h(x) > t\}$ is open for all $t \in \mathbb{R}$, which yields that, for almost every $t \in \mathbb{R}$,

$$\partial\{x \in (a,b) : h(x) > t\} = \{x \in (a,b) : h(x) = t\}.$$

Thus, we obtain that

$$\mathcal{H}^0\big(\{x \in (a,b) : h(x) = t\}\big) < +\infty,$$

for almost every $t \in \mathbb{R}$. Let $D \subset \mathbb{R}$ defined as

$$D \coloneqq \big\{t > 0 : \mathcal{H}^0\big(\{x \in (a,b) : h(x) = t\}\big) = +\infty\big\}.$$

By definition, we have that $|D| = 0$. Let $r > 0$, and, for every $t > 0$, set

$$G(t) \coloneqq \{rj + t : j \in \mathbb{Z}\}.$$

We now claim that

$$|\{t \in [0,r) : G(t) \cap D \neq \emptyset\}| = 0.$$

First, note that if $s, t \in [0,r)$, with $s \neq t$, we have $G(t) \cap G(s) = \emptyset$. Now, define

$$\begin{aligned} D_j &\coloneqq D \cap [rj, (r+1)j], \\ \widetilde{D}_j &\coloneqq D_j - rj. \end{aligned}$$

By definition $\widetilde{D}_j \subset [0,r)$ and $|D_j| = |\widetilde{D}_j| = 0$, for every $j \in \mathbb{Z}$. In conclusion, we notice that

$$\{t \in (0,r) : G(t) \cap D \neq \emptyset\} = \bigcup_{j \in \mathbb{Z}} \widetilde{D}_j.$$

The claim follows from the above equality.
By proving the claim, we infer the existence of $v_2 \in \mathbb{R}$ such that

$$\mathcal{H}^0\big(\Gamma \cap \big[\bigcup_{j \in \mathbb{N}} \partial\big((0, v_2) + Q^j\big)\big]\big) < +\infty.$$

In conclusion the translation $v \coloneqq (v_1, v_2)$ is the one we were looking for.

We now prove part (b). Let $v \in \mathbb{R}^2$ be the vector found above, and let $\widetilde{Q}^j$ be the translated squares. If the graph of h is contained in a single square $\widetilde{Q}^j$, then there is nothing to prove. Thus, we assume that this is not the case.

Fix $j \in \mathbb{N}$ such that

$$\mathcal{H}^1(\Gamma \cap \widetilde{Q}^j) \neq 0.$$

We will prove that there exists $\bar{k}(j) \in \mathbb{N}$ such that

$$\mathcal{H}^1(\Gamma_k \cap \widetilde{Q}^j) \neq 0.$$

for all $k \geqslant \bar{k}(j)$. Let $x \in \Gamma \cap \widetilde{Q}^j$. By the Kuratowski convergence, there exists $(x_k)_k$ with $x_k \in \Gamma_k$ for all $k \in \mathbb{N}$ such that $x_k \to x$ as $k \to \infty$. Since $\widetilde{Q}^j$ is open, there exists $\bar{k}(j) \in \mathbb{N}$ (depending also on x, but this is not a problem) such that $x_k \in \Gamma_k \cap \widetilde{Q}^j$ for all $k \geqslant \bar{k}(j)$. Using the fact that the graph of h is not entirely contained in the open square Q^j, and that the extended graph of h_k is a connected curve, we obtain that

$$\mathcal{H}^1(\Gamma_k \cap \widetilde{Q}^j) \neq 0$$

as desired. Since $h \in \mathrm{BV}(a,b)$, it is bounded, and hence contained in a finite number of squares. In the following, we will also need to consider $\bar{k}_1 \in \mathbb{N}$, the maximum of the $\bar{k}(j)$'s.

We now prove the opposite implication. Let $j \in \mathbb{N}$ be such that

$$\mathcal{H}^1(\Gamma \cap \widetilde{Q}^j) = 0.$$

Then, by Kuratowski convergence and the fact that $\widetilde{Q}^j$ is open, we infer that there exists $\widetilde{k}(j) \in \mathbb{N}$ such that for all $k \geqslant \widetilde{k}(j)$ it holds

$$\mathcal{H}^1(\Gamma_k \cap \widetilde{Q}^j) = 0.$$

Again, let $\widetilde{k}_2 \in \mathbb{N}$ be the maximum of the $\widetilde{k}(j)$'s.

Setting $\bar{k} := \max\{\bar{k}_1, \widetilde{k}_2\}$, we get the desired result. □

Finally, we prove a result about the so called wriggling process. This was introduced in [18, Lemma 4] to exploit the quantitative loss of lower semi-continuity of the perimeter in order to recover the relaxed energy density from ψ. The difference with this latter is that, in our case, only vertical perturbations are allowed. Moreover, we impose the oscillating profiles to stay below the given function.

Proposition 3. Let $h : [\alpha, \beta] \to \mathbb{R}$ be a non-negative Lipschitz function and let $r \geqslant 1$. Then, there exists a sequence of non-negative Lipschitz functions $(h_k)_k$ such that:

(*i*) $\mathcal{H}^1(\Gamma_k) = r\mathcal{H}^1(\Gamma)$;

(*ii*) $h(\alpha) = h_k(\alpha)$, and $h(\beta) = h_k(\beta)$, for every k;

(*iii*) $h \leqslant h_k$, for every k;

(*iv*) $h_k \to h$ uniformly as $k \to \infty$;

(*v*) $\mathcal{H}^1 \llcorner \Gamma_k \overset{*}{\rightharpoonup} r\mathcal{H}^1 \llcorner \Gamma$, as $k \to \infty$,

Proof. Step 1. Fix $\alpha \leqslant p \leqslant q \leqslant \beta$. We prove the existence of a sequence $(\xi_k)_k$ of Lipschitz functions $\xi_k : [p, q] \to [0, +\infty)$, that satisfies

(*i'*) $\mathcal{H}^1(\Gamma_{\xi_k}) = r\mathcal{H}^1(\Gamma)$;

(*ii'*) $h(p) = \xi_k(p)$, and $h(q) = \xi_k(q)$, for every k;

(*iii'*) $h \leqslant \xi_k$, for every k;

(*iv'*) $\xi_k \to h$ uniformly on $[p, q]$, as $k \to \infty$,

Notice that if $r = 1$ it is enough to consider the constant sequence $\xi_k = h$, for each k. Thus, fix $r > 1$. Let $(\lambda_k)_k \subset (0, 1)$ be an infinitesimal sequence such that $0 < \lambda_k < q - p$ for each $k \in \mathbb{N}$, and $k\lambda_k \to \infty$ as $k \to \infty$. For each $k \in \mathbb{N}$, define the function $\eta_k \in \mathcal{C}\big([p, q]\big)$ as

$$\eta_k(x) := \begin{cases} \dfrac{x - p}{\lambda_k} & x \in [p, p + \lambda_k), \\ 1 & x \in [p + \lambda_k, q - \lambda_k], \\ -\dfrac{x - q}{\lambda_k} & x \in (q - \lambda_k, q]. \end{cases}$$

For each $k \in \mathbb{N}$, let $t_k \geqslant 0$ that will be chosen later, and define the non-negative Lipschitz function $\xi_k : [p,q] \to [0,+\infty)$ as

$$\xi_k(x) \coloneqq h(x) + \Big(\frac{2}{k} - \frac{1}{k}|\sin(t_k x)|\Big)\eta_k(x). \tag{27}$$

First of all, note that $\xi_k \to h$ uniformly as $k \to \infty$. Moreover, from (27), we get that

$$0 \leqslant h \leqslant \xi_k, \qquad h(p) = \xi_k(p), \qquad h(q) = \xi_k(q).$$

We claim that it is possible to choose $t_k > 0$ such that $\mathcal{H}^1(\Gamma_{\xi_k}) = r\mathcal{H}^1(\Gamma)$, for every $k \in \mathbb{N}$. In order to show that, for each $k \in \mathbb{N}$, let $f_k : [0,+\infty) \to (0,+\infty)$ be defined as

$$f_k(t) \coloneqq \int_p^q \sqrt{1 + \partial_x H_k(x,t)^2}\,\mathrm{d}x,$$

where

$$H_k(x,t) \coloneqq h(x) + \Big(\frac{2}{k} - \frac{1}{k}|\sin(tx)|\Big)\eta_k(x). \tag{28}$$

We claim that:

(a) $\lim_{t\to+\infty} f_k(t) = +\infty$, for every $k \in \mathbb{N}$;

(b) $\lim_{k\to\infty} f_k(0) = \mathcal{H}^1(\Gamma)$.

Therefore, since f_k is continuous for every $k \in \mathbb{N}$, and $r > 1$, it is possible to chose $t_k > 0$ such that $f_k(t_k) = \mathcal{H}^1(\Gamma_{\xi_k}) = r\mathcal{H}^1(\Gamma)$, for every $k \in \mathbb{N}$. We now prove claims (a) and (b) in two separate sub-steps.

Step 1.1. We now prove claim (a). First, notice that

$$f_k(t) = \int_p^q \sqrt{1 + \partial_x H_k(x,t)^2}\,\mathrm{d}x \geqslant \int_{p+\lambda_k}^{q-\lambda_k} \sqrt{1 + \partial_x H_k(x,t)^2}\,\mathrm{d}x.$$

Now, fix $k \in \mathbb{N}$ and consider the set

$$Z_t \coloneqq \{x \in (p+\lambda_k, q-\lambda_k) : \cos(tx) \geqslant 1/2\}.$$

We now prove that

$$\inf_{t>0} |Z_t| > 0. \tag{29}$$

In order to do so, we first show that $|Z_n| > 0$, for $n \in \mathbb{N}$. Set $I := (p + \lambda_k, q - \lambda_k)$ and consider the function $g : I \to \{0, 1\}$ defined as

$$g(x) := \mathbb{1}_{\{\cos(y) \geqslant 1/2\}}(x),$$

and extend it periodically on $\mathbb{R}$. Notice that, for $n \in \mathbb{N}$,

$$g(nx) = \mathbb{1}_{\{\cos(ny) \geqslant 1/2\}}(x).$$

By applying the Riemann-Lebesgue Lemma, we get that

$$\begin{aligned} |Z_n| &= |\{\cos(nx) \geqslant 1/2\} \cap I| \\ &= \int_I g(nx)\,\mathrm{d}x \to \frac{1}{|I|}\int_I g(x)\,\mathrm{d}x > 0, \end{aligned} \tag{30}$$

as $n \to \infty$. Now, we use the above result to show (29). Let $t \in (n, n+1)$. We have that

$$|Z_t| = |\{\cos(tx) \geqslant 1/2\} \cap I|$$

and that

$$\int_I g(tx)\,\mathrm{d}x = \frac{1}{t}\int_{tI} g(z)\,\mathrm{d}z.$$

As

$$g(z) = \sum_{m\in\mathbb{Z}} \mathbb{1}_{\{-\frac{\pi}{3}+2m\pi \leqslant y \leqslant \frac{\pi}{3}+2m\pi\}}(z), \tag{31}$$

we can define the following families of intervals. Set

$$\mathcal{A}_t := \{J \subset \mathbb{R} : J \cap tI \neq \emptyset\} \qquad \mathcal{B}_t := \{J \subset \mathbb{R} : J \subset tI\}.$$

Then, by (31), we have

$$\frac{2\pi}{3t}\mathcal{H}^0(\mathcal{B}_t) \leqslant |Z_t| \leqslant \frac{2\pi}{3t}\mathcal{H}^0(\mathcal{A}_t). \tag{32}$$

Since $t \in (n, n+1)$ and by (30) and (32), we get that

$$|Z_t| \geqslant \frac{2\pi}{3(n+1)}\mathcal{H}^0(\mathcal{B}_n) = \frac{2\pi}{3(n+1)}(\mathcal{H}^0(\mathcal{A}_n) - 2)$$

$$\geqslant \int_I g(nx)\,\mathrm{d}x - \frac{4\pi}{3(n+1)} > C - \frac{4\pi}{3(n+1)},$$

where $C > 0$ is a constant independent of n. We conclude our claim by letting $n \to \infty$.

Note that for every $t > 0$, on Z_t we have $\eta_k(x) = 1$ and $\cos(tx) > 1/2$. Thus, we get that

$$\begin{aligned} f_k(t) &\geqslant \int_{Z_t} \sqrt{1 + h'(x)^2 + \frac{t}{k}\cos(tx)\left[\frac{t}{k}\cos(tx) - 2\ell\right]}\,\mathrm{d}x \\ &\geqslant \int_{Z_t} \sqrt{1 + h'(x)^2 + \frac{t}{k}\cos(tx)\left[\frac{t}{2k} - 2\ell\right]}\,\mathrm{d}x \\ &\geqslant \int_{Z_t} \sqrt{1 + \frac{t}{k}\cos(tx)\left[\frac{t}{2k} - 2\ell\right]}\,\mathrm{d}x, \end{aligned} \tag{33}$$

where ℓ is the Lipschitz constant of h. By choosing $t > 0$ such that

$$t > 4k\ell,$$

from (33), and from $\cos(tx) > 1/2$ on Z_t, we obtain

$$f_k(t) \geqslant \int_{Z_t} \sqrt{1 + \frac{t}{2k}\left[\frac{t}{2k} - 2\ell\right]}\,\mathrm{d}x. \tag{34}$$

Thus, from (29) and (34), we conclude that

$$\lim_{t\to+\infty} f_k(t) = +\infty.$$

Step 1.2 Now we prove claim (b). Notice that

$$\begin{aligned} f_k(0) = &\int_p^{p+\lambda_k} \sqrt{1 + \left(h'(x) + \frac{2}{k\lambda_k}\right)^2}\,\mathrm{d}x + \int_{p+\lambda_k}^{q-\lambda_k} \sqrt{1 + h'(x)^2}\,\mathrm{d}x \\ &+ \int_{q-\lambda_k}^{q} \sqrt{1 + \left(h'(x) + \frac{2}{k\lambda_k}\right)^2}\,\mathrm{d}x. \end{aligned} \tag{35}$$

Since the sequence $(\lambda_k)_k$ is such that $k\lambda_k \to \infty$, and $\|h'\|_{L^\infty} < +\infty$ since h is Lipschitz, it holds that

$$\sup_{k\in\mathbb{N}} \sup_{x\in[p,q]} \left| h'(x) + \frac{2}{k\lambda_k} \right| < +\infty.$$

Thus, letting $k \to \infty$ in (35), we obtain

$$\lim_{k\to\infty} f_k(0) = \mathcal{H}^1(\Gamma).$$

This concluses the proof of (b).

Step 2. We now prove the statement of the Lemma. Fix $r > 1$, otherwise the statement is trivial. For $k \in \mathbb{N}$, divide the interval $[\alpha, \beta]$ into k subintervals $\big([\alpha_i^k, \alpha_{i+1}^k]\big)_{i=1}^k$, where $\alpha_k^1 = \alpha$ and $\alpha_{k+1}^k = \beta$. Assume that $|\alpha_{i+1}^k - \alpha_i^k| < 2/k$. Thanks to Step 1, for each $k \in \mathbb{N}$, and each $i \in \{1, \dots, k\}$, there exists a function $\xi_i^k : [\alpha_i^k, \alpha_{i+1}^k] \to [0, +\infty)$ such that

$$\xi_1^k(\alpha) = h(\alpha), \qquad \xi_i^k(\alpha_{i+1}^k) = \xi_{i+1}^k(\alpha_{i+1}^k), \qquad \xi_{k+1}^k(\beta) = h(\beta),$$

for all $i \in \{2, \dots, k\}$, with

$$\|\xi_i^k - h\|_{\mathcal{C}^0(\mathbb{R})} \leqslant \frac{1}{k},$$

and such that

$$\mathcal{H}^1(\mathrm{graph}(\xi_i^k)) = r\mathcal{H}^1(\Gamma \llcorner [\alpha_i^k, \alpha_{i+1}^k] \times \mathbb{R}),$$

for all $i \in \{1, \dots, k\}$, and all $k \in \mathbb{N}$. Define $h_k : [\alpha, \beta] \to [0, +\infty)$ as

$$h_k(x) \coloneqq \xi_i^k(x),$$

for $x \in [\alpha_i^k, \alpha_{i+1}^k]$. Note that h_k is Lipschitz, $h \leqslant h_k$ for all $k \in \mathbb{N}$, $h_k \to h$ uniformly in k, and

$$\mathcal{H}^1(\Gamma_k) = \sum_{i=1}^k \mathcal{H}^1\big(\mathrm{graph}(\xi_i^k)\big) = r\sum_{i=1}^k \mathcal{H}^1(\Gamma \llcorner [\alpha^i, \alpha^{i+1}] \times \mathbb{R}) = r\mathcal{H}^1(\Gamma).$$

It remains to prove property (v). To do so, fix $\varphi \in C_c(\mathbb{R}^2)$ and $\varepsilon > 0$. Thanks to the uniform continuity of φ, there exists $\bar{k} \in \mathbb{N}$ such that for $k \geqslant \bar{k}$ the following holds: if $x_i \in [\alpha_i^k, \alpha_{i+1}^k]$, then

$$|\varphi\big(x, h_k(x)\big) - \varphi\big(x_i, h_k(x_i)\big)| \leqslant \varepsilon. \tag{36}$$

Moreover, from the fact that h_k is converging uniformly to the continuous function h, up to increasing the value of $\bar{k}$, we can also assume that

$$|\varphi\big(x_i, h_k(x_i)\big) - \varphi\big(x_i, h(x_i)\big)| \leqslant \varepsilon. \tag{37}$$

Using (36), we get

$$\begin{aligned}
&\int_{\Gamma_k} \varphi(\mathbf{x})\, \mathrm{d}\mathcal{H}^1 - r\int_{\Gamma} \varphi(\mathbf{x})\, \mathrm{d}\mathcal{H}^1 \\
&= \sum_{i=1}^{k} \int_{\alpha^i}^{\alpha^{i+1}} \Big[\varphi\big(x, h_k(x)\big)\sqrt{1+h_k'(x)^2} - r\varphi\big(x, h(x)\big)\sqrt{1+h'(x)^2}\Big]\mathrm{d}x \\
&\leqslant \sum_{i=1}^{k} \Big[\varepsilon \int_{\alpha^i}^{\alpha^{i+1}} \Big(\sqrt{1+h_k'(x)^2} + r\sqrt{1+h'(x)^2}\Big)\mathrm{d}x \\
&\qquad + \int_{\alpha^i}^{\alpha^{i+1}} \Big(\varphi\big(x_i, h_k(x_i)\big)\sqrt{1+h_k'(x)^2} \\
&\qquad\qquad - r\varphi\big(x_i, h(x_i)\big)\sqrt{1+h'(x)^2}\Big)\mathrm{d}x\Big] \\
&\leqslant \varepsilon \sum_{i=1}^{k} \int_{\alpha^i}^{\alpha^{i+1}} \Big(\sqrt{1+h_k'(x)^2} + r\sqrt{1+h'(x)^2}\Big)\mathrm{d}x \\
&\qquad + \varphi\big(x_i, h(x_i)\big) \sum_{i=1}^{k} \Big[\int_{\alpha^i}^{\alpha^{i+1}} \Big(\sqrt{1+h_k'(x)^2} - r\sqrt{1+h'(x)^2}\Big)\mathrm{d}x\Big] \\
&= \varepsilon \sum_{i=1}^{k} \int_{\alpha^i}^{\alpha^{i+1}} \Big(\sqrt{1+h_k'(x)^2} + r\sqrt{1+h'(x)^2}\Big)\mathrm{d}x, \qquad (38)
\end{aligned}$$

where in the previous to last step we used (37), while last step follows from $\mathcal{H}^1(\Gamma_k) = r\mathcal{H}^1(\Gamma)$. Thus, from (38) we obtain

$$\int_{\Gamma_k} \varphi(\mathbf{x})\, \mathrm{d}\mathcal{H}^1 - r\int_{\Gamma} \varphi(\mathbf{x})\, \mathrm{d}\mathcal{H}^1 \leqslant 2r\mathcal{H}^1(\Gamma)\varepsilon.$$

Thus, since ε is arbitrary, we get that $\mathcal{H}^1 \llcorner \Gamma_k \overset{*}{\rightharpoonup} r\mathcal{H}^1 \llcorner \Gamma$ as $k \to \infty$. $\square$

Remark 7. From the above proof, we can infer the following facts:

(i) Following (34),

$$r\mathcal{H}^1(\Gamma) \geqslant \int_{Z_{t_k}} \sqrt{1 + h'(x)^2 + \frac{t_k}{2k}\Big[\frac{t_k}{2k} - 2\ell\Big]}\,\mathrm{d}x \geqslant \mu\sqrt{\frac{t_k}{2k}\Big[\frac{t_k}{2k} - 2\ell\Big]},$$

where $\mu \coloneqq \inf_{t\geqslant 0}|Z_t|$. This leads us to

$$\Big(\frac{t_k}{2k}\Big)^2 - 2\ell\Big(\frac{t_k}{2k}\Big) \leqslant \frac{1}{\mu^2}r^2\mathcal{H}^1(\Gamma)^2.$$

If we solve for $t/2k$ we get

$$\frac{t_k}{k} \leqslant C, \tag{39}$$

where

$$C \coloneqq 2\Big(\ell + \sqrt{\ell^2 + \frac{r^2\mathcal{H}^1(\Gamma)^2}{\mu^2}}\Big).$$

(ii) We claim that $t_k \to +\infty$ as $k \to \infty$. Assume by contradiction this is not the case, namely that

$$\sup_k t_k \leqslant \tau,$$

for some $\tau > 0$. Thus, we have that

$$\begin{aligned} h_k'(x) &= h'(x) - \frac{t_k}{k}\cos(t_k x)\frac{|\sin(t_k x)|}{\sin(t_k x)}\eta_k(x) \\ &\quad + \Big(\frac{2}{k} - \frac{1}{k}|\sin(t_k x)|\Big)\eta_k'(x) \\ &\leqslant h'(x) + \frac{\tau}{k} + \frac{2\eta_k'(x)}{k}, \end{aligned}$$

for every k. From the inequality

$$|h_k'(x) - h'(x)| \leqslant \frac{\tau}{k} + \frac{2\eta_k'(x)}{k}$$

we infer that

$$\mathcal{H}^1(\Gamma_k) \to \mathcal{H}^1(\Gamma). \tag{40}$$

From step 1 we know that

$$\mathcal{H}^1(\Gamma_k) = r\mathcal{H}^1(\Gamma) > \mathcal{H}^1(\Gamma), \tag{41}$$

with $r > 1$ and for every k. By putting together (40) and (41) we get a contradiction.

(*iii*) From the expression of h'_k, we can actually choose the sequence $(\lambda_k)_k$ such that the sequence $(h_k)_k$ is uniformly Lipschitz. Indeed, on $[\alpha, \alpha + \lambda_k]$ we have

$$|h'_k(x)| \leqslant \ell + \frac{t_k}{k} + \frac{2}{k\lambda_k}.$$

As t_k/k is bounded and $(\lambda_k)_k$ is chosen such in such a way that $k\lambda_k \to +\infty$ as $k \to \infty$, we can conclude.

1.6 Setting

In this section we give the rigorous definitions of the objects discussed in the introduction. We start with the set of admissible configurations.

Definition 15. We say that the triplet (Ω, v, μ) is an *admissible sharp configuration* if Ω is the subgraph of $h \in \mathrm{BV}(a, b)$, $v \in H^1(\Omega; \mathbb{R}^2)$ and $\mu = u\mathcal{H}^1 \llcorner \Gamma + \mu^s$, with $u \in L^1(\Gamma)$. We denote the set of admissible sharp configurations by $\mathcal{A}$.

Definition 16. An admissible sharp configuration (Ω, v, μ) is called *regular* if Ω is the subgraph of a Lipschitz function and $\mu = u\mathcal{H}^1 \llcorner \Gamma$ with $u \in L^1(\Gamma)$. The set of such configurations is denoted by $\mathcal{A}_r$.

The next definition introduces configurations which satisfy the mass constraints.

Definition 17. Given $M, m > 0$, we say that $(\Omega, v, \mu) \in \mathcal{A}(m, M)$ if $(\Omega, v, \mu) \in \mathcal{A}$ and

$$\int_a^b h(x)\,\mathrm{d}x = M \quad \text{and} \quad \mu(\mathbb{R}^2) = m. \tag{42}$$

In a similar way we say that $(\Omega, v, \mu) \in \mathcal{A}_r(m, M)$ if $(\Omega, v, \mu) \in \mathcal{A}_r$ and (42) holds.

Now, we are in position to give the definitions of the functional treated in [28]. Let $\psi : [0, \infty) \to (0, \infty)$ be a Borel function with $\inf_{s \geqslant 0} \psi(s) > 0$ and set

$$\mathcal{E}(\Omega, v, u) \coloneqq \int_\Omega W\big(E(v) - E_0(y)\big)\,\mathrm{d}\mathbf{x} + \int_\Gamma \psi(u)\,\mathrm{d}\mathcal{H}^1,$$

Definition 18. We define the functional $\mathcal{H} : \mathcal{A} \to [0, +\infty]$ as

$$\mathcal{H}(\Omega, v, \mu) \coloneqq \begin{cases} \mathcal{E}(\Omega, v, u) & (\Omega, v, \mu) \in \mathcal{A}_r, \\ \\ +\infty & \text{else.} \end{cases}$$

and $\mathcal{H}^{m,M} : \mathcal{A} \to [0, +\infty]$ as

$$\mathcal{H}^{m,M}(\Omega, v, \mu) \coloneqq \begin{cases} \mathcal{E}(\Omega, v, u) & (\Omega, v, \mu) \in \mathcal{A}_r(m, M), \\ \\ +\infty & \text{else.} \end{cases}$$

Definition 19. We define the functional $\mathcal{F} : \mathcal{A} \to [0, +\infty]$ as

$$\mathcal{F}(\Omega, v, \mu) \coloneqq \begin{cases} \mathcal{G}(\Omega, v, \mu) & \text{if } (\Omega, v, \mu) \in \mathcal{A}, \\ +\infty & \text{else,} \end{cases}$$

where

$$\mathcal{G}(\Omega, v, \mu) \coloneqq \int_\Omega W\big(E(v) - E_0(y)\big)\,\mathrm{d}\mathbf{x} + \int_{\widetilde{\Gamma}} \widetilde{\psi}(u)\,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma^c} \psi^c(u)\,\mathrm{d}\mathcal{H}^1 + \theta\mu^s(\mathbb{R}^2),$$

where we recall that

$$\theta = \lim_{s \to +\infty} \frac{\widetilde{\psi}(s)}{s},$$

and that $\widetilde{\psi}$, ψ^c are given in Definitions 11 and 12 respectively. Moreover, we define the functional $\mathcal{F}^{m,M} : \mathcal{A} \to [0, +\infty]$ as

$$\mathcal{F}^{m,M}(\Omega, v, \mu) := \begin{cases} \mathcal{G}(\Omega, v, \mu) & \text{if } (\Omega, v, \mu) \in \mathcal{A}(m, M), \\ +\infty & \text{else.} \end{cases}$$

We now define the notion of convergence that we are going to use to study our functionals.

Definition 20. We say that a sequence $(\Omega_k, v_k, \mu_k)_k \subset \mathcal{A}$ converges to $(\Omega, v, \mu) \in \mathcal{A}$ if the following three conditions are satisfied:

(i) $\mathbb{R}^2 \setminus \Omega_k \xrightarrow{H} \mathbb{R}^2 \setminus \Omega$ in the Hausdorff convergence of sets;

(ii) $v_k \rightharpoonup v$ weakly in $H^1_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$,

(iii) $\mu_k \overset{*}{\rightharpoonup} \mu$ weakly* in the sense of measures;

as $k \to \infty$. We will write $(\Omega_k, v_k, \mu_k) \to (\Omega, v, \mu)$ to denote the above convergence.

Remark 8. Note that, if $K \subset \Omega$ is a compact set, then there exists $k_0 \in \mathbb{N}$ such that $K \subset \Omega_k$ for all $k \geqslant k_0$. Therefore, the convergence of the functions v_k's is well defined.

1.7 Liminf inequality

We now present the main ideas of the proof of the liminf inequality, contained in the following theorem. One of the issues that we take into account is the fact that our final configuration Γ, is the graph of a BV function which might have a dense cut set. In particular, this is a problem since in our argument we deal with what is happening on the left and on the right of every cut in Γ. This is not doable in case the cut set is dense. One possible way to go around, is to split the energy on Γ^c. By fixing $\varepsilon > 0$, since h is a BV function, the cuts in Γ^c whose lengths are larger than ε is necessarily finite. For those amount of cuts we do the liminf inequality by using the result contained in [17]. Finally, for the portion of the cut in Γ^c with length smaller than ε, we prove that the energy there can be made arbitrarily small as $\varepsilon \to 0$.

Theorem 9. For every sharp configuration $(\Omega, v, \mu) \in \mathcal{A}$ and for every sequence of regular sharp configurations $(\Omega_k, v_k, \mu_k)_k \subset \mathcal{A}_r$ such that $(\Omega_k, v_k, \mu_k) \to (\Omega, v, \mu)$ as $k \to \infty$, we have

$$\mathcal{F}(\Omega, v, \mu) \leqslant \liminf_{k\to\infty} \mathcal{H}(\Omega_k, v_k, \mu_k).$$

Proof. Fix $\varepsilon > 0$ and consider the set

$$C_\varepsilon \coloneqq \{\mathbf{x} = (x, y) \in \Gamma^c : h^-(x) - y < \varepsilon\}.$$

By a standard measure theory argument, it is possible to choose ε such that $\mu(\Gamma \cap \partial C_\varepsilon) = 0$. As a consequence, from Lemma 2, we have that $\Gamma^c \setminus C_\varepsilon$ consists of a finite number of vertical segments, whose projections on the x-axis corresponds to the set $(x^i)_{i=1}^N$. Recalling the definition of Γ^c (see Definition 6), it holds that C_ε is monotonically converging to the empty set, as $\varepsilon \to 0$. Therefore, we get that

$$\mu(C_\varepsilon) \to 0, \qquad\qquad \mu(\Gamma^c \setminus C_\varepsilon) \to \mu(\Gamma^c), \tag{43}$$

as $\varepsilon \to 0$. Let $\delta = \delta(\varepsilon) > 0$ such that $\delta \to 0$, as $\varepsilon \to 0$, and $\delta < |x^i - x^j|$, for every $i, j = 1, \dots, N$. As we have a finite number of cuts, in order to simplify the notation, we do the following construction as we had only one cut point, and then we repeat it for each other one.

Fix $i \in 1, \dots, N$. Since $\mathbb{R}^2 \setminus \Omega_k \overset{H}{\to} \mathbb{R}^2 \setminus \Omega$, for every cut point $\big(x^i, h(x^i)\big)$, there is a sequence of the form $\big(x_k, h_k(x_k)\big)_k$ such that $(x_k)_k \subset (x^i - \delta, x^i + \delta)$ and $\big(x_k, h_k(x_k)\big) \to \big(x^i, h(x^i)\big)$ as $k \to \infty$. Indeed, by Proposition 2 there is a sequence $(x_k, y_k)_k \subset \mathbb{R}^2 \setminus \Omega_k$ such that $(x_k, y_k) \to \big(x^i, h(x^i)\big)$. By definition, we have that $h_k(x_k) \leqslant y_k$, up to a subsequence (not relabelled), we have that $\big(x_k, h_k(x_k)\big) \to (x^i, z^i)$, for some $z^i \in \mathbb{R}$. We would like to have $z^i = h(x^i)$. If we had $z^i > h(x^i)$, then

$$\lim_{k\to\infty} h_k(x_k) \leqslant h(x^i) < z^i,$$

which contradicts our convergence above. Vice versa, if $z^i < h(x^i)$, then $(x^i, z^i) \notin \mathbb{R}^2 \setminus \Omega$. In conclusion we have $z^i = h(x^i)$ and thus $\big(x_k, h_k(x_k)\big) \to \big(x^i, h(x^i)\big)$, as $k \to \infty$.

Around each vertical cut, we set, for each $k \in \mathbb{N}$ (see Figure 1.4),

$$R_k^\ell \coloneqq (x^i - \delta, x_k) \times (-\delta, h^-(x^i) - \varepsilon), \qquad R_k^r \coloneqq (x_k, x^i + \delta) \times (-\delta, h^-(x^i) - \varepsilon),$$

and

$$\mathcal{R}_\delta^\varepsilon \coloneqq R_k^\ell \cup R_k^r \cup \big[\{x_k\} \times (-\delta, h^-(x^i) - \varepsilon)\big],$$

Thanks to the existence of the right and left limits of h at every point (see Theorem 5), up to further reducing δ, we can assume that

$$\mathcal{R}_\delta^\varepsilon \cap \Gamma = \{x_k\} \times [h(x^i), h^-(x^i) - \varepsilon).$$

Now we split the energy in the following way. Take any $(\Omega_k, v_k, \mu_k)_k \subset \mathcal{A}_r$ such that $(\Omega_k, v_k, \mu_k) \to (\Omega, v, \mu)$ as $k \to \infty$. We have

$$\begin{aligned}
\liminf_{k\to\infty} \Big[\int_{\Omega_k} W\big(E(v_k) - E_0(y)\big)\,\mathrm{d}\mathbf{x} + \int_{\Gamma_k} \psi(u_k)\ \mathrm{d}\mathcal{H}^1 \Big] \\
\geqslant \liminf_{k\to\infty} \int_{\Omega_k} W\big(E(v_k) - E_0(y)\big)\,\mathrm{d}\mathbf{x} \\
+ \liminf_{k\to\infty} \int_{\Gamma_k \setminus \mathcal{R}_\delta^\varepsilon} \psi(u_k)\ \mathrm{d}\mathcal{H}^1 + \liminf_{k\to\infty} \int_{\Gamma_k \cap \mathcal{R}_\delta^\varepsilon} \psi(u_k)\ \mathrm{d}\mathcal{H}^1.
\end{aligned} \tag{44}$$

We are going to estimate each term on the right-hand side of (44) separately.

Step 1. Here we estimate the bulk term on the right-hand side of (44). Since $v_k \rightharpoonup v$ in $H^1_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ as $k \to \infty$, for every compactly contained set $K \subset \Omega$, we get

$$\liminf_{k\to\infty} \int_{\Omega_k} W\big(E(v_k) - E_0(y)\big)\,\mathrm{d}\mathbf{x} \geqslant \liminf_{k\to\infty} \int_K W\big(E(v_k) - E_0(y)\big)\,\mathrm{d}\mathbf{x}$$

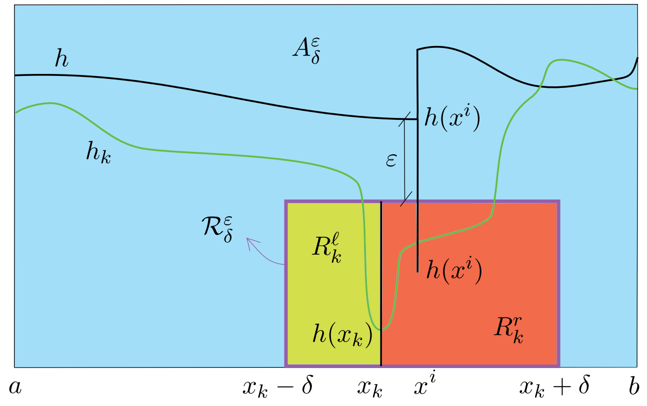

Figure 1.4: The rectangles we are using for the estimate of the liminf, in case $h(x^i) > 0$. In particular, the set A^ε_δ is the light blue, while the boundary of the rectangle $\mathcal{R}^\varepsilon_\delta$ is the one in purple. We remark that, in general, we need to consider rectangles that go below $\{y = 0\}$, as a cut might touch the y-axes and a singular measure (for instance a Dirac delta) might be present at the endpoint of such a cut.

$$\geqslant \int_K W\big(E(v) - E_0(y)\big)\, \mathrm{d}\mathbf{x},$$

as $E(v_k) \rightharpoonup E(v)$ weakly in $L^2(K)$, as $k \to \infty$, and $W(\cdot)$ is convex. Since K is arbitrary, we can conclude by taking an increasing sequence $(K_j)_j$ of sets compactly contained in Ω with $|\Omega \setminus K_j| \to 0$ as $k \to \infty$. Thus, sending $j \to \infty$ and by using the Monotone Convergence Theorem, we obtain

$$\liminf_{k\to\infty} \int_{\Omega_k} W\big(E(v_k) - E_0(y)\big)\, \mathrm{d}\mathbf{x} \geqslant \int_\Omega W\big(E(v) - E_0(y)\big)\, \mathrm{d}\mathbf{x}. \tag{45}$$

Therefore, we get the liminf inequality for the bulk term.

Step 2. For the second term on the right-hand side of (44), we would like to apply Theorem 8. Fix $\varepsilon > 0$. By knowing that for each $k \in \mathbb{N}$ we have $|h_k| \leqslant M$, we define the open set

$$A^\varepsilon_\delta := \big([a, b] \times [0, M]\big) \setminus \overline{\mathcal{R}}^\varepsilon_\delta.$$

We have that $A^\varepsilon_\delta \cap \Omega_k \to A^\varepsilon_\delta \cap \Omega$ in L^1 as $k \to \infty$. From Lemma 1, we have that

$$\mathcal{H}^1\big((\partial^*\Omega \cap A^\varepsilon_\delta)\Delta\widetilde{\Gamma}\big) = 0.$$

By definition, we can write

$$u_k\mathcal{H}^1 \llcorner (\partial\Omega_k \cap A^\varepsilon_\delta) \overset{*}{\rightharpoonup} \mu \llcorner A^\varepsilon_\delta = u\mathcal{H}^1 \llcorner \widetilde{\Gamma} + \mu^s \llcorner A^\varepsilon_\delta + u\mathcal{H}^1 \llcorner C_\varepsilon,$$

as $k \to \infty$, and, by applying Theorem 8, we have

$$\begin{aligned}\liminf_{k\to\infty} \int_{\partial\Omega_k \cap A^\varepsilon_\delta} \psi(u_k)\ \mathrm{d}\mathcal{H}^1 \geqslant \int_{\widetilde{\Gamma}\cap A^\varepsilon_\delta} \widetilde{\psi}(u)\ \mathrm{d}\mathcal{H}^1 \\ + \theta\mu^s(A^\varepsilon_\delta) + \theta \int_{C_\varepsilon} u\ \mathrm{d}\mathcal{H}^1,\end{aligned} \tag{46}$$

as desired.

Step 3. We now deal with the third term on the right-hand side of (44). Define

$$E^\ell_k \coloneqq \Omega_k \cap R^\ell_k \quad \text{and} \quad E^r_k \coloneqq \Omega_k \cap R^r_k. \tag{47}$$

Using Lemma 4 we obtain that

$$\begin{aligned} E^\ell_k \to R^\ell = \coloneqq (x^i - \delta, x^i) \times (-\delta, h^-(x^i) - \varepsilon), \\ E^\ell_k \to R^r \coloneqq (x^i, x^i + \delta) \times (-\delta, h^-(x^i) - \varepsilon), \end{aligned}$$

as $k \to \infty$ in L^1. Note that, for every k large enough, both $E^\ell_k \neq \emptyset$ and $E^r_k \neq \emptyset$. Furthermore, notice that

$$\begin{aligned} \partial E^\ell_k \cap R^\ell = \big(\Gamma_k \cap R^\ell_k\big) \cup \big[\{x_k\} \times \big(-\delta, h_k(x_k)\big)\big], \\ \partial E^r_k \cap R^r = \big(\Gamma_k \cap R^r_k\big) \cup \big[\{x_k\} \times \big(-\delta, h_k(x_k)\big)\big]. \end{aligned}$$

We now define the densities

$$u^\ell_k(\mathbf{x}) \coloneqq \begin{cases} u_k(\mathbf{x}) & \mathbf{x} \in \Gamma_k \cap R^\ell_k, \\ 0 & \mathbf{x} \in \{x_k\} \times (-\delta, h_k(x_k)), \end{cases}$$

$$u_k^r(\mathbf{x}) := \begin{cases} u_k(\mathbf{x}) & \mathbf{x} \in \Gamma_k \cap R_k^r, \\ 0 & \mathbf{x} \in \{x_k\} \times (-\delta, h_k(x_k)). \end{cases}$$

We now prove that that

$$\mu_k^\ell := u_k^\ell \mathcal{H}^1 \llcorner (\partial E_k^\ell \cap R^\ell) \overset{*}{\rightharpoonup} \mu^\ell := f \mathcal{H}^1 \llcorner (\Gamma^c \setminus C_\varepsilon) + (\mu^\ell)^s,$$
$$\mu_k^r := u_k^r \mathcal{H}^1 \llcorner (\partial E_k^r \cap R^r) \overset{*}{\rightharpoonup} \mu^r := g \mathcal{H}^1 \llcorner (\Gamma^c \setminus C_\varepsilon) + (\mu^r)^s,$$

for some $f, g \in L^1(\Gamma^c \setminus C_\varepsilon)$ such that

$$f + g = u_{|\Gamma^c \setminus C_\varepsilon}, \tag{48}$$

and

$$(\mu^\ell)^s + (\mu^r)^s = \mu^s, \tag{49}$$

where $(\mu^\ell)^s$ and $(\mu^r)^s$ are supported in $\Gamma^c \setminus C_\varepsilon$. Notice that

$$\mu_k^\ell\big(\{x_k\} \times (-\delta, h_k(x_k))\big) = \mu_k^r\big(\{x_k\} \times (-\delta, h_k(x_k))\big) = 0$$

holds for every $k \in \mathbb{N}$. By definition we have $\mu_k^\ell + \mu_k^r = \mu_k$, for every $k \in \mathbb{N}$. Moreover, for every set A, measurable with respect to μ_k (thus also for μ_k^ℓ and μ_k^r), we have

$$\mu_k^\ell(A) \leqslant \mu_k(A) = \int_{\Gamma_k \cap A} u_k \, \mathrm{d}\mathcal{H}^1 = ||u_k||_{L^1(\Gamma_k \cap A)} \leqslant L,$$

where L is a constant independent of A, and is given by the fact that the sequence $(\mu_k)_k$ is weakly* converging. The same bound for μ_k^r also holds. We have that, up to a subsequence (not relabelled), there are two Radon measures μ^ℓ and μ^r such that

$$\mu_k^\ell \overset{*}{\rightharpoonup} \mu^\ell \quad \text{and} \quad \mu_k^r \overset{*}{\rightharpoonup} \mu^r,$$

as $k \to \infty$.

We claim that $\mathrm{supp}(\mu^\ell) \subset \Gamma^c \setminus C_\varepsilon$ and $\mathrm{supp}(\mu^r) \subset \Gamma^c \setminus C_\varepsilon$. Indeed, take any set A such that $\mu\big((\Gamma^c \setminus C_\varepsilon) \cap \partial A\big) = 0$ and $A \cap (\Gamma^c \setminus C_\varepsilon) = \emptyset$.

Then $\mu\big((\Gamma^c \setminus C_\varepsilon) \cap A\big) = 0$. If we had $\mu^\ell\big((\Gamma^c \setminus C_\varepsilon) \cap A\big) > \mu\big((\Gamma^c \setminus C_\varepsilon) \cap A\big)$, we would have

$$\begin{aligned} \mu\big((\Gamma^c \setminus C_\varepsilon) \cap A\big) &= \lim_{k\to\infty} \mu_k\big((\Gamma^c \setminus C_\varepsilon) \cap A\big) \\ &\geqslant \lim_{k\to\infty} \mu_k^\ell\big((\Gamma^c \setminus C_\varepsilon) \cap A\big) \\ &= \mu^\ell\big((\Gamma^c \setminus C_\varepsilon) \cap A\big), \end{aligned}$$

and this implies that $\mu^\ell\big((\Gamma^c \setminus C_\varepsilon) \cap A\big) = 0$. Thus $\mu^\ell \leqslant \mu$ and if $\mu\big((\Gamma^c \setminus C_\varepsilon) \cap A\big) = 0$, then also $\mu^\ell\big((\Gamma^c \setminus C_\varepsilon) \cap A\big) = 0$. As the same holds for μ^r, we conclude our claim.

Then, there are $f, g \in L^1(\Gamma^c \setminus C_\varepsilon)$ for which we can write

$$\mu^\ell = f\mathcal{H}^1 \llcorner (\Gamma^c \setminus C_\varepsilon) + \ (\mu^\ell)^s \quad \text{and} \quad \mu^r = g\mathcal{H}^1 \llcorner (\Gamma^c \setminus C_\varepsilon) + \ (\mu^r)^s,$$

with $(\mu^\ell)^s$ and $(\mu^r)^s$ are singular measures with respect to $f\mathcal{H}^1 \llcorner (\Gamma^c \setminus C_\varepsilon)$ and $g\mathcal{H}^1 \llcorner (\Gamma^c \setminus C_\varepsilon)$ respectively. We now prove that $\mu = \mu^\ell + \mu^r$. Notice that for every $\varphi \in C_c(\mathbb{R}^2)$,

$$\int_{\partial E_k^\ell \cup \partial E_k^r} \varphi \, \mathrm{d}\mu_k \to \int_{\Gamma^c \setminus C_\varepsilon} \varphi \, \mathrm{d}\mu,$$

as $k \to \infty$, from the fact that $\mu_k \overset{*}{\rightharpoonup} \mu$. On the other hand we have

$$\begin{aligned} \int_{\partial E_k^\ell \cup \partial E_k^r} \varphi \, \mathrm{d}\mu_k = \int_{\partial E_k^\ell \cup \partial E_k^r} \varphi \, \mathrm{d}(\mu_k^\ell + \mu_k^r) = \int_{\partial E_k^\ell} \varphi \, \mathrm{d}\mu_k^\ell + \int_{\partial E_k^r} \varphi \, \mathrm{d}\mu_k^r \\ \underset{k\to\infty}{\longrightarrow} \int_{\Gamma^c \setminus C_\varepsilon} \varphi \, \mathrm{d}\mu^\ell + \int_{\Gamma^c \setminus C_\varepsilon} \varphi \, \mathrm{d}\mu^r. \end{aligned}$$

Since $\varphi \in C_c(\mathbb{R}^2)$ is arbitrary, we get $\mu = \mu^\ell + \mu^s$. In particular, we obtain (48) and (49).

We now prove the convergence of the energy. Set

$$S_k := \{x_k\} \times \big(-\delta, h_k(x_k)\big) \quad \text{and} \quad S := \{x^i\} \times \big(-\delta, h(x^i)\big).$$

We notice that $\mathcal{H}^1(S_k) \to \mathcal{H}^1(S)$ as $k \to \infty$. In particular, this implies that

$$\lim_{k\to\infty} \int_{S_k} \psi(0)\ \mathrm{d}\mathcal{H}^1 = \int_S \psi(0)\ \mathrm{d}\mathcal{H}^1. \tag{50}$$

Now, we want to apply Theorem 8. Recalling Definition 47 of the sets E_k^ℓ and E_k^r, we obtain

$$\begin{aligned}
\liminf_{k\to\infty} \int_{\Gamma_k\cap\mathcal{R}_\delta^\varepsilon} & \psi(u_k)\ \mathrm{d}\mathcal{H}^1 + 2\int_S \psi(0)\ \mathrm{d}\mathcal{H}^1 \\
&= \liminf_{k\to\infty} \left[\int_{\Gamma_k\cap\mathcal{R}_\delta^\varepsilon} \psi(u_k)\ \mathrm{d}\mathcal{H}^1 + 2\int_{S_k} \psi(0)\ \mathrm{d}\mathcal{H}^1 \right] \\
&= \liminf_{k\to\infty} \left[\int_{\partial E_k^\ell\cap\mathcal{R}_\delta^\varepsilon} \psi(u_k^\ell)\ \mathrm{d}\mathcal{H}^1 + \int_{\partial E_k^r\cap\mathcal{R}_\delta^\varepsilon} \psi(u_k^r)\ \mathrm{d}\mathcal{H}^1 \right] \\
&\geqslant \liminf_{k\to\infty} \int_{\partial E_k^\ell\cap\mathcal{R}_\delta^\varepsilon} \psi(u_k^\ell)\ \mathrm{d}\mathcal{H}^1 + \liminf_{k\to\infty} \int_{\partial E_k^r\cap\mathcal{R}_\delta^\varepsilon} \psi(u_k^r)\ \mathrm{d}\mathcal{H}^1 \\
&\geqslant \int_{\partial R^\ell\cap\mathcal{R}_\delta^\varepsilon} \widetilde{\psi}(f)\ \mathrm{d}\mathcal{H}^1 + \theta(\mu^\ell)^s(\partial R^\ell \cap \mathcal{R}_\delta^\varepsilon) \\
&\qquad + \int_{\partial R^r\cap\mathcal{R}_\delta^\varepsilon} \widetilde{\psi}(g)\ \mathrm{d}\mathcal{H}^1 + \theta(\mu^r)^s(\partial R^r \cap \mathcal{R}_\delta^\varepsilon) \\
&= \int_{\Gamma^c\setminus C_\varepsilon} \widetilde{\psi}(f)\ \mathrm{d}\mathcal{H}^1 + \theta(\mu^\ell)^s(\Gamma^c \setminus C_\varepsilon) \\
&\qquad + \int_{\Gamma^c\setminus C_\varepsilon} \widetilde{\psi}(g)\ \mathrm{d}\mathcal{H}^1 + \theta(\mu^r)^s(\Gamma^c \setminus C_\varepsilon) + 2\int_S \psi(0)\ \mathrm{d}\mathcal{H}^1 \\
&\geqslant \int_{\Gamma^c\setminus C_\varepsilon} \psi^c(u)\ \mathrm{d}\mathcal{H}^1 + \theta\mu^s(\Gamma^c \setminus C_\varepsilon) + 2\int_S \psi(0)\ \mathrm{d}\mathcal{H}^1,
\end{aligned}$$

where the last inequality follows from (48) together with the definition of ψ^c. Thus,

$$\liminf_{k\to\infty} \int_{\Gamma_k\cap\mathcal{R}_\delta^\varepsilon} \psi(u_k)\ \mathrm{d}\mathcal{H}^1 \geqslant \int_{\Gamma^c\setminus C_\varepsilon} \psi^c(u)\ \mathrm{d}\mathcal{H}^1 + \theta\mu^s(\Gamma^c \setminus C_\varepsilon), \tag{51}$$

for all $\varepsilon > 0$

Step 5. Using (44), (45), (46) and (51) we obtain

$$
\begin{aligned}
\liminf_{k\to\infty} \Big[\int_{\Omega_k} W\big(E(v_k) - E_0(y)\big)\,\mathrm{d}\mathbf{x} + \int_{\Gamma_k} \psi(u_k)\,\mathrm{d}\mathcal{H}^1 \Big]& \\
\geqslant \int_{\Omega} W\big(E(v) - E_0(y)\big)\,\mathrm{d}\mathbf{x}& \\
+ \int_{\widetilde{\Gamma}\cap A^{\varepsilon}_{\delta}} \widetilde{\psi}(u)\,\mathrm{d}\mathcal{H}^1 + \theta\mu^s(A^{\varepsilon}_{\delta}) + \theta \int_{C_\varepsilon} u\,\mathrm{d}\mathcal{H}^1& \\
+ \int_{\Gamma^c\setminus C_\varepsilon} \psi^c(u)\,\mathrm{d}\mathcal{H}^1 + \theta\mu^s(\Gamma^c \setminus C_\varepsilon).&
\end{aligned}
$$

By letting $\varepsilon \to 0$, and using (43) we get the desired liminf inequality. $\square$

1.8 Limsup inequality

The goal of this section is to prove the limsup inequality for the mass constrained problem. We recall that the classes $\mathcal{A}_r(m, M)$ and $\mathcal{A}_r(m, M)$ are given in Definition 17.

Theorem 10. Let $m, M > 0$ and $(\Omega, v, \mu) \in \mathcal{A}(m, M)$. Then, there exists a sequence of regular sharp configurations $(\Omega_k, v_k, \mu_k)_k \subset \mathcal{A}_r(m, M)$ such that

$$
\limsup_{k\to\infty} \mathcal{H}(\Omega_k, v_k, \mu_k) \leqslant \mathcal{F}(\Omega, v, \mu),
$$

and such that $(\Omega_k, v_k, \mu_k) \to (\Omega, v, \mu)$ as $k \to \infty$.

The proof is long and therefore it will be divided in several steps, each proved in a separate result. Therefore, we explain the steps of the strategy that we will use in order to prove Theorem 10.

Step 1: For any sharp configuration $(\Omega, v, \mu) \in \mathcal{A}(m, M)$, we find a sequence $(u_k)_k \subset L^1(\Gamma)$ where each u_k is a grid constant function (see Definition 8), such that $\mu_k := u_k \mathcal{H}^1 \llcorner \Gamma \xrightharpoonup{*} \mu$ as $k \to \infty$, $(\Omega, v, \mu_k) \in \mathcal{A}(m, M)$ for all $k \in \mathbb{N}$, and

$$\lim_{k\to\infty} \mathcal{F}(\Omega, v, \mu_k) \leqslant \mathcal{F}(\Omega, v, \mu).$$

This will be proved in Theorem 11;

Step 2: Let $(\Omega, v, \mu) \in \mathcal{A}(m, M)$, be such that $\mu = u\mathcal{H}^1 \llcorner \Gamma$, and $u \in L^1(\Gamma)$ is grid constant. In Theorem 12, we construct a sequence $\big(\Omega_k, v_k, \mu_k\big)_k \subset \mathcal{A}_r(m, M)$, where $\mu_k = u_k \mathcal{H}^1 \llcorner \Gamma_k$ and $u_k \in L^1(\Gamma_k)$ is grid constant, such that $(\Omega_k, v_k, \mu_k) \to (\Omega, v, \mu)$ as $k \to \infty$, and

$$\lim_{k\to\infty} \mathcal{F}(\Omega_k, v_k, \mu_k) = \mathcal{F}(\Omega, v, \mu);$$

Step 3: For every sharp configuration $(\Omega, v, \mu) \in \mathcal{A}_r$, in Theorem 13 we build a sequence $\big(\Omega_k, v_k, \mu_k\big)_k \subset \mathcal{A}_r$ where $\mu_k = u_k \mathcal{H}^1 \llcorner \Gamma_k$ and $u_k \in L^1(\Gamma_k)$ is grid constant, such that $(\Omega_k, v_k, \mu_k) \to (\Omega, v, \mu)$ as $k \to \infty$, and

$$\lim_{k\to\infty} \mathcal{H}(\Omega_k, v_k, \mu_k) = \mathcal{F}(\Omega, v, \mu);$$

Step 4: From Theorems 11, 12 and 13 and a diagonalization argument we get the limsup inequality.

Remark 9. Using Theorem 10 with Theorem 9, we have proved Theorem 1 and Theorem 2.

We now carry on Step 1: approximate any admissible configuration with a sequence of configurations where the density is grid constant.

Theorem 11. Let $(\Omega, v, \mu) \in \mathcal{A}(m, M)$. Then, there exists a sequence $(u_k)_k \subset L^1(\Gamma)$, with $u_k \in L^1(\Gamma)$ grid constant, such that $(\Omega, v, \mu_k) \to (\Omega, v, \mu)$, as $k \to \infty$, and

$$\lim_{k\to\infty} \mathcal{F}(\Omega, v, \mu_k) \leqslant \mathcal{F}(\Omega, v, \mu),$$

where $\mu_k := u_k \mathcal{H}^1 \llcorner \Gamma$. Moreover, $(\Omega, v, \mu_k) \in \mathcal{A}(m, M)$.

Proof. Step 1. Given $(\Omega, v, \mu) \in \mathcal{A}(m, M)$, with $\mu = u\mathcal{H}^1 \llcorner \Gamma + \mu^s$, we would like to approximate μ^s with a finite number of Dirac deltas. Given $k \in \mathbb{N}$, consider an $1/k$-admissible covering of Γ. Let $Q^1, \dots, Q^{N_k}$ be those cubes that intersect with Γ. For each $i = 1, \dots, N_k$, let $x_k^i \in Q^i \cap \Gamma$. We define

$$m_k^i := \mu^s(Q_k^i)$$

and set

$$\mu_k := u\mathcal{H}^1 \llcorner \Gamma + \sum_{i=1}^{N_k} m_k^i \delta_{x_k^i},$$

where, for every $k \in \mathbb{N}$, N_k is finite. It is possible to see that $\mu_k(\Gamma) = m$ and $\mu_k \overset{*}{\rightharpoonup} \mu$ as $k \to \infty$. Furthermore, the fact that $\mu^s(\Gamma) = \sum_{i=1}^{N_k} m_k^i$, for every $k \in \mathbb{N}$, implies that

$$\mathcal{F}(\Omega, v, \mu_k) = \mathcal{F}(\Omega, v, \mu),$$

for every $k \in \mathbb{N}$.

Step 2. Now, consider $(\Omega, v, \mu) \in \mathcal{A}(m, M)$, with $\mu = u\mathcal{H}^1 \llcorner \Gamma + \sum_{i=1}^{N} m^i \delta_{x_i}$, with $x_i \in \Gamma$ and $m^i > 0$ as defined in step 1, for every $i = 1, \dots N$. We now construct an admissible covering in order to define a suitable density on Γ.
For $k \in \mathbb{N}$, consider $(Q_k^j)_{j=1}^{L_k}$, an $1/k$-admissible covering for Γ. Consider the covering of Γ given by

$$\Big(\bigcup_{i=1}^{N} Q(x^i, 1/k)\Big) \cup \Big[\Big(\bigcup_{j=1}^{L_k} Q_k^j\Big) \setminus \Big(\bigcup_{i=1}^{N} Q(x^i, 1/k)\Big)\Big]. \tag{52}$$

We notice that $\big(\bigcup_{j=1}^{L_k} Q_k^j\big) \setminus \big(\bigcup_{i=1}^{N} Q(x^i, 1/k)\big)$ can be divided N_k rectangles whose sides does not exceed $1/k$. Thus, up to a further subdivision in rectangles, we consider (52) as a $1/k$-admissible covering of Γ. In order to simplify the notation, we denote as Q_k^j any rectangle contained in (52). Furthermore, by reordering the rectangles in (52), we assume that for $j = 1, \dots, N$, $Q_k^j \subset \bigcup_{i=1}^{N} Q(x^i, 1/k)$ and for $j = N+1, \dots, N+N_k$,

we have $Q_k^j \subset \big(\bigcup_{j=1}^{L_k} Q_k^j\big) \setminus \big(\bigcup_{i=1}^{N} Q(x^i, 1/k)\big)$.
Fix $\varepsilon > 0$. Since

$$\lim_{k\to\infty} \frac{\mu^s(\widetilde{\Gamma}\cap Q_k^j)}{\mathcal{H}^1(\widetilde{\Gamma}\cap Q_k^j)} = +\infty, \quad \text{and} \quad \lim_{k\to\infty} \frac{\mu^s(\Gamma^c\cap Q_k^j)}{\mathcal{H}^1(\Gamma^c\cap Q_k^j)} = +\infty,$$

for all $j = 1, \dots, N$, there is $\bar{k} \in \mathbb{N}$ such that, for every $k \geqslant \bar{k}$, we have

$$\left| \frac{\mathcal{H}^1(\Gamma\cap Q_k^j)}{\mu^s(\Gamma\cap Q_k^j)} \widetilde{\psi}\left(\frac{\mu^s(\Gamma\cap Q_k^j)}{\mathcal{H}^1(\Gamma\cap Q_k^j)} \right) - \theta \right| < \varepsilon \tag{53}$$

and

$$\left| \frac{\mathcal{H}^1(\Gamma\cap Q_k^j)}{\mu^s(\Gamma\cap Q_k^j)} \psi^c\left(\frac{\mu^s(\Gamma\cap Q_k^j)}{\mathcal{H}^1(\Gamma\cap Q_k^j)} \right) - \theta \right| < \varepsilon. \tag{54}$$

We now define a density on Γ. For $\mathbf{x} \in Q_k^j$, we define $u_k : \Gamma \to \mathbb{R}$ as

$$u_k(\mathbf{x}) := \begin{cases} \dfrac{\mu(\widetilde{\Gamma}\cap Q_k^j)}{\mathcal{H}^1(\widetilde{\Gamma}\cap Q_k^j)} & \text{if } \mathbf{x} \in \widetilde{\Gamma},\ \widetilde{\Gamma}\cap Q_k^j \neq \emptyset, \\[2ex] \dfrac{\mu(\Gamma^c\cap Q_k^j)}{\mathcal{H}^1(\Gamma^c\cap Q_k^j)} & \text{if } \mathbf{x} \in \Gamma^c,\ \Gamma^c\cap Q_k^j \neq \emptyset. \end{cases}$$

Note that the function $u_k \in L^1(\Gamma)$ is $1/k$-grid constant by definition. For each $k \in \mathbb{N}$, define the measure

$$\mu_k := u_k \mathcal{H}^1 \llcorner \Gamma. \tag{55}$$

By definition, it follows directly that the mass constraint is satisfied, namely that $(\Omega, v, \mu_k) \in \mathcal{A}(m, M)$.

Step 3. We now prove that $\mu_k \overset{*}{\rightharpoonup} \mu$ as $k \to \infty$. Take $\varphi \in \mathcal{C}_c(\mathbb{R}^2)$. Fix $\varepsilon > 0$. Using the uniform continuity of φ, there exists $\bar{k} \in \mathbb{N}$ such that for every $k \geqslant \bar{k}$ we have that

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{x}_k^i)| < \varepsilon,$$

for every $\mathbf{x} \in Q_k^j$, where $\mathbf{x}_k^i$ is the intersection point of the diagonals of Q_k^j. First, we write

$$
\begin{aligned}
\Big|\int_\Gamma \varphi \,\mathrm{d}\mu_k - \int_\Gamma \varphi \,\mathrm{d}\mu\Big| &\leqslant \Big|\int_{\widetilde{\Gamma}} \varphi \,\mathrm{d}\mu_k - \int_{\widetilde{\Gamma}} \varphi \,\mathrm{d}\mu\Big| + \Big|\int_{\Gamma^c} \varphi \,\mathrm{d}\mu_k - \int_{\Gamma^c} \varphi \,\mathrm{d}\mu\Big| \\
&\leqslant \sum_{j=1}^{N+N_k} \Big|\int_{\widetilde{\Gamma}\cap Q_k^j} \varphi \,\mathrm{d}\mu_k - \int_{\widetilde{\Gamma}\cap Q_k^j} \varphi \,\mathrm{d}\mu\Big| \\
&\quad + \sum_{j=1}^{N+N_k} \Big|\int_{\Gamma^c\cap Q_k^j} \varphi \,\mathrm{d}\mu_k - \int_{\Gamma^c\cap Q_k^j} \varphi \,\mathrm{d}\mu\Big|, \qquad (56)
\end{aligned}
$$

and we estimate the two terms on the right-hand side of (56) separately. We have that

$$
\begin{aligned}
\sum_{j=1}^{N+N_k} \Big|\int_{\widetilde{\Gamma}\cap Q_k^j} \varphi \,\mathrm{d}\mu_k - \int_{\widetilde{\Gamma}\cap Q_k^j} \varphi \,\mathrm{d}\mu\Big| &\leqslant \sum_{j=1}^{N+N_k} \Big[\int_{\widetilde{\Gamma}\cap Q_k^j} |\varphi(\mathbf{x}) - \varphi(\mathbf{x}_k^i)| \,\mathrm{d}\mu_k \\
&\quad + \int_{\widetilde{\Gamma}\cap Q_k^j} |\varphi(\mathbf{x}) - \varphi(\mathbf{x}_k^i)| \,\mathrm{d}\mu \\
&\quad + |\varphi(\mathbf{x}_k^i)| \big|\mu(\widetilde{\Gamma}\cap Q_k^j) - \mu_k(\widetilde{\Gamma}\cap Q_k^j)\big|\Big] \\
&\leqslant 2m\varepsilon \|\varphi\|_{C^0(\mathbb{R}^2)}, \qquad (57)
\end{aligned}
$$

where we used the fact that $\mu(\widetilde{\Gamma}\cap Q_k^j) = \mu_k(\widetilde{\Gamma}\cap Q_k^j)$ for each $j = 1, \dots, M + N_k$ and every $k \in \mathbb{N}$, by definition of μ_k. Using similar computations, we also get that the second term on the right-hand side of (56) can be estimated as

$$
\sum_{j=1}^{N+N_k} \Big|\int_{\Gamma^c\cap Q_k^j} \varphi \,\mathrm{d}\mu_k - \int_{\Gamma^c\cap Q_k^j} \varphi \,\mathrm{d}\mu\Big| \leqslant 2m\varepsilon \|\varphi\|_{C^0(\mathbb{R}^2)}, \qquad (58)
$$

Finally, from (56), (57) and (58), we get

$$
\Big|\int_\Gamma \varphi \,\mathrm{d}\mu_k - \int_\Gamma \varphi \,\mathrm{d}\mu\Big| \leqslant 4m\varepsilon \|\varphi\|_{C^0(\mathbb{R}^2)}.
$$

As $\varepsilon > 0$ is arbitrary, we can conclude that $\mu_k \stackrel{*}{\rightharpoonup} \mu$ as $k \to \infty$.

Step 4. We now prove the convergence of the energy. We will prove that

$$\limsup_{k\to\infty} \mathcal{F}(\Omega, v, \mu_k) \leqslant \mathcal{F}(\Omega, v, \mu).$$

Since the bulk term of the energy is unchanged, we estimate the other contributions. We have that

$$\begin{aligned}
&\int_{\widetilde{\Gamma}} \widetilde{\psi}(u_k)\,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma^c} \psi^c(u_k)\,\mathrm{d}\mathcal{H}^1 \\
&\quad = \sum_{j=1}^{N+N_k} \left[\int_{\widetilde{\Gamma}\cap Q_k^j} \widetilde{\psi}(u_k)\,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma^c\cap Q_k^j} \psi^c(u_k)\,\mathrm{d}\mathcal{H}^1 \right] \\
&\quad = \sum_{j=1}^{N+N_k} \left[\mathcal{H}^1(\widetilde{\Gamma}\cap Q_k^j)\widetilde{\psi}\left(\frac{\mu(\widetilde{\Gamma}\cap Q_k^j)}{\mathcal{H}^1(\widetilde{\Gamma}\cap Q_k^j)} \right) \right. \\
&\qquad\qquad \left. + \mathcal{H}^1(\Gamma^c\cap Q_k^j)\psi^c\left(\frac{\mu(\Gamma^c\cap Q_k^j)}{\mathcal{H}^1(\Gamma^c\cap Q_k^j)} \right) \right] \\
&\quad = \sum_{j=1}^{N} \left[\mathcal{H}^1(\widetilde{\Gamma}\cap Q_k^j)\widetilde{\psi}\left(\fint_{\widetilde{\Gamma}\cap Q_k^j} u\,\mathrm{d}\mathcal{H}^1 + \frac{\mu^s(\widetilde{\Gamma}\cap Q_k^j)}{\mathcal{H}^1(\widetilde{\Gamma}\cap Q_k^j)} \right) \right. \\
&\qquad\qquad \left. + \mathcal{H}^1(\Gamma^c\cap Q_k^j)\psi^c\left(\fint_{\Gamma^c\cap Q_k^j} u\,\mathrm{d}\mathcal{H}^1 + \frac{\mu^s(\Gamma^c\cap Q_k^j)}{\mathcal{H}^1(\Gamma^c\cap Q_k^j)} \right) \right] \\
&\qquad + \sum_{j=N}^{N+N_k} \left[\mathcal{H}^1(\widetilde{\Gamma}\cap Q_k^j)\widetilde{\psi}\Big(\fint_{\widetilde{\Gamma}\cap Q_k^j} u\,\mathrm{d}\mathcal{H}^1 \Big) \right. \\
&\qquad\qquad \left. + \mathcal{H}^1(\Gamma^c\cap Q_k^j)\psi^c\Big(\fint_{\Gamma^c\cap Q_k^j} u\,\mathrm{d}\mathcal{H}^1 \Big) \right] \\
&\quad \leqslant \sum_{j=1}^{N} \left[\mathcal{H}^1(\widetilde{\Gamma}\cap Q_k^j)\widetilde{\psi}\Big(\fint_{\widetilde{\Gamma}\cap Q_k^j} u\,\mathrm{d}\mathcal{H}^1 \Big) + \mathcal{H}^1(\widetilde{\Gamma}\cap Q_k^j)\widetilde{\psi}\left(\frac{\mu^s(\widetilde{\Gamma}\cap Q_k^j)}{\mathcal{H}^1(\widetilde{\Gamma}\cap Q_k^j)} \right) \right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \mathcal{H}^1(\Gamma^c \cap Q_k^j)\psi^c\Big(\fint_{\Gamma^c \cap Q_k^j} u \, \mathrm{d}\mathcal{H}^1 \Big) \\
&\qquad + \mathcal{H}^1(\Gamma^c \cap Q_k^j)\psi^c\left(\frac{\mu^s(\Gamma^c \cap Q_k^j)}{\mathcal{H}^1(\Gamma^c \cap Q_k^j)} \right)\Bigg] \\
&\qquad + \sum_{j=N}^{N+N_k} \Bigg[\mathcal{H}^1(\widetilde{\Gamma} \cap Q_k^j)\widetilde{\psi}\Big(\fint_{\widetilde{\Gamma} \cap Q_k^j} u \, \mathrm{d}\mathcal{H}^1 \Big) \\
&\qquad + \mathcal{H}^1(\Gamma^c \cap Q_k^j)\psi^c\Big(\fint_{\Gamma^c \cap Q_k^j} u \, \mathrm{d}\mathcal{H}^1 \Big)\Bigg] \\
&\leqslant \sum_{j=1}^{N} \Bigg[\int_{\widetilde{\Gamma} \cap Q_k^j} \widetilde{\psi}(u) \, \mathrm{d}\mathcal{H}^1 + \mathcal{H}^1(\widetilde{\Gamma} \cap Q_k^j)\widetilde{\psi}\left(\frac{\mu^s(\widetilde{\Gamma} \cap Q_k^j)}{\mathcal{H}^1(\widetilde{\Gamma} \cap Q_k^j)} \right) \\
&\qquad + \int_{\Gamma^c \cap Q_k^j} \psi^c(u) \, \mathrm{d}\mathcal{H}^1 + \mathcal{H}^1(\Gamma^c \cap Q_k^j)\psi^c\left(\frac{\mu^s(\Gamma^c \cap Q_k^j)}{\mathcal{H}^1(\Gamma^c \cap Q_k^j)} \right)\Bigg] \\
&\qquad + \sum_{j=N}^{N+N_k} \Bigg[\int_{\widetilde{\Gamma} \cap Q_k^j} \widetilde{\psi}(u) \, \mathrm{d}\mathcal{H}^1 + \int_{\Gamma^c \cap Q_k^j} \psi^c(u) \, \mathrm{d}\mathcal{H}^1 \Bigg] \\
&= \sum_{j=1}^{N} \Bigg[\int_{\widetilde{\Gamma} \cap Q_k^j} \widetilde{\psi}(u) \, \mathrm{d}\mathcal{H}^1 + \mu^s(\widetilde{\Gamma} \cap Q_k^j) \frac{\mathcal{H}^1(\widetilde{\Gamma} \cap Q_k^j)}{\mu^s(\widetilde{\Gamma} \cap Q_k^j)} \widetilde{\psi}\left(\frac{\mu^s(\widetilde{\Gamma} \cap Q_k^j)}{\mathcal{H}^1(\widetilde{\Gamma} \cap Q_k^j)} \right) \\
&\qquad + \mu^s(\Gamma^c \cap Q_k^j) \frac{\mathcal{H}^1(\Gamma^c \cap Q_k^j)}{\mu^s(\Gamma^c \cap Q_k^j)} \psi^c\left(\frac{\mu^s(\Gamma^c \cap Q_k^j)}{\mathcal{H}^1(\Gamma^c \cap Q_k^j)} \right)\Bigg] \\
&\qquad + \sum_{j=N}^{N+N_k} \Bigg[\int_{\widetilde{\Gamma} \cap Q_k^j} \widetilde{\psi}(u) \, \mathrm{d}\mathcal{H}^1 + \int_{\Gamma^c \cap Q_k^j} \psi^c(u) \, \mathrm{d}\mathcal{H}^1 \Bigg] \\
&\qquad + \int_{\Gamma^c \cap Q_k^j} \psi^c(u) \, \mathrm{d}\mathcal{H}^1, \qquad (59)
\end{aligned}
$$

where in the first inequality we used the sub-additivity of $\widetilde{\psi}$ and ψ^c, while in the previous to last step we used Jensen's inequality.

By construction, we have that (53) and (54) hold. Thus, from (59), we obtain

$$
\begin{aligned}
&\int_{\widetilde{\Gamma}} \widetilde{\psi}(u_k)\ \mathrm{d}\mathcal{H}^1 + \int_{\Gamma^c} \psi^c(u_k)\ \mathrm{d}\mathcal{H}^1 \\
&\qquad \leqslant \sum_{j=1}^{N} \Bigg[\int_{\widetilde{\Gamma}\cap Q_k^j} \widetilde{\psi}(u)\ \mathrm{d}\mathcal{H}^1 + \mu^s(\widetilde{\Gamma}\cap Q_k^j)(\theta+\varepsilon) \\
&\qquad\qquad + \int_{\Gamma^c\cap Q_k^j} \psi^c(u) + \mu^s(\Gamma^c \cap Q_k^j)(\theta+\varepsilon) \Bigg] \\
&\qquad\qquad + \sum_{j=N}^{N+N_k} \Bigg[\int_{\widetilde{\Gamma}\cap Q_k^j} \widetilde{\psi}(u)\ \mathrm{d}\mathcal{H}^1 + \int_{\Gamma^c\cap Q_k^j} \psi^c(u)\ \mathrm{d}\mathcal{H}^1 \Bigg] \\
&\qquad = \sum_{j=1}^{N+N_k} \Bigg[\int_{\widetilde{\Gamma}\cap Q_k^j} \widetilde{\psi}(u)\ \mathrm{d}\mathcal{H}^1 \\
&\qquad\qquad + \int_{\Gamma^c\cap Q_k^j} \psi^c(u)\ \mathrm{d}\mathcal{H}^1 + \theta\mu^s(\Gamma\cap Q_k^j) + \varepsilon\mu^s(\Gamma\cap Q_k^j) \Bigg] \\
&\qquad \leqslant \int_{\widetilde{\Gamma}} \widetilde{\psi}(u)\ \mathrm{d}\mathcal{H}^1 + \int_{\Gamma^c} \psi^c(u)\ \mathrm{d}\mathcal{H}^1 + \theta\mu^s(\Gamma) + \varepsilon\mu^s(\Gamma). \qquad (60)
\end{aligned}
$$

From (60), since ε is arbitrary, we can conclude

$$\limsup_{k\to\infty} \mathcal{F}(\Omega, v, \mu_k) \leqslant \mathcal{F}(\Omega, v, \mu).$$

This concludes the proof. $\square$

We proceed our analysis with the second step, which will allows us to reduce to the case of a Lipschitz profile and a grid constant adatom density.

Theorem 12. Let $(\Omega, v, \mu) \subset \mathcal{A}(m, M)$ be such that u is grid constant. Then, there exists a sequence $(\Omega_k, v_k, \mu_k)_k \subset \mathcal{A}_r(m, M)$, where $\mu_k = u_k\mathcal{H}^1 \llcorner \Gamma_k$ with each u_k grid constant, such that

$$\lim_{k\to\infty} \mathcal{F}(\Omega_k, v_k, \mu_k) = \mathcal{F}(\Omega, v, \mu),$$

and $(\Omega_k, v_k, \mu_k) \to (\Omega, v, \mu)$, as $k \to \infty$.

Proof. The strategy of the proof is the following. In *Step 1* we show that it suffices to build the required sequence in case h has finitely many cut points. In *Step 2* we build the recovery sequence. Finally in *Step 3* we show the convergence of the energy.

Step 1. In this first step we are going to show that it suffices to prove the result in the case h has a finite number of cuts. Namely, we prove that there exist sequences $(\Omega_{g_k}, w_k, \nu_k)_k \subset \mathcal{A}(m, M)$ where each g_k has a finite number of cuts, and ν_k is grid constant, such that

$$\lim_{k\to\infty} \mathcal{F}(\Omega_{g_k}, w_k, \nu_k) = \mathcal{F}(\Omega, v, \mu),$$

and $(\Omega_{g_k}, w_k, \nu_k) \to (\Omega, v, \mu)$ as $k \to \infty$.

The following construction is inspired by [37, Theorem 2.8]. For $k \in \mathbb{N}$, define (see Figure 1.5)

$$\hat{g}_k(x) \coloneqq \min\{\max\{h^-(x) - 1/k, 0\}, h(x)\},$$

for every $x \in (a, b)$. It is possible to see that, for each k, the function $\hat{g}_k$ is lower semicontinuous, of bounded variation, and such that $\hat{g}_k \leqslant h$. Moreover, thanks to Lemma 2, we have that $\hat{g}_k$ has finitely many cuts. We then define

$$g_k(x) \coloneqq \hat{g}_k(x) + \varepsilon_k, \tag{61}$$

for each k, where

$$\varepsilon_k \coloneqq \frac{1}{b-a}\Big(M - \int_a^b \hat{g}_k(x)\, \mathrm{d}x\Big) > 0.$$

Set $\Gamma_k \coloneqq \Gamma_{g_k}$, and note that

$$\lim_{k\to\infty} \mathcal{H}^1(\Gamma_k) = \mathcal{H}^1(\Gamma). \tag{62}$$

We now need, for each $k \in \mathbb{N}$, to define the displacement v_k and the adatom density u_k. For the former, by fixing a $y_0 < 0$ such that $v(\cdot, y_0) \in H^1\big((a, b); \mathbb{R}^2\big)$, we define

$$w_k(\mathbf{x}) \coloneqq \begin{cases} v(x, y - \varepsilon_k) & \text{if } y > y_0 + \varepsilon_k, \\ v(x, y_0) & \text{if } y_0 < y \leqslant y_0 + \varepsilon_k, \\ v(x, y) & \text{if } y \leqslant y_0. \end{cases} \tag{63}$$

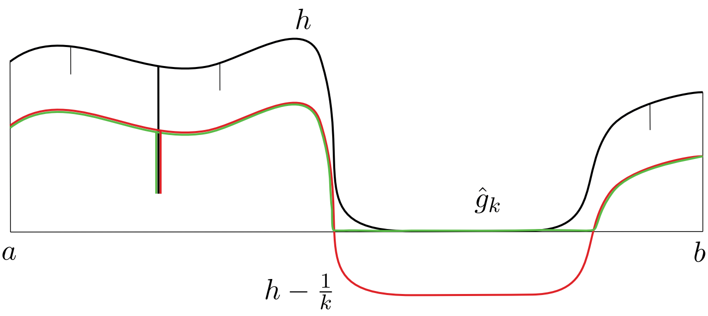

Figure 1.5: In order to reduce to a finite number of cuts, we do the following: first, we shift down by $1/k$ the regular part of the graph of h (not the cuts), getting the red graph. In this process, some parts of the graph might have gone below zero. Thus, we get the function $\hat{g}_k$ by cutting them, and by adding the remaining part of the original cuts.

For $k \in \mathbb{N} \setminus \{0\}$, and $\mathbf{x} \in \Gamma_k$, we define

$$z_k(\mathbf{x}) := \begin{cases} u\big(x, y + 1/k\big) & \text{if } (x, y+1/k) \in \widetilde{\Gamma}, \text{ and } h(x) > 1/k, \\ u(x,y) & \text{if } \mathbf{x} \in \Gamma^c, \\ u(x,0) & \text{if } h(x) = 0. \end{cases}$$

For each $k \in \mathbb{N} \setminus \{0\}$, we then define the measure

$$\nu_k := (z_k + r_k)\mathcal{H}^1 \llcorner \Gamma_k,$$

where

$$r_k := \frac{1}{\mathcal{H}^1(\Gamma_k)} \left[\int_\Gamma u \, \mathrm{d}\mathcal{H}^1 - \int_{\Gamma_k} z_k \, \mathrm{d}\mathcal{H}^1 \right].$$

We notice that, by using (62),

$$\lim_{k \to \infty} r_k = 0. \tag{64}$$

Step 1.1 Note that, by definition, the sequences $(g_k)_k$ and $(\nu_k)_k$ satisfy the mass and the density constraint as in Theorem 2, and thus $(\Omega_{g_k}, w_k, \nu_k)_k \subset \mathcal{A}(m, M)$.

Step 1.2 We now prove that $(\Omega_{g_k}, w_k, \nu_k) \to (\Omega, v, \mu)$ as $k \to \infty$. By using the definition, it is possible to see that $\mathbb{R}^2 \setminus \Omega_{g_k} \xrightarrow{H} \mathbb{R}^2 \setminus \Omega$, and $w_k \rightharpoonup v$ in $H^1_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ as $k \to \infty$. In particular, we have that $\mathcal{H}^1(\Gamma_k) \to \mathcal{H}^1(\Gamma)$ as $k \to \infty$.
We now prove that $\nu_k \overset{*}{\rightharpoonup} \mu$ as $k \to \infty$. Take any $\varphi \in C_c(\mathbb{R}^2)$ and fix $\varepsilon > 0$. By the uniform continuity of φ we find $\delta > 0$ such that, if $|(x, y - 1/k) - (x, y)| < \delta$, we have

$$|\varphi(x, y - 1/k) - \varphi(x, y)| < \varepsilon.$$

Then, for k large enough,

$$\begin{aligned}
&\Big| \int_{\Gamma_k} \varphi z_k \,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma_k} r_k \varphi \,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma} \varphi u \,\mathrm{d}\mathcal{H}^1 \Big| \\
&\qquad \leqslant \Big| \int_{\widetilde{\Gamma} \cap \{h > 1/k\}} \varphi(x, y - 1/k) u \,\mathrm{d}\mathcal{H}^1 \\
&\qquad\qquad + \int_{\widetilde{\Gamma} \cap \{h = 0\}} \varphi u(x, 0) \,\mathrm{d}\mathcal{H}^1 - \int_{\widetilde{\Gamma}} \varphi u \,\mathrm{d}\mathcal{H}^1 \Big| \\
&\qquad\qquad + \Big| \int_{\Gamma_k^c} \varphi u \,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma^c} \varphi u \,\mathrm{d}\mathcal{H}^1 \Big| + ||\varphi||_{\mathcal{C}^0(\mathbb{R}^2)} \mathcal{H}^1(\Gamma_k) r_k \\
&\qquad \leqslant \varepsilon ||u||_{L^1(\widetilde{\Gamma})} + \Big| \int_{\widetilde{\Gamma} \cap \{h > 1/k\}} \varphi u \,\mathrm{d}\mathcal{H}^1 - \int_{\widetilde{\Gamma} \cap \{h > 0\}} \varphi u \,\mathrm{d}\mathcal{H}^1 \Big| \\
&\qquad\qquad + \Big| \int_{\Gamma^c \setminus \Gamma_k^c} \varphi u \,\mathrm{d}\mathcal{H}^1 \Big| + ||\varphi||_{\mathcal{C}^0(\mathbb{R}^2)} \mathcal{H}^1(\Gamma_k) r_k.
\end{aligned}$$

Here we notice that $\Gamma^c \setminus \Gamma_k^c \to \emptyset$, $r_k \to 0$ and that $\widetilde{\Gamma} \cap \{h > 1/k\} \to \widetilde{\Gamma} \cap \{h > 0\}$ as $k \to \infty$. From these considerations, as ε is arbitrary, we infer that $\nu_k \overset{*}{\rightharpoonup} \mu$ as $k \to \infty$.

Step 1.3 Finally, we prove the convergence of the energy. First, by a standard argument, we can reduce to the case $u \in L^\infty(\Gamma)$. Thus, we have

$$|\mathcal{F}(\Omega_{g_k}, w_k, \nu_k) - \mathcal{F}(\Omega_{g_k}, v, \mu)| \leqslant$$

$$\Big|\int_{\Omega_k} W\big(E(w_k) - E_0(y)\big)\,\mathrm{d}\mathbf{x} - \int_{\Omega} W\big(E(v) - E_0(y)\big)\,\mathrm{d}\mathbf{x}\Big|$$

$$+\Big|\int_{\widetilde{\Gamma}_k} \widetilde{\psi}(z_k + r_k)\,\mathrm{d}\mathcal{H}^1 - \int_{\widetilde{\Gamma}} \widetilde{\psi}(u)\,\mathrm{d}\mathcal{H}^1\Big|$$

$$+\Big|\int_{\Gamma_k^c} \psi^c(z_k + r_k)\,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma^c} \psi^c(u)\,\mathrm{d}\mathcal{H}^1\Big|. \qquad (65)$$

Regarding the bulk term on the right-hand side of (65), we have that $w_k \to v$ in $H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$ as $k \to \infty$. Remember that, by construction, $\Omega_{g_k} \subset \Omega$. From the fact that $\Omega_{g_k} \to \Omega$ in L^1 as $k \to \infty$, we can find $\bar{k} \in \mathbb{N}$ such that for every $k \geqslant \bar{k}$, we have $|\Omega \setminus \Omega_{g_k}| < \varepsilon$. Then, for $k \geqslant \bar{k}$, we have

$$\Big|\int_{\Omega_{g_k}} W\big(E(w_k) - E_0(y)\big)\,\mathrm{d}\mathbf{x} - \int_{\Omega} W\big(E(v) - E_0(y)\big)\,\mathrm{d}\mathbf{x}\Big|$$

$$\leqslant \int_{\Omega_{g_k}\cap\{y>y_0+\varepsilon_k\}} \big|W\big(E(w_k)) - E_0(y)\big) - W\big(E(v) - E_0(y)\big)\big|\,\mathrm{d}\mathbf{x}$$

$$+\int_{\Omega_{g_k}\cap\{y_0<y<y_0+\varepsilon_k\}} \big|W\big(E(w_k) - E_0(y)\big) - W\big(E(v) - E_0(y)\big)\big|\,\mathrm{d}\mathbf{x}$$

$$+\Big|\int_{\Omega\setminus\Omega_{g_k}} W\big(E(v) - E_0(y)\big)\,\mathrm{d}\mathbf{x}\Big|. \qquad (66)$$

Notice that the first term on the right-hand side of (66) goes to 0 as $k \to \infty$ by a continuity argument for integral functional of convex integrand. The second term is infinitesimal as well, by noticing that the domain of integration is converging to a negligible set. Finally, for the last term we can apply the Dominated Convergence Theorem to conclude the estimate for the bulk term.

We now consider the surface terms on the right-hand side of (65). From (64), we can choose k large enough so that $r_k \leqslant 1$. Since $u \in L^\infty(\Gamma)$, we have that $\widetilde{\psi}$ and ψ^c are uniformly continuous in $[0, \|u\|_{L^\infty} + 1]$. Then, for every $\varepsilon > 0$, there is $\bar{k} \in \mathbb{N}$ such that, for every $k \geqslant \bar{k}$,

$$|\widetilde{\psi}(u + r_k) - \widetilde{\psi}(u)| < \varepsilon \quad \text{and} \quad |\psi^c(u + r_k) - \psi^c(u)| < \varepsilon. \qquad (67)$$

For the first term, we get

$$\Big| \int_{\widetilde{\Gamma}_k} \widetilde{\psi}(z_k + r_k)\ \mathrm{d}\mathcal{H}^1 - \int_{\widetilde{\Gamma}} \widetilde{\psi}(u)\ \mathrm{d}\mathcal{H}^1 \Big| = \Big| \int_{\widetilde{\Gamma}} \big[\widetilde{\psi}(u + r_k) - \widetilde{\psi}(u)\big]\ \mathrm{d}\mathcal{H}^1 \Big| \\ + \Big| \int_{\widetilde{\Gamma}\cap\{0<h<1/k\}} \widetilde{\psi}(z_k + r_k)\ \mathrm{d}\mathcal{H}^1 \Big|. \tag{68}$$

Now we use (67), together with

$$\widetilde{\Gamma} \cap \{0 < h < 1/k\} \to \emptyset,$$

and we conclude the convergence to 0 of the surface term in (68), as $k \to \infty$. Regarding the second surface term on the right-hand side of (65), we have that

$$\Big| \int_{\Gamma^c_k} \psi^c(u{+}r_k)\ \mathrm{d}\mathcal{H}^1 - \int_{\Gamma^c} \psi^c(u)\ \mathrm{d}\mathcal{H}^1 \Big| \leqslant \Big| \int_{\Gamma^c} \big[\psi^c(u + r_k) - \psi^c(u)\big]\ \mathrm{d}\mathcal{H}^1 \Big| \\ + \Big| \int_{\Gamma^c\cap\{h^-(x)-1/k<y<h^-(x)\}} \psi^c(u)\ \mathrm{d}\mathcal{H}^1 \Big| \tag{69}$$

From (67) and since

$$\Gamma^c \cap \{h^-(x) - 1/k < y < h^-(x)\} \to \emptyset$$

for $k \to \infty$, we conclude our estimate on the cut part.

By putting together (66), (68) and (69) in (65), we get that

$$\lim_{k\to\infty} \mathcal{F}(\Omega_{g_k}, w_k, \nu_k) = \mathcal{F}(\Omega, v, \mu).$$

Step 2. Now, consider $h \in \mathrm{BV}(a, b)$ with a finite number of cuts. Let $(c^i)_{i=1}^n \subset (a, b)$ be the orthogonal projection on the x-axes of the cuts. Set

$$\varepsilon_0 := \min\{|c^i - c^j| : \quad i \neq j = 1, \dots, n\}. \tag{70}$$

In order to lighten the notation, and since we are considering a function h which has a finite number of cut points, we can work as h had a single cut and then repeating the following construction for the general case. So let c be the cut point of h.

The idea of the construction is to use a Yosida-Moreau transform far from the cut point $a < c < b$ and, around the cut, we use an interpolation in $[c - \varepsilon_0/k, c + \varepsilon_0/k]$ in order to get the Hausdorff convergence to the vertical cut. We need to apply the Yosida-Moreau transform of h with maximal slope k beforehand because we need the mass constraint to be satisfied, as we want to use the same procedure as in (61), which requires a sequence that lies below h. Moreover, since we use the Yosida-Moreau transform of h with maximal slope k far form the cut point, thanks to [37, Lemma 2.7], we have the Hausdorff convergence to our configuration as well as the convergence of the length of the graph.

We define, for each $k \in \mathbb{N}$, $h_k^\ell : (a, c) \mapsto [0, \infty)$ as the Yosida-Moreau transform of h with maximal slope k on (a, c) and $h_k^r : (c, b) \mapsto [0, \infty)$ as the Yosida transform of h on (c, b). Namely

$$h_k^\ell(x) \coloneqq \inf\{h(z) + k|x - z| : z \in (a, c)\},$$

$$h_k^r(x) \coloneqq \inf\{h(z) + k|x - z| : z \in (c, b)\}.$$

We have that both h_k^ℓ and h_k^r are k-Lipschitz functions such that $h_k^\ell \leqslant h$ and $h_k^r \leqslant h$. Furthermore, by [37, Lemma 2.7] we have that $\Omega_{h_k^\ell} \to \Omega \cap \big[(a, c) \times \mathbb{R}\big]$ and $\Omega_{h_k^r} \to \Omega \cap \big[(c, b) \times \mathbb{R}\big]$ as $k \to \infty$, together with their convergence of the length of their respective graph, namely

$$\mathcal{H}^1(\Gamma_{h_k^\ell}) \to \mathcal{H}^1\big(\Gamma \cap ((a, c) \times \mathbb{R})\big),$$

$$\mathcal{H}^1(\Gamma_{h_k^r}) \to \mathcal{H}^1\big(\Gamma \cap ((c, b) \times \mathbb{R})\big),$$

as $k \to \infty$. We can also extend by continuity h_k^ℓ and h_k^r at c, as we have both right and left limit of h at c. We are going to use the following notation

$$S_k \coloneqq \left[c - \frac{\varepsilon_0}{k}, c + \frac{\varepsilon_0}{k}\right] \times \mathbb{R},$$

$$S_k^\ell \coloneqq \left[c - \frac{\varepsilon_0}{k}, c\right] \times \mathbb{R},$$

$$S_k^r \coloneqq \left[c, c + \frac{\varepsilon_0}{k}\right] \times \mathbb{R},$$

where ε_0 is defined in (70). The definition of our sequence $(h_k)_k$ uses the definition of h_k^ℓ and h_k^r outside S_k whereas in S_k we have a linear interpolation from the cut point $(c, h(c))$ and the points $(c-\varepsilon_0/k, h_k^\ell(c-\varepsilon_0/k)$ and $(c+\varepsilon_0/k, h_k^r(c+\varepsilon_0/k))$. We define our Lipschitz sequence as

$$\hat{h}_k(x) := \begin{cases} h_k^\ell(x) & x \in (a, c-\varepsilon_0/k), \\ m_k^\ell x + q_k^\ell & x \in S_k^\ell, \\ m_k^r x + q_k^r & x \in S_k^r, \\ h_k^r(x) & x \in (c+\varepsilon_0/k, b), \end{cases}$$

with suitable coefficients $m_k^\ell, q_k^\ell, m_k^r, q_k^r \in \mathbb{R}$ such that we have linear interpolation from $\big(c-\varepsilon_0/k, h_k^\ell(c-\varepsilon_0/k)\big)$ and $\big(c+\varepsilon_0/k, h_k^r(c+\varepsilon_0/k)\big)$ to the point $(c, h(c))$. Notice that, by definition, $\hat{h}_k(c) = \hat{h}(c)$ and h_k is continuous. Moreover, thanks to Theorem 5, for k large enough, it holds that $\hat{h}_k \leqslant h$. Now, following the same path as in (61), we set

$$h_k(x) := \hat{h}_k(x) + \varepsilon_k,$$

where

$$\varepsilon_k := \frac{1}{b-a}\Big(M - \int_a^b \hat{h}_k(x)\,\mathrm{d}x\Big).$$

We then have that the sequence $(h_k)_k$ satisfies the mass constraint, namely,

$$\int_a^b h_k(x)\,\mathrm{d}x = M.$$

Step 2.1. For every $k \in \mathbb{N}$, let Ω_k be the subgraph of h_k. We prove that $\mathbb{R}^2 \setminus \Omega_k \xrightarrow{H} \mathbb{R}^2 \setminus \Omega$ as $k \to \infty$. We use again the equivalence of the Hausdorff convergence with the Kuratowski convergence (see Proposition 2). Take $\bar{\mathbf{x}} = (\bar{x}, \bar{y}) \in \mathbb{R}^2 \setminus \Omega$. We first want to prove that there exists a sequence $(x_k, y_k)_k \subset \mathbb{R}^2 \setminus \Omega_k$ such that $(x_k, y_k) \to \bar{x}$. Then, we have different cases depending on whether $\bar{\mathbf{x}} \in S_k$ or not. In case $\bar{\mathbf{x}} \notin S_k$, as the sequence $(h_k)_k$ is defined as the Yosida-Moreau transform of h, away from the cut point we can use Lemma 2.7 of [37] and we have already the Hausdorff convergence desired.

Next we deal the case in which $\bar{\mathbf{x}} \in S_k$. If $\bar{x} = c$ and $\bar{y} \leqslant h^-(c)$, consider the sequence

$$(x_k, y_k) := \left(\frac{\bar{y} - q_k^\ell}{m_k^\ell}, \bar{y} \right),$$

for every $k \in \mathbb{N}$. We obtain $(x_k, y_k) \to (c, \bar{y})$ as $k \to \infty$.

In case $\bar{x} = c$ and $\bar{y} \geqslant h^-(c)$ or in case $\bar{x} \neq c$, it is enough to consider the constant sequence $(x_k, y_k) := (c, \bar{y})$, since by definition $h_k \leqslant h$ and thus we have that $(\bar{x}, \bar{y}) \in \mathbb{R}^2 \setminus \Omega_k$, for every $k \in \mathbb{N}$.

We are left to check the second condition of the Kuratowski convergence. Take a sequence $(x_k, y_k)_k \subset \mathbb{R}^2 \setminus (\Omega_k \cap S_k)$ and suppose that $(x_k, y_k) \to (x, y)$ as $k \to \infty$. We need to prove that $(x, y) \in \mathbb{R}^2 \setminus \Omega$. Since $(x_k, y_k) \in S_k$ and the vertical strip S_k is shrinking to the vertical line $c \times \mathbb{R}$, then we must have that $x = c$ thus the point $(x, y) \in \mathbb{R}^2 \setminus \Omega$.

In case our sequence $(x_k, y_k)_k$ is laying both in $\mathbb{R}^2 \setminus (\Omega_k \cap S_k)$ and in $\mathbb{R}^2 \setminus (\Omega_k \setminus S_k)$, as it is converging, it is enough consider k large enough and we get that (x_k, y_k) is only in one of the two sets. Then we can proceed as before.

Thus, we can conclude that $\mathbb{R}^2 \setminus \Omega_k \xrightarrow{H} \mathbb{R}^2 \setminus \Omega$ as $k \to \infty$.

Step 2.2. We are going to define a density on Γ_k. Since u is grid constant we can consider a family of squares $(Q^j)_{j \in J}$, with $J = \{1, \ldots N\}$, such that on each square Q^j we have

$$u_{|Q^j \cap \Gamma} = u^j \in \mathbb{R}.$$

We now define two index sets

$$A_k := \big\{ j \in J : Q^j \cap S_k = \emptyset \big\}, \qquad B_k := J \setminus A_k. \tag{71}$$

In order to define what follows, we recall Lemma 8. The density is then defined as $u_k : \Gamma_k \to \mathbb{R}$

$$
u_k(\mathbf{x}) := \begin{cases}
u^j \dfrac{\mathcal{H}^1(\widetilde{\Gamma} \cap Q^j)}{\mathcal{H}^1(\Gamma_k \cap Q^j)} & \mathbf{x} \in \Gamma_k \cap Q^j, \ j \in A_k, \\[2ex]
a^j \dfrac{\mathcal{H}^1(\Gamma^c \cap Q^j)}{\mathcal{H}^1(\Gamma_k \cap Q^j \cap S_k^\ell)} & \mathbf{x} \in \Gamma_k \cap Q^j \cap S_k^\ell, \ j \in B_k, \\[2ex]
b^j \dfrac{\mathcal{H}^1(\Gamma^c \cap Q^j)}{\mathcal{H}^1(\Gamma_k \cap Q^j \cap S_k^r)} & \mathbf{x} \in \Gamma_k \cap Q^j \cap S_k^r, \ j \in B_k, \\[2ex]
u^j \dfrac{\mathcal{H}^1\big((\widetilde{\Gamma} \cap Q^j) \setminus S_k\big)}{\mathcal{H}^1\big((\Gamma_k \cap Q^j) \setminus S_k\big)} & \mathbf{x} \in (\Gamma_k \cap Q^j) \setminus S_k, \ j \in B_k,
\end{cases}
$$

where a^j, b^j are such that

$$
a^j + b^j = u^j \tag{72}
$$

and

$$
\psi^c(u^j) = \widetilde{\psi}(a^j) + \widetilde{\psi}(b^j). \tag{73}
$$

As the size of the squares is fixed, we take k large enough such that the vertical strip S_k is contained in a single vertical column of squares.

For each $k \in \mathbb{N}$, define the measure $\mu_k := u_k \mathcal{H}^1 \llcorner \Gamma_k$. We have that μ_k satisfies the density constraint. Indeed,

$$
\begin{aligned}
\int_{\Gamma_k} u_k \, \mathrm{d}\mathcal{H}^1 &= \sum_{j \in A_k} \int_{\Gamma_k \cap Q^j} u^j \frac{\mathcal{H}^1(\widetilde{\Gamma} \cap Q^j)}{\mathcal{H}^1(\Gamma_k \cap Q^j)} \, \mathrm{d}\mathcal{H}^1 \\
&\quad + \sum_{j \in B_k} \Big(\int_{\Gamma_k \cap Q^j \cap S_k^\ell} a^j \frac{\mathcal{H}^1(\Gamma^c \cap Q^j)}{\mathcal{H}^1(\Gamma_k \cap Q^j \cap S_k^\ell)} \, \mathrm{d}\mathcal{H}^1 \\
&\quad + \int_{\Gamma_k \cap Q^j \cap S_k^r} b^j \frac{\mathcal{H}^1(\Gamma^c \cap Q^j)}{\mathcal{H}^1(\Gamma_k \cap Q^j \cap S_k^r)} \, \mathrm{d}\mathcal{H}^1 \\
&\quad + \int_{(\Gamma_k \cap Q^j) \setminus S_k} u_j \frac{\mathcal{H}^1\big((\widetilde{\Gamma} \cap Q^j) \setminus S_k\big)}{\mathcal{H}^1\big((\Gamma_k \cap Q^j) \setminus S_k\big)} \, \mathrm{d}\mathcal{H}^1 \Big) \\
&= \sum_{j \in A_k} u^j \mathcal{H}^1(\widetilde{\Gamma} \cap Q^j) + \sum_{j \in B_k} \Big(a_j \mathcal{H}^1(\Gamma^c \cap Q^j) + b_j \mathcal{H}^1(\Gamma^c \cap Q^j)
\end{aligned}
$$

$$+ u^j \mathcal{H}^1\big((\widetilde{\Gamma} \cap Q^j) \setminus S_k)\Big)$$

$$= \sum_{j=1}^{N} \int_{\Gamma \cap Q^j} u^j \; \mathrm{d}\mathcal{H}^1 = m,$$

where in the previous to last step we used (72).

Step 2.3. We prove that $\mu_k \overset{*}{\rightharpoonup} \mu$. Take any $\varphi \in C_c(\mathbb{R}^2)$. For every $\varepsilon > 0$, we can find $\bar{k} \in \mathbb{N}$ such that for every $k \geqslant \bar{k}$ we have $|\varphi(\mathbf{x}) - \varphi(\mathbf{x}^j)| \leqslant \varepsilon$ for all $\mathbf{x} \in Q^j$, where $\mathbf{x}^j$ denotes the center of the square Q^j. From Lemma 8 we have

$$\begin{aligned}
\Big| \int_{\Gamma_k} & \varphi u_k \; \mathrm{d}\mathcal{H}^1 - \int_{\Gamma} \varphi u \; \mathrm{d}\mathcal{H}^1 \Big| \\
&\leqslant \sum_{A_k} \Big| \int_{\Gamma_k \cap Q^j} \varphi u_k \; \mathrm{d}\mathcal{H}^1 - \int_{\widetilde{\Gamma} \cap Q^j} \varphi u^j \; \mathrm{d}\mathcal{H}^1 \Big| \\
&\quad + \sum_{B_k} \Big(\Big| \int_{\Gamma_k \cap Q^j \cap S_k} \varphi u_k \; \mathrm{d}\mathcal{H}^1 - \int_{\Gamma \cap Q^j \cap S_k} \varphi u^j \; \mathrm{d}\mathcal{H}^1 \Big| \\
&\quad + \Big| \int_{(\Gamma_k \cap Q^j) \setminus S_k} \varphi u_k \; \mathrm{d}\mathcal{H}^1 - \int_{(\widetilde{\Gamma} \cap Q^j) \setminus S_k} \varphi u^j \; \mathrm{d}\mathcal{H}^1 \Big| \Big). \qquad (74)
\end{aligned}$$

We now compute first the sum over the indexes in A_k on the right-hand side of (74). By summing and subtracting $\varphi(\mathbf{x}^j)$ inside each of the integral, it holds that

$$\begin{aligned}
\sum_{j \in A_k} \Big| \int_{\Gamma_k \cap Q^j} \varphi u^j \frac{\mathcal{H}^1(\widetilde{\Gamma} \cap Q^j)}{\mathcal{H}^1(\Gamma_k \cap Q^j)} \; \mathrm{d}\mathcal{H}^1 - \int_{\widetilde{\Gamma} \cap Q^j} \varphi u^j \; \mathrm{d}\mathcal{H}^1 \Big| \\
&\leqslant 2 \sum_{j \in A_k} \big|\varphi(\mathbf{x}) - \varphi(\mathbf{x}^j)\big| |u^j| \mathcal{H}^1(\widetilde{\Gamma} \cap Q^j) \\
&\leqslant 2\varepsilon \sum_{j \in A_k} \mathcal{H}^1(\widetilde{\Gamma} \cap Q^j) |u^j| \\
&\leqslant 2\varepsilon ||u||_{L^1(\widetilde{\Gamma})}. \qquad (75)
\end{aligned}$$

We now estimate the sum over B_k on the right-hand side of (74). Note that, up taking a larger $\bar{k} \in \mathbb{N}$, we can assume that

$$\sum_{j \in B_j} |\mathcal{H}^1(\Gamma \cap Q^j \cap S_k) - \mathcal{H}^1(\Gamma \cap Q^j)| \leqslant 4\varepsilon,$$

for all $k \geqslant \bar{k}$. Bearing in mind that for every $j \in \mathbb{N}$ it holds $a^j + b^j = u^j$, we get

$$\begin{aligned}
&\sum_{j \in B_k} \Big| \int_{\Gamma_k \cap Q^j \cap S_k} \varphi u_k \,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma \cap Q^j \cap S_k} \varphi u^j \,\mathrm{d}\mathcal{H}^1 \Big| \\
&\qquad = \sum_{j \in B_k} \Big| \int_{\Gamma_k \cap Q^j \cap S_k^\ell} \varphi a^j \frac{\mathcal{H}^1(\Gamma^c \cap Q^j)}{\mathcal{H}^1(\Gamma_k \cap Q^j \cap S_k^\ell)} \,\mathrm{d}\mathcal{H}^1 \\
&\qquad\qquad + \int_{\Gamma_k \cap Q^j \cap S_k^r} \varphi b^j \frac{\mathcal{H}^1(\Gamma^c \cap Q^j)}{\mathcal{H}^1(\Gamma_k \cap Q^j \cap S_k^r)} \,\mathrm{d}\mathcal{H}^1 \\
&\qquad\qquad - \int_{\Gamma \cap Q^j \cap S_k} \varphi u^j \,\mathrm{d}\mathcal{H}^1 \Big| \\
&\qquad \leqslant 2\varepsilon \sum_{j \in B_k} u^j \big(\mathcal{H}^1(\Gamma^c \cap Q^j) + \mathcal{H}^1(\Gamma \cap Q^j \cap S_k)\big) \\
&\qquad\qquad + |\varphi(\mathbf{x}^j)| u^j \Big| \mathcal{H}^1(\Gamma^c \cap Q^j) - \mathcal{H}^1(\Gamma \cap Q^j \cap S_k) \Big| \\
&\qquad \leqslant 2\varepsilon(2||u||_{L^\infty(\Gamma)} + 4\varepsilon) + 4\varepsilon \|\varphi\|_{\mathcal{C}^0(\mathbb{R}^2)} \|u\|_{L^\infty(\Gamma)}.
\end{aligned} \tag{76}$$

In the same way, we can obtain the estimate for last two terms of the sum over B_k on the right-hand side of (74),

$$\sum_{j \in B_k} \Big| \int_{(\Gamma_k \cap Q^j) \setminus S_k} \varphi u^j \,\mathrm{d}\mathcal{H}^1 - \int_{(\widetilde{\Gamma} \cap Q^j) \setminus S_k} \varphi u^j \,\mathrm{d}\mathcal{H}^1 \Big| \leqslant C\varepsilon ||u||_{L^1(\widetilde{\Gamma})}, \tag{77}$$

for some constant $C > 0$. In conclusion, if we put together (74), (75), (76), (77), we obtain that

$$\Big| \int_{\Gamma_k} \varphi u_k \,\mathrm{d}\mathcal{H}^1 - \int_\Gamma \varphi u \,\mathrm{d}\mathcal{H}^1 \Big| < C'\varepsilon,$$

with $C' > 0$. Since ε is arbitrary, we get that $\mu_k \overset{*}{\rightharpoonup} \mu$ as $k \to \infty$.

Step 2.5. Arguing as in (63), we can define the displacement sequence $(v_k)_k$, with $v_k \in H^1(\Omega_k;\mathbb{R}^2)$ such that $v_k \rightharpoonup v$ in $H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$ as $k \to \infty$.

Step 2.6. It remains to prove the convergence of the energy. By using the index sets in (71), we have that

$$
\begin{aligned}
&|\mathcal{F}(\Omega_k, v_k, \mu_k) - \mathcal{F}(\Omega, v, \mu)| \\
&\leqslant \Big| \int_{\Omega_k} W\big(E(v_k) - E_0(y)\big)\,\mathrm{d}\mathbf{x} - \int_{\Omega} W\big(E(v) - E_0(y)\big)\,\mathrm{d}\mathbf{x} \Big| \\
&\qquad + \sum_{j\in A_k} \Big| \int_{\Gamma_k \cap Q^j} \widetilde{\psi}(u^j)\,\mathrm{d}\mathcal{H}^1 - \int_{\widetilde{\Gamma}\cap Q^j} \widetilde{\psi}(u^j)\,\mathrm{d}\mathcal{H}^1 \Big| \\
&\qquad + \sum_{j\in B_k} \Big| \int_{\Gamma_k \cap Q^j \cap S_k} \widetilde{\psi}(u^j)\,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma^c \cap Q^j_k} \psi^c(u^j)\,\mathrm{d}\mathcal{H}^1 \Big| \\
&\qquad + \sum_{j\in B_k} \Big| \int_{(\Gamma_k \cap Q^j)\setminus S_k} \widetilde{\psi}(u_j)\,\mathrm{d}\mathcal{H}^1 - \int_{(\widetilde{\Gamma}\cap Q^j)\setminus S_k} \widetilde{\psi}(u^j)\,\mathrm{d}\mathcal{H}^1 \Big|.
\end{aligned} \tag{78}
$$

We will estimate the four terms on the right-hand side of (78) separately. For the bulk term, we can use the same method as in (66) and we conclude that

$$
\Big| \int_{\Omega_k} W\big(E(v_k) - E_0(y)\big)\,\mathrm{d}\mathbf{x} - \int_{\Omega} W\big(E(v) - E_0(y)\big)\,\mathrm{d}\mathbf{x} \Big| \to 0, \tag{79}
$$

as $k \to \infty$.

We now consider the first sum on the right hand side of (78).We have that

$$
\begin{aligned}
&\sum_{j\in A_k} \Big| \int_{\Gamma_k \cap Q^j} \widetilde{\psi}\left(u^j \frac{\mathcal{H}^1(\widetilde{\Gamma}\cap Q^j)}{\mathcal{H}^1(\Gamma_k \cap Q^j)} \right) \mathrm{d}\mathcal{H}^1 - \int_{\widetilde{\Gamma}\cap Q^j} \widetilde{\psi}(u^j)\,\mathrm{d}\mathcal{H}^1 \Big| \\
&\qquad \leqslant \sum_{j\in A_k} \left| \widetilde{\psi}\left(u^j \frac{\mathcal{H}^1(\widetilde{\Gamma}\cap Q^j)}{\mathcal{H}^1(\Gamma_k \cap Q^j)} \right) \mathcal{H}^1(\Gamma_k \cap Q^j) - \widetilde{\psi}(u^j)\mathcal{H}^1(\Gamma_k \cap Q^j) \right|
\end{aligned}
$$

$$+\sum_{j\in A_k}\left|\widetilde{\psi}(u^j)\mathcal{H}^1(\Gamma_k\cap Q^j)-\widetilde{\psi}(u^j)\mathcal{H}^1(\widetilde{\Gamma}\cap Q^j)\right|. \tag{80}$$

From the fact that $\widetilde{\psi}$ is continuous and since $\mathcal{H}^1(\Gamma_k\cap Q^j)\to\mathcal{H}^1(\widetilde{\Gamma}\cap Q^j)$ as $k\to\infty$, for every $\varepsilon>0$, there is $\bar{k}\in\mathbb{N}$ such that for every $k\geqslant\bar{k}$ we have

$$|\mathcal{H}^1(\Gamma_k\cap Q^j)-\mathcal{H}^1(\widetilde{\Gamma}\cap Q^j)|<\varepsilon.$$

and

$$\left|\widetilde{\psi}\left(u^j\frac{\mathcal{H}^1(\widetilde{\Gamma}\cap Q^j)}{\mathcal{H}^1(\Gamma_k\cap Q^j)}\right)-\widetilde{\psi}(u^j)\right|<\varepsilon.$$

Then, from (80) we have that

$$\sum_{j\in A_k}\Big|\int_{\Gamma_k\cap Q^j}\widetilde{\psi}\left(u^j\frac{\mathcal{H}^1(\widetilde{\Gamma}\cap Q^j)}{\mathcal{H}^1(\Gamma_k\cap Q^j)}\right)\,\mathrm{d}\mathcal{H}^1-\int_{\widetilde{\Gamma}\cap Q^j}\widetilde{\psi}(u^j)\,\mathrm{d}\mathcal{H}^1\Big|$$

$$\leqslant\varepsilon\sum_{j\in A_k}\mathcal{H}^1(\Gamma_k\cap Q^j)+\varepsilon\sum_{j\in A_k}\widetilde{\psi}(u^j). \tag{81}$$

As ε is arbitrary, we can conclude our estimate.

Regarding the second sum on the right-hand side of (78), we use the a similar method as in (76). Now, for the first two terms can be estimated as follows,

$$\sum_{j\in B_k}\Big|\int_{\Gamma_k\cap Q^j\cap S_k}\widetilde{\psi}(u^j)\,\mathrm{d}\mathcal{H}^1-\int_{\Gamma^c\cap Q^j}\psi^c(u^j)\,\mathrm{d}\mathcal{H}^1\Big|$$

$$=\sum_{j\in B_k}\Big|\widetilde{\psi}\left(a^j\frac{\mathcal{H}^1(\Gamma^c\cap Q^j)}{\mathcal{H}^1(\Gamma_k\cap Q^j\cap S_k^\ell)}\right)\mathcal{H}^1(\Gamma_k\cap Q^j\cap S_k^\ell)$$

$$+\widetilde{\psi}\left(b^j\frac{\mathcal{H}^1(\Gamma^c\cap Q^j)}{\mathcal{H}^1(\Gamma_k\cap Q^j\cap S_k^r)}\right)\mathcal{H}^1(\Gamma_k\cap Q^j\cap S_k^r)$$

$$-\psi^c(u^j)\mathcal{H}^1(\Gamma^c\cap Q^j)\Big|. \tag{82}$$

By using the same argument that led us to (81), consider $\varepsilon > 0$ as before, then, for k large enough, we have

$$|\mathcal{H}^1(\Gamma_k \cap Q^j \cap S_k^\ell) - \mathcal{H}^1(\Gamma^c \cap Q^j)| < \varepsilon,$$
$$|\mathcal{H}^1(\Gamma_k \cap Q^j \cap S_k^r) - \mathcal{H}^1(\Gamma^c \cap Q^j)| < \varepsilon,$$

and, by the continuity of $\widetilde{\psi}$,

$$\left|\widetilde{\psi}\left(a^j \frac{\mathcal{H}^1(\Gamma^c \cap Q^j)}{\mathcal{H}^1(\Gamma_k \cap Q^j \cap S_k^\ell)}\right) - \widetilde{\psi}(a^j)\right| < \varepsilon,$$
$$\left|\widetilde{\psi}\left(b^j \frac{\mathcal{H}^1(\Gamma^c \cap Q^j)}{\mathcal{H}^1(\Gamma_k \cap Q^j \cap S_k^r)}\right) - \widetilde{\psi}(b^j)\right| < \varepsilon.$$

As a consequence, from (82) we get

$$\begin{aligned}
\sum_{j \in B_k} \Big| \int_{\Gamma_k \cap Q^j \cap S_k} & \widetilde{\psi}(u^j)\ \mathrm{d}\mathcal{H}^1 - \int_{\Gamma^c \cap Q^j} \psi^c(u^j)\ \mathrm{d}\mathcal{H}^1 \Big| \\
&\leqslant \varepsilon \sum_{j \in B_k} \big(\mathcal{H}^1(\Gamma_k \cap Q^j \cap S_k^\ell) + \mathcal{H}^1(\Gamma_k \cap Q^j \cap S_k^r)\big) \\
&\quad + \sum_{j \in B_k} |\widetilde{\psi}(a_j)\mathcal{H}^1(\Gamma_k \cap Q^j \cap S_k^\ell) \\
&\quad + \widetilde{\psi}(b_j)\mathcal{H}^1(\Gamma_k \cap Q^j \cap S_k^r) - \psi^c(u^j)\mathcal{H}^1(\Gamma^c \cap Q^j)| \\
&= \varepsilon \sum_{j \in B_k} \mathcal{H}^1(\Gamma_k \cap Q^j \cap S_k) + \varepsilon \sum_{j \in B_k} \big(\widetilde{\psi}(a^j) + \widetilde{\psi}(b^j)\big) \\
&\quad + \sum_{j \in B_k} |\widetilde{\psi}(a^j) + \widetilde{\psi}(b^j) - \psi^c(u^j)|\mathcal{H}^1(\Gamma^c \cap Q^j)
\end{aligned} \tag{83}$$

Now, we conclude our estimate by using (73) and the fact that ε is arbitrary.

The third sum in the right hand side of (78) can be treated in the same way as before. Consider $\varepsilon > 0$ as above, then, for k large enough we have

$$|\mathcal{H}^1\big((\widetilde{\Gamma} \cap Q^j) \setminus S_k\big) - \mathcal{H}^1(\widetilde{\Gamma} \cap Q^j)| < \varepsilon$$

and

$$\left|\widetilde{\psi}\left(u_j\frac{\mathcal{H}^1\big((\widetilde{\Gamma}\cap Q^j)\setminus S_k\big)}{\mathcal{H}^1\big((\Gamma_k\cap Q^j)\setminus S_k\big)}\right)-\widetilde{\psi}(u^j)\right|<\varepsilon.$$

Thus, we have

$$\begin{aligned}\sum_{j\in B_k}\Big|\int_{(\Gamma_k\cap Q^j)\setminus S_k}&\widetilde{\psi}\left(u_j\frac{\mathcal{H}^1\big((\widetilde{\Gamma}\cap Q^j)\setminus S_k\big)}{\mathcal{H}^1\big((\Gamma_k\cap Q^j)\setminus S_k\big)}\right)\,\mathrm{d}\mathcal{H}^1-\int_{(\widetilde{\Gamma}\cap Q^j)\setminus S_k}\widetilde{\psi}(u^j)\,\mathrm{d}\mathcal{H}^1\Big|\\ &\leqslant\varepsilon\sum_{j\in B_k}\mathcal{H}^1\big((\Gamma_k\cap Q^j)\setminus S_k\big)+\varepsilon\sum_{j\in B_k}\widetilde{\psi}(u^j)\mathcal{H}^1\big((\Gamma_k\cap Q^j)\setminus S_k\big)\\ &\qquad+\sum_{j\in B_k}\big|\widetilde{\psi}(u^j)\mathcal{H}^1\big((\Gamma_k\cap Q^j)\setminus S_k\big)-\widetilde{\psi}(u^j)\mathcal{H}^1(\widetilde{\Gamma}\cap Q^j)\big|.\end{aligned}\tag{84}$$

Since ε is arbitrary and from the fact that $\mathcal{H}^1\big((\Gamma_k\cap Q^j)\setminus S_k\big)\to\mathcal{H}^1(\widetilde{\Gamma}\cap Q^j)$ as $k\to\infty$, we can conclude the last estimate.

By putting together (79), (81), (83) and (84) in (78), we conclude that

$$\lim_{k\to\infty}\mathcal{F}(\Omega_k,v_k,\mu_k)=\mathcal{F}(\Omega,v,\mu).$$

□

Theorem 13. Let (Ω,v,μ) be such that h is a non-negative Lipschitz function, $v\in H^1(\Omega;\mathbb{R}^2)$ and $\mu=u\mathcal{H}^1\llcorner\Gamma$, with $u\in L^1(\Gamma)$ a grid constant density. Then, there is a sequence $(\Omega_k,v_k,\mu_k)_k\subset\mathcal{A}_r(m,M)$, with $\mu_k=u_k\mathcal{H}^1\llcorner\Gamma_k$ and $u_k\in L^1(\mathcal{H}^1\llcorner\Gamma_k)$ grid constant, such that

$$\lim_{k\to\infty}\mathcal{H}(\Omega_k,v_k,\mu_k)=\mathcal{F}(\Omega,v,\mu),$$

and $(\Omega_k,v_k,\mu_k)\to(\Omega,v,\mu)$, as $k\to\infty$.

Proof. Step 1. Denote by ψ^{cvx} the convex envelope of ψ, namely,

$$\psi^{\mathrm{cvx}}:=\{\rho:\rho\text{ is convex and }\rho\leqslant\psi\}.$$

It is well known (see, for instance, [40, Theorem 5.32 and Remark 5.33]) that for any given density $w \in L^1(\Gamma_g)$, with g a Lipschitz function, then there is a sequence $(w_m)_m \subset L^1(\Gamma_g)$ such that $w_m \rightharpoonup w$ in $L^1(\Gamma_g)$ and

$$\lim_{m\to\infty} \int_{\Gamma_g} \psi(w_m)\, \mathrm{d}\mathcal{H}^1 = \int_{\Gamma_g} \psi^{\mathrm{cvx}}(w)\, \mathrm{d}\mathcal{H}^1.$$

In particular, $w_m\mathcal{H}^1 \llcorner \Gamma_g \overset{*}{\rightharpoonup} w\mathcal{H}^1 \llcorner \Gamma_g$ as $k \to \infty$. Therefore, if we prove the statement of the proposition for ψ convex we also have it for ψ Borel. Thus, from now on, in order to enlighten the notation, we will assume ψ to be a convex function.

Step 2. Take any configuration (Ω, v, μ), where h is a Lipschitz function, $v \in H^1(\Omega;\mathbb{R}^2)$ and $\mu = u\mathcal{H}^1 \llcorner \Gamma$ is a grid constant density. Then, we can consider a finite grid of open squares $(Q^j)_{j\in J}$ such that

$$u_{|Q^j\cap\Gamma} = u^j \in \mathbb{R}.$$

for each $j \in J$. By construction, there are finitely many points $a = x^0 < x^1 < \cdots < x^n = b$ such that $u = u^i \in \mathbb{R}$ on

$$\mathrm{graph}(h) \cap [(x^i, x^{i+1}) \times \mathbb{R}],$$

for every $i = 0, \ldots, n$ (see Figure 1.6).

Define the index sets

$$\begin{aligned} A &\coloneqq \{i = 1, \ldots, n : u^i \leqslant s_0\}, \\ B &\coloneqq \{i = 1, \ldots, n\} \setminus A, \end{aligned} \tag{85}$$

where s_0 is given by Lemma 5. In such a way, we are going to apply the wriggling process for $i \in B$. By Lemma 3, for every $i \in B$, we choose $r^i > 1$ such that

$$u^i = r^i s_0.$$

and we have, on each interval (x^i, x^{i+1}), a Lipschitz sequence $(\bar{h}^i_k)_k$, that verifies the following properties:

(*i*) $\mathcal{H}^1(\Gamma^i_k) = r^i\mathcal{H}^1(\Gamma \cap [(x^i, x^{i+1}) \times \mathbb{R}])$, where $\Gamma^i_k \coloneqq \mathrm{graph}(\bar{h}^i_k)$,

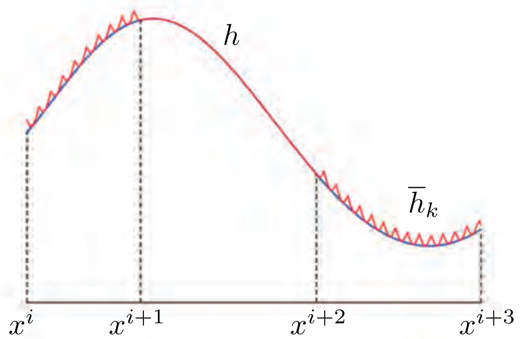

Figure 1.6: On each interval $[x^i, x^{i+1}]$, depending on whether $u^j > s_0$ or not, we will apply the wriggling process and change the density to s_0, or do not change anything.

(ii) $h(x^i) = \bar{h}^i_k(x^i)$, and $h(x^{i+1}) = \bar{h}^i_k(x^{i+1})$,

(iii) $h_{|(x^i,x^{i+1})} \leqslant \bar{h}^i_k$,

(iv) $\bar{h}^i_k \to h_{|(x^i,x^{i+1})}$ uniformly as $k \to \infty$,

(v) $\mathcal{H}^1 \llcorner \Gamma^i_k \overset{*}{\rightharpoonup} r^i \mathcal{H}^1 \llcorner (\Gamma \cap (x^i, x^{i+1}) \times \mathbb{R})$, as $k \to \infty$.

Then, we define the Lipschitz sequence $(\bar{h}_k)_k$ as

$$\bar{h}_{k|(x^i,x^{i+1})} := \begin{cases} \bar{h}^i_k & u^i > s_0, \\ h_{|(x^i,x^{i+1})} & u^i \leqslant s_0, \end{cases}$$

By setting $\bar{\Gamma}_k := \operatorname{graph}(\bar{h}_k)$, we define the density $\bar{u}_k$ on $\bar{\Gamma}_k$ as

$$\bar{u}_{k|(x^i,x^{i+1})\times\mathbb{R}} := \begin{cases} s_0 & u^i > s_0 \\ u^i & u^i \leqslant s_0, \end{cases}$$

We have that the sequence $(\bar{u}_k)_k$ define above satisfies the density constraint. Indeed, by considering the index set defined in (85), we have

$$
\begin{aligned}
\int_{\overline{\Gamma}_k} \bar{u}_k \, \mathrm{d}\mathcal{H}^1 &= \sum_{i\in A} \int_{\overline{\Gamma}_k \cap [(x^i, x^{i+1})\times\mathbb{R}]} u^i \, \mathrm{d}\mathcal{H}^1 + \sum_{i\in B} \int_{\overline{\Gamma}_k \cap [(x^i, x^{i+1})\times\mathbb{R}]} s_0 \, \mathrm{d}\mathcal{H}^1 \\
&= \sum_{i\in A} u^i \mathcal{H}^1(\overline{\Gamma}_k \cap [(x^i, x^{i+1}) \times \mathbb{R}]) \\
&\qquad + \sum_{i\in B} s_0 \mathcal{H}^1(\overline{\Gamma}_k \cap [(x^i, x^{i+1}) \times \mathbb{R}]) \\
&= \sum_{i\in A} u^i \mathcal{H}^1(\Gamma \cap [(x^i, x^{i+1}) \times \mathbb{R}]) \\
&\qquad + + \sum_{i\in B} s_0 r^i \mathcal{H}^1(\Gamma \cap [(x^i, x^{i+1}) \times \mathbb{R}]) \\
&= \sum_{i=1}^{n} \int_{\Gamma \cap [(x^i, x^{i+1})\times\mathbb{R}]} u^i \, \mathrm{d}\mathcal{H}^1 \\
&= m,
\end{aligned}
$$

where in the third to last step we used the fact that

$$
\mathcal{H}^1(\Gamma_k \cap [(x^i, x^{i+1}) \times \mathbb{R}]) = r^i \mathcal{H}^1(\Gamma \cap [(x^i, x^{i+1}) \times \mathbb{R}]), \tag{86}
$$

for every $i \in B$.

Step 3. Since in general $h \leqslant \bar{h}_k$, we have that $M = |\Omega| \leqslant |\overline{\Omega}_k|$, where $\overline{\Omega}_k$ is the subgraph of $\bar{h}_k$, for each $k \in \mathbb{N}$. In order to fix the mass constraint we set

$$
\gamma_k := \frac{M}{|\overline{\Omega}_k|} \leqslant 1,
$$

and we have that $\gamma_k \to 1$ as $k \to \infty$. Define, for each $k \in \mathbb{N}$,

$$
h_k := \gamma_k \bar{h}_k.
$$

Now the sequence $(h_k)_k$ satisfies the mass constraint, indeed

$$
\int_a^b h_k \, \mathrm{d}x = \int_a^b \gamma_k \bar{h}_k \, \mathrm{d}x = \gamma_k |\overline{\Omega}_k| = M.
$$

Now, let $\Gamma_k := \mathrm{graph}(h_k)$. Since in general, for every $k \in \mathbb{N}$, $\mathcal{H}^1(\Gamma_k) \leqslant \mathcal{H}^1(\overline{\Gamma}_k)$, we need to adjust the density constraint. By knowing that

$$\int_{\overline{\Gamma}_k} \bar{u}_k \, \mathrm{d}\mathcal{H}^1 = m,$$

we need to define a new sequence of density $(u_k)_k$ on Γ_k such that, for every $k \in \mathbb{N}$,

$$\int_{\Gamma_k} u_k \, \mathrm{d}\mathcal{H}^1 = m.$$

Thus we set, for each $k \in \mathbb{N}$,

$$u_k := \frac{\bar{u}_k}{t_k},$$

with

$$t_k := \frac{\mathcal{H}^1(\Gamma_k)}{\mathcal{H}^1(\overline{\Gamma}_k)} \leqslant 1.$$

Notice that $t_k \to 1$ as $k \to \infty$. We have that the sequence $(u_k)_k$ satisfies the density constraint. Indeed,

$$\int_{\Gamma_k} u_k \, \mathrm{d}\mathcal{H}^1 = \frac{\bar{u}_k}{t_k}\mathcal{H}^1(\Gamma_k) = \bar{u}_k \mathcal{H}^1(\overline{\Gamma}_k) = \int_{\overline{\Gamma}_k} \bar{u}_k \, \mathrm{d}\mathcal{H}^1 = m.$$

Step 3. We now prove the convergence of the density, namely $u_k\mathcal{H}^1 \llcorner \Gamma_k \stackrel{*}{\rightharpoonup} u\mathcal{H}^1 \llcorner \Gamma$. To do so, we first prove that $\bar{u}_k\mathcal{H}^1 \llcorner \Gamma_k \stackrel{*}{\rightharpoonup} u\mathcal{H}^1 \llcorner \Gamma$, and then we conclude by triangle inequality.

Take any $\varphi \in C_c(\mathbb{R}^2)$ and consider $\varepsilon > 0$. We can find $\delta > 0$ such that, if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ satisfy

$$|\mathbf{y} - \mathbf{x}| < \delta,$$

then

$$|\varphi(\mathbf{y}) - \varphi(\mathbf{x})| < \varepsilon. \tag{87}$$

Up to refining the intervals (x^i, x^{i+1}), we can assume that

$$|x^i - x^{i+1}| < \frac{\delta}{\sqrt{2}}.$$

Let $K > 0$ such that for every $k \in \mathbb{N}$ we have $h_k \leqslant K$ and $h \leqslant K$. This is possible, as our sequence is uniformly bounded by definition and h is bounded. Consider a finite partition of $[0, K]$ given by $y^0 = 0, y^1, \dots, y^m = K$, such that for every $l = 1, \dots, m$ we have

$$|y^l - y^{l+1}| < \frac{\delta}{\sqrt{2}}.$$

Moreover, for every l, consider $\bar{y}^l \in [y^l, y^{l+1}]$. Then, from (87), for every $\mathbf{x} \in [x^i, x^{i+1}] \times [y^l, y^{l+1}]$, we have

$$|\varphi(\mathbf{x}) - \varphi(\bar{x}^i, \bar{y}^l)| < \varepsilon.$$

We then have

$$\begin{aligned}
&\Big| \int_{\Gamma_k} \bar{u}_k \varphi \, \mathrm{d}\mathcal{H}^1 - \int_{\Gamma} u\varphi \, \mathrm{d}\mathcal{H}^1 \Big| \\
&\quad = \Big| \sum_{i\in A} \int_{\Gamma_k \cap [(x^i, x^{i+1}) \times \mathbb{R}]} u^i \varphi \, \mathrm{d}\mathcal{H}^1 + \sum_{i \in B} \int_{\Gamma_k \cap [(x^i, x^{i+1}) \times \mathbb{R}]} s_0 \varphi \, \mathrm{d}\mathcal{H}^1 \\
&\qquad\quad - \sum_{i=0}^{n} \int_{\Gamma \cap [(x^i, x^{i+1}) \times \mathbb{R}]} u^i \varphi \, \mathrm{d}\mathcal{H}^1 \Big| \\
&\quad = \sum_{i \in B} \Big| \int_{\Gamma_k \cap [(x^i, x^{i+1}) \times \mathbb{R}]} s_0 \varphi \, \mathrm{d}\mathcal{H}^1 - \int_{\Gamma \cap [(x^i, x^{i+1}) \times \mathbb{R}]} u^i \varphi \, \mathrm{d}\mathcal{H}^1 \Big| \\
&\quad = \sum_{l=0}^{m} \sum_{i \in B} \Big| \int_{\Gamma_k \cap [(x^i, x^{i+1}) \times (y^l, y^{l+1})]} s_0 \big[\varphi(\mathbf{x}) - \varphi\big(\bar{x}^i, \bar{y}^l\big)\big] \, \mathrm{d}\mathcal{H}^1 \Big| \\
&\qquad\quad + \sum_{l=0}^{m} \sum_{i \in B} \Big| \int_{\Gamma \cap [(x^i, x^{i+1}) \times (y^l, y^{l+1})]} u^i \big[\varphi(\mathbf{x}) - \varphi\big(\bar{x}^i, \bar{y}^l\big)\big] \, \mathrm{d}\mathcal{H}^1 \Big| \\
&\qquad\quad + \sum_{l=0}^{m} \sum_{i \in B} \big| s_0 \varphi(\bar{x}^i, \bar{y}^l) \mathcal{H}^1(\overline{\Gamma}_k \cap [(x^i, x^{i+1}) \times (y^l, y^{l+1})])
\end{aligned}$$

$$
\begin{aligned}
&- u^i\varphi(\bar{x}^i,\bar{y}^l)\mathcal{H}^1(\Gamma\cap[(x^i,x^{i+1})\times(y^l,y^{l+1})])\big|\\
\leqslant\ &\varepsilon s_0\sum_{l=0}^{m}\sum_{i\in B}\mathcal{H}^1(\overline{\Gamma}_k\cap[(x^i,x^{i+1})\times(y^l,y^{l+1})])\\
&+\varepsilon u^i\sum_{l=0}^{m}\sum_{i\in B}\mathcal{H}^1(\Gamma\cap[(x^i,x^{i+1})\times(y^l,y^{l+1})])\\
&+||\varphi||_{\mathcal{C}^0(\mathbb{R}^2)}\sum_{l=0}^{m}\sum_{i\in B}|s_0\mathcal{H}^1(\overline{\Gamma}_k\cap[(x^i,x^{i+1})\times(y^l,y^{l+1})])\\
&- u^i\mathcal{H}^1(\Gamma\cap[(x^i,x^{i+1})\times(y^l,y^{l+1})])|\\
\leqslant\ &\varepsilon s_0\sum_{i\in B}\mathcal{H}^1(\overline{\Gamma}_k\cap[(x^i,x^{i+1})\times\mathbb{R}])\\
&+\varepsilon u^i\sum_{i\in B}\mathcal{H}^1(\Gamma\cap[(x^i,x^{i+1})\times\mathbb{R}])\\
&+||\varphi||_{\mathcal{C}^0(\mathbb{R}^2)}\sum_{i\in B}|s_0\mathcal{H}^1(\overline{\Gamma}_k\cap[(x^i,x^{i+1})\times\mathbb{R}])\\
&- u^i\mathcal{H}^1(\Gamma\cap[(x^i,x^{i+1})\times\mathbb{R}])|
\end{aligned}
$$

Now, by using condition (86) we get

$$
\Big|\int_{\overline{\Gamma}_k}\bar{u}_k\varphi\ \mathrm{d}\mathcal{H}^1-\int_{\Gamma}u\varphi\ \mathrm{d}\mathcal{H}^1\Big|\leqslant 2\varepsilon||u||_{L^1(\Gamma)}, \tag{88}
$$

where we can conclude as ε was arbitrary.

In order to prove that $u_k\mathcal{H}^1\llcorner\Gamma_k\overset{*}{\rightharpoonup}u\mathcal{H}^1\llcorner\Gamma$, we can use (88) together with the triangle inequality and the following estimates. We fix φ and ε as in (87), so we have

$$
\begin{aligned}
&\Big|\int_{\Gamma_k}u_k\varphi\ \mathrm{d}\mathcal{H}^1-\int_{\overline{\Gamma}_k}\bar{u}_k\varphi\ \mathrm{d}\mathcal{H}^1\Big|\\
&=\Big|\int_a^b\Big(\frac{\bar{u}_k}{t_k}\varphi(x,h_k(x))\sqrt{1+\gamma_k^2\bar{h}_k'(x)^2}-\bar{u}_k\varphi(x,\bar{h}_k(x))\sqrt{1+\bar{h}_k'(x)^2}\Big)\ \mathrm{d}x\Big|
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \Big| \int_a^b \Big[\Big(\frac{1}{t_k} - 1 \Big) \bar{u}_k \varphi(x, h_k(x)) \sqrt{1 + \gamma_k^2 \bar{h}'_k(x)^2} \\
&\quad + \bar{u}_k \varphi(x, h_k(x)) \sqrt{1 + \gamma_k^2 \bar{h}'_k(x)^2} \\
&\quad - \bar{u}_k \varphi(x, \bar{h}_k(x)) \sqrt{1 + \bar{h}'_k(x)^2} \Big] \, \mathrm{d}x \Big|. \qquad (89)
\end{aligned}$$

Regarding the first term on the right hand side of (89), we have that the sequence $(\bar{h}_k)_k$ is uniformly Lipschitz, as stated in Remark 7. Then there is $L > 0$ such that $|\bar{h}'_k| \leqslant L$. Furthermore, we have that, for every $k \in \mathbb{N}$, $|\bar{u}_k| \leqslant C$, with $C > 0$, and we get

$$\begin{aligned}
&\Big| \int_a^b \Big(\frac{1}{t_k} - 1 \Big) \bar{u}_k \varphi(x, h_k(x)) \sqrt{1 + \gamma_k^2 \bar{h}'_k(x)^2} \, \mathrm{d}x \Big| \\
&\qquad \leqslant \Big| \frac{1}{t_k} - 1 \Big| C ||\varphi||_{\mathcal{C}^0(\mathbb{R}^2)} \sqrt{1 + \gamma_k^2 L^2}, \qquad (90)
\end{aligned}$$

Now, we estimate the remaining two terms on the right-hand side of (89). Let $\varepsilon' > 0$. There is $k' \in \mathbb{N}$ such that for $k \geqslant k'$ we have

$$|\gamma_k - 1| \leqslant \varepsilon'.$$

Since the function $x \mapsto \sqrt{1 + x^2}$ is Lipschitz we have

$$\begin{aligned}
\Big| \sqrt{1 + \gamma_k^2 \bar{h}'_k(x)^2} - \sqrt{1 + \bar{h}'_k(x)^2} \Big| &\leqslant 2 |\gamma_k \bar{h}'_k(x) - \bar{h}'_k(x)| \\
&\leqslant 2L |\gamma_k - 1| \\
&\leqslant 2L\varepsilon'. \qquad (91)
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\int_a^b \Big| \bar{u}_k \varphi(x, h_k(x)) \sqrt{1 + \gamma_k^2 \bar{h}'_k(x)^2} - \bar{u}_k \varphi(x, \bar{h}_k(x)) \sqrt{1 + \bar{h}'_k(x)^2} \Big| \, \mathrm{d}x \\
&\quad \leqslant \int_a^b \Big| \bar{u}_k \varphi(x, h_k(x)) \sqrt{1 + \gamma_k^2 \bar{h}'_k(x)^2} - \bar{u}_k \varphi(x, h_k(x)) \sqrt{1 + \bar{h}'_k(x)^2} \Big| \, \mathrm{d}x \\
&\qquad + \int_a^b \Big| \bar{u}_k \varphi(x, h_k(x)) \sqrt{1 + \bar{h}'_k(x)^2}
\end{aligned}$$

$$- \bar{u}_k \varphi(x, \bar{h}_k(x)) \sqrt{1 + \bar{h}_k'(x)^2} \Big| \, \mathrm{d}x. \tag{92}$$

Then, the first term on the right-hand side of (92) can be estimated by using (91) and we get

$$\int_a^b \Big| \bar{u}_k \varphi(x, h_k(x)) \sqrt{1 + \gamma_k^2 \bar{h}_k'(x)^2} - \bar{u}_k \varphi(x, h_k(x)) \sqrt{1 + \bar{h}_k'(x)^2} \Big| \, \mathrm{d}x$$
$$\leqslant K' \varepsilon', \tag{93}$$

where $K' := 2LC(b-a)||\varphi||_{\mathcal{C}^0(\mathbb{R}^2)}$.
The second term on the right-hand side of (92) is estimated by using the uniform continuity of φ. Since there is $C' > 0$ such that $|h_k| < C'$, for every $k \in \mathbb{N}$, we also have

$$|h_k(x) - \bar{h}_k(x)| = |\gamma_k - 1||\bar{h}_k(x)| \leqslant \varepsilon' C'.$$

As a consequence, by using a similar approach as in (87), we get

$$\int_a^b \Big| \bar{u}_k \varphi(x, h_k(x)) \sqrt{1 + \bar{h}_k'(x)^2} - \bar{u}_k \varphi(x, \bar{h}_k(x)) \sqrt{1 + \bar{h}_k'(x)^2} \Big| \, \mathrm{d}x$$
$$\leqslant K'' \varepsilon, \tag{94}$$

where $K'' := (b-a)C\sqrt{1 + L^2}$.

By putting (93) and (94) in (91), we get that

$$\int_a^b \Big| \bar{u}_k \varphi(x, h_k(x)) \sqrt{1 + \gamma_k^2 \bar{h}_k'(x)^2} - \bar{u}_k \varphi(x, \bar{h}_k(x)) \sqrt{1 + \bar{h}_k'(x)^2} \Big| \, \mathrm{d}x$$
$$\leqslant K' \varepsilon' + K'' \varepsilon. \tag{95}$$

Now, by putting (90) and (95) in (89) we get

$$\Big| \int_{\Gamma_k} u_k \varphi \, \mathrm{d}\mathcal{H}^1 - \int_{\bar{\Gamma}_k} \bar{u}_k \varphi \, \mathrm{d}\mathcal{H}^1 \Big| \leqslant \Big| \frac{1}{t_k} - 1 \Big| C ||\varphi||_{\mathcal{C}^0(\mathbb{R}^2)} \sqrt{1 + \gamma_k^2 L^2}$$
$$+ K' \varepsilon' + K'' \varepsilon. \tag{96}$$

Finally, by using (88) and (96) we get

$$\Big| \int_{\Gamma_k} u_k \varphi \, \mathrm{d}\mathcal{H}^1 - \int_{\Gamma} u \varphi \, \mathrm{d}\mathcal{H}^1 \Big| \leqslant \Big| \int_{\Gamma_k} u_k \varphi \, \mathrm{d}\mathcal{H}^1 - \int_{\bar{\Gamma}_k} \bar{u}_k \varphi \, \mathrm{d}\mathcal{H}^1 \Big|$$

$$
\begin{aligned}
&\qquad + \Big| \int_{\overline{\Gamma}_k} \bar{u}_k \varphi \, \mathrm{d}\mathcal{H}^1 - \int_{\Gamma} u\varphi \, \mathrm{d}\mathcal{H}^1 \Big| \\
&\leqslant 2\varepsilon ||u||_{L^1(\Gamma)} + K'\varepsilon' + K''\varepsilon \\
&\qquad + \Big| \frac{1}{t_k} - 1 \Big| C ||\varphi||_{\mathcal{C}^0(\mathbb{R}^2)} \sqrt{1+\gamma_k^2 L^2}.
\end{aligned}
$$

we can conclude since ε and ε' were arbitrary and by letting $k \to \infty$.

Step 4. Regarding the displacement, set

$$
v_k(x,y) := v(x, \gamma_k y).
$$

The definition of the v_k's is well posed, indeed $(x, \gamma_k y) \in \Omega_k$ if and only if $y \leqslant \bar{h}_k(x)$. In particular $h \leqslant \bar{h}_k$, hence $v(x, \gamma_k y)$ is well defined at every point. Notice that, since $h_k \geqslant 0$, we have that for $y \leqslant 0$ it holds $v_k = v$. Thus, denote the bounded open set

$$
\Omega^+ := \Omega \cap \{y > 0\},
$$

and note that the set

$$
\Omega_k^+ := \{(x, \gamma_k y) : (x,y) \in \Omega^+\}.
$$

is also open and bounded.

We now prove that $v_k \rightharpoonup v$ in $H^1_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$, as $k \to \infty$. Indeed, take $\varphi \in \mathcal{C}_c(\mathbb{R}^2)$. Fix $\varepsilon > 0$ and since φ is uniformly continuous, we have that $|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| < \varepsilon$, every time $|\mathbf{x} - \mathbf{y}| < \delta$ for some $\delta > 0$. In particular, since $\gamma_k \to 1$, if k is large enough, we have

$$
\Big| \varphi\Big(x, \frac{y}{\gamma_k}\Big) - \varphi(x,y) \Big| < \varepsilon.
$$

By using the above fact, we get

$$
\begin{aligned}
\Big| \int_{\mathbb{R}^2} v_k \varphi \, \mathrm{d}\mathbf{x} - \int_{\mathbb{R}^2} v\varphi \, \mathrm{d}\mathbf{x} \Big| &= \Big| \int_{\Omega_k^+} v_k \varphi \, \mathrm{d}\mathbf{x} - \int_{\Omega^+} v\varphi \, \mathrm{d}\mathbf{x} \Big| \\
&= \Big| \frac{1}{\gamma_k} \int_{\Omega^+} v(x,y) \varphi\Big(x, \frac{y}{\gamma_k}\Big) \, \mathrm{d}x\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{\Omega^+} v(x,y)\varphi(x,y)\ \mathrm{d}x\mathrm{d}y\Big| \\
&\leqslant \frac{1}{\gamma_k}\Big|\int_{\Omega^+} v(x,y)\Big[\varphi\Big(x,\frac{y}{\gamma_k}\Big)-\varphi(x,y)\Big]\ \mathrm{d}x\mathrm{d}y\Big| \\
&\qquad + \Big(\frac{1}{\gamma_k}-1\Big)\int_{\Omega^+} v(x,y)\varphi(x,y)\ \mathrm{d}x\mathrm{d}y\Big| \\
&\leqslant \frac{\varepsilon}{\gamma_k}||v||_{L^1(\Omega)} + \Big|\frac{1}{\gamma_k}-1\Big|||v||_{L^2(\Omega)}||\varphi||_{L^2(\Omega)}.
\end{aligned}
$$

By letting $\varepsilon \to 0$ and $k \to \infty$ we conclude the first estimate. Here, we used the Sobolev embedding for $H^1(\Omega^+;\mathbb{R}^2)$.

Now we prove the convergence of the gradient. First we note that the gradients are uniformly bounded, namely it can be verified that

$$
||\nabla v_k||_{L^2(\Omega)} \leqslant C||\nabla v||_{L^2(\Omega)},
$$

for some positive uniform constant $C > 0$. Thus, we have

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^2} \nabla v_k \cdot \nabla\varphi\ \mathrm{d}\mathbf{x} - \int_{\mathbb{R}^2} \nabla v \cdot \nabla\varphi\ \mathrm{d}\mathbf{x}\Big| \\
&\qquad = \Big|\int_{\Omega_k^+} \nabla v_k \cdot \nabla\varphi\ \mathrm{d}\mathbf{x} - \int_{\Omega^+} \nabla v \cdot \nabla\varphi\ \mathrm{d}\mathbf{x}\Big| \\
&\qquad = \frac{1}{\lambda_k}\int_{\Omega^+} \partial_x v(x,y)\partial_x\varphi\Big(x,\frac{y}{\lambda_k}\Big)\ \mathrm{d}x\mathrm{d}y \\
&\qquad\qquad + \int_{\Omega^+} \partial_y v(x,y)\partial_y\varphi\Big(x,\frac{y}{\lambda_k}\Big)\ \mathrm{d}x\mathrm{d}y,
\end{aligned}
$$

and, from similar estimates as before, together with the uniform boundedness of the gradients, we can conclude that $v_k \rightharpoonup v$ in $H^1(\Omega^+;\mathbb{R}^2)$, as $k \to \infty$.

Step 5. It remains to prove the convergence of the energy. Set $\mu_k := u_k\mathcal{H}^1 \llcorner \Gamma_k$. We have

$$
\mathcal{H}(\Omega_k, v_k, \mu_k) - \mathcal{F}(\Omega, v, \mu)
$$

$$= \int_{\Omega_k} W\big(E(v_k) - E_0(y)\big)\ \mathrm{d}\mathbf{x} - \int_{\Omega} W\big(E(v) - E_0(y)\big)\ \mathrm{d}\mathbf{x}$$

$$+ \int_{\Gamma_k} \psi(u_k)\ \mathrm{d}\mathcal{H}^1 - \int_{\Gamma} \widetilde{\psi}(u)\ \mathrm{d}\mathcal{H}^1 \tag{97}$$

Step 5.1 We now prove the convergence of the bulk term in (97).

$$\int_{\Omega_k} W\big(E(v_k) - E_0(y)\big)\ \mathrm{d}\mathbf{x} - \int_{\Omega} W\big(E(v) - E_0(y)\big)\ \mathrm{d}\mathbf{x}$$

$$= \int_{\Omega_k} W\big(E(v(x,\gamma_k y)) - E_0(y)\big)\ \mathrm{d}\mathbf{x} - \int_{\Omega} W\big(E(v) - E_0(y)\big)\ \mathrm{d}\mathbf{x}$$

$$= \frac{1}{\gamma_k}\Big[\int_{\overline{\Omega}_k} W\Big(E(v) - E_0\Big(\frac{z}{\gamma_k}\Big)\Big)\ \mathrm{d}\mathbf{x} - \int_{\Omega} W\big(E(v) - E_0(z)\big)\ \mathrm{d}\mathbf{x}\Big]$$

$$+ \Big(\frac{1}{\gamma_k} - 1\Big)\int_{\Omega} W\big(E(v) - E_0(z)\big)\ \mathrm{d}\mathbf{x} \tag{98}$$

By noticing that $E_0(z) = E_0(z/\gamma_k)$, fix $\varepsilon' > 0$ such that, if k is large enough, $|\Omega_k \setminus \Omega| \leqslant \varepsilon'$. In the first two terms on the right-hand side of (98), we have that, for every k, $\overline{\Omega} \subset \Omega_k$, and then we can proceed as in (66), and we get

$$\frac{1}{\gamma_k}\Big[\int_{\Omega_k} W\big(E(v) - E_0(z)\big)\ \mathrm{d}\mathbf{x} - \int_{\Omega} W\big(E(v) - E_0(z)\big)\ \mathrm{d}\mathbf{x}\Big]$$

$$= \frac{1}{\gamma_k}\int_{\Omega_k\setminus\Omega} W\big(E(v) - E_0(y)\big)\ \mathrm{d}\mathbf{x}.$$

From here we conclude by Dominated Convergence Theorem. Notice that the second term on the right-hand side of (98) is going to zero, since $\gamma_k \to 1$ as $k \to \infty$.
From here we conclude the convergence of the bulk term in (97).

Step 5.2 We now consider the surface terms in (97). Using the index sets defined in (85), we get

$$\int_{\Gamma_k} \psi(u_k)\ \mathrm{d}\mathcal{H}^1 = \sum_{i\in A}\int_{\Gamma_k\cap[(x^i,x^{i+1})\times\mathbb{R}]} \psi\Big(\frac{u^j}{t_k}\Big)\ \mathrm{d}\mathcal{H}^1$$

$$+\sum_{i\in B}\int_{\Gamma_k\cap[(x^i,x^{i+1})\times\mathbb{R}]}\psi\Big(\frac{s_0}{t_k}\Big)\,\mathrm{d}\mathcal{H}^1.$$

By using the fact that ψ is continuous (as we are in the convexity assumption stated in Step 1) and from the fact that, for every $i\in B$,

$$\psi\Big(\frac{s_0}{t_k}\Big)\mathcal{H}^1(\Gamma_k\cap[(x^i,x^{i+1})\times\mathbb{R}])=r^it_k\psi\Big(\frac{s_0}{t_k}\Big)\mathcal{H}^1(\Gamma\cap[(x^i,x^{i+1})\times\mathbb{R}]),$$

we get

$$\begin{aligned}
\lim_{k\to\infty}&\int_{\Gamma_k}\psi(u_k)\,\mathrm{d}\mathcal{H}^1\\
&=\lim_{k\to\infty}\Big[\sum_{i\in A}\psi\Big(\frac{u^j}{t_k}\Big)\mathcal{H}^1(\Gamma\cap[(x^i,x^{i+1})\times\mathbb{R}])\\
&\qquad+\sum_{i\in B}r^it_k\psi\Big(\frac{s_0}{t_k}\Big)\mathcal{H}^1(\Gamma\cap[(x^i,x^{i+1})\times\mathbb{R}])\Big]\\
&=\sum_{i\in A}\psi(u^j)\mathcal{H}^1(\Gamma\cap[(x^i,x^{i+1})\times\mathbb{R}])\\
&\qquad+\sum_{i\in B}r^i\psi(s_0)\mathcal{H}^1(\Gamma\cap[(x^i,x^{i+1})\times\mathbb{R}])\\
&=\sum_{i\in A}\widetilde{\psi}(u^j)\mathcal{H}^1(\Gamma\cap[(x^i,x^{i+1})\times\mathbb{R}])\\
&\qquad+\sum_{i\in B}\widetilde{\psi}(u^j)\mathcal{H}^1(\Gamma\cap[(x^i,x^{i+1})\times\mathbb{R}])\\
&=\int_\Gamma\widetilde{\psi}(u^j)\,\mathrm{d}\mathcal{H}^1.
\end{aligned}$$

This concludes the estimate for the surface term in (97).

Step 6. By putting all the steps together, we then conclude that

$$\lim_{k\to\infty}\mathcal{H}(\Omega_k,v_k,\mu_k)=\mathcal{F}(\Omega,v,\mu).$$

This completes the proof of Theorem 10. □

Chapter 2

A phase-field formulation for epitaxial growth with adatoms

This Chapter is the continuation of the study contained in the previous one and is based on the paper [35]. Here we will develop a phase-field formulation for the functional $\mathcal{F}$ defined in (19). Phase-field formulations are, in general, useful for their flexibility in the numerical analysis. Indeed, a sharp interface such as the surface of a thin film (e.g. the one studied in Chapter 1) is hard to tackle numerically. Therefore, we aim to replace the free interface Γ in $\mathcal{F}$ with a diffuse region that in the limit can approximate it, in some sense that will be clarified, in the analysis performed in this Chapter.
The following analysis will use the notations and definitions introduced in Chapter 1. Moreover, in what follows, we will use the convergence of sequences of the form $(\varepsilon_n)_{n\in\mathbb{N}}$. Every time we would write $\varepsilon_n \to 0$ as $n \to \infty$, in order to enlighten the notation, we write $\varepsilon \to 0$ instead. However, since the main results of this paper do not depend on a subsequence, we can replace the discrete parameter with a continuous one and get the same results.

2.1 The model

In this section we introduce the functionals that are object of our study. We notice that the formulation we are going to propose resembles the Modica-Mortola (also known as Cahn-Hilliard) and the Ambrosio-Tortorelli energies. We start with the definition of a double-well potential.

Definition 21. Let $P : \mathbb{R} \to \mathbb{R}^+$ be a continuous function such that

(i) $P^{-1}(0) = \{0, 1\}$;

(ii) There exist $r, C > 0$ such that

$$P(t) \geqslant C\,|t|\,,$$

for every $|t| > r$. A function that satisfies those properties is referred as *double-well* potential.

Consider

$$Q := (a, b) \times \mathbb{R} \quad \text{and} \quad Q^+ := Q \cap \{y > 0\}.$$

We define the phase field variable $w \in H^1(Q^+; [0, 1])$ and the admissible configurations that will be used in the phase-field formulation.

Definition 22. We say that the triplet (w, v, u) is an *admissible phase-field configuration* if $w \in H^1(Q^+)$ is such that $0 \leqslant w \leqslant 1$, $v \in H^1(Q; \mathbb{R}^2)$ and $u \in L^1(\mathbb{R}^2)$ with associated measure $\mu := u\mathcal{L}^2 \llcorner Q^+$. We denote the set of admissible phase-field configurations by $\mathcal{A}_p$.
Given $M, m > 0$, we say that $(w, v, u) \in \mathcal{A}_p(m, M)$ if $(w, v, u) \in \mathcal{A}_p$ and if

$$\int_{Q^+} w \,\mathrm{d}\mathbf{x} = M, \qquad \int_{Q^+} u \,\mathrm{d}\mathbf{x} = m \tag{99}$$

hold.

The phase-field functional is given in the following definition.

Definition 23. We define the functional $\mathcal{F}_\varepsilon : H^1(Q^+) \times H^1(Q;\mathbb{R}^2) \times L^1(\mathbb{R}^2) \to [0,+\infty]$ as

$$\mathcal{F}_\varepsilon(w,v,u) := \begin{cases} \mathcal{G}_\varepsilon(w,v,u) & \text{if } (w,v,u) \in \mathcal{A}_p, \\ +\infty & \text{else,} \end{cases}$$

where

$$\mathcal{G}_\varepsilon(w,v,u) := \int_{Q^+} (w+\eta_\varepsilon) W\big(E(v) - E_0(y)\big)\ \mathrm{d}\mathbf{x} \\ + \frac{1}{\sigma} \int_{Q^+} \Big[\varepsilon|\nabla w|^2 + \frac{1}{\varepsilon} P(w)\Big] \psi(u)\ \mathrm{d}\mathbf{x},$$

with $\eta_\varepsilon = o(\varepsilon)$, as $\varepsilon \to 0$, and

$$\sigma := 2\int_0^1 \sqrt{P(t)}\ \mathrm{d}t.$$

Additionally, we define the functional with the mass constraint $\mathcal{F}_\varepsilon^{m,M} : H^1(Q^+) \times H^1(Q;\mathbb{R}^2) \times L^1(\mathbb{R}^2) \to [0,+\infty]$, as

$$\mathcal{F}_\varepsilon^{m,M}(w,v,u) := \begin{cases} \mathcal{G}_\varepsilon(w,v,u) & \text{if } (w,v,u) \in \mathcal{A}_p(m,M), \\ +\infty & \text{else.} \end{cases}$$

For the reader's convenience, we recall the definitions of the Γ-limiting functionals, that are already mentioned in Definition 19.

Definition 24. We define the functional $\mathcal{F} : \mathcal{A} \to [0,+\infty]$ as

$$\mathcal{F}(\Omega,v,\mu) := \begin{cases} \mathcal{G}(\Omega,v,\mu) & \text{if } (\Omega,v,\mu) \in \mathcal{A}, \\ +\infty & \text{else,} \end{cases}$$

where

$$\mathcal{G}(\Omega,v,\mu) := \int_\Omega W\big(E(v) - E_0(y)\big)\ \mathrm{d}\mathbf{x} \\ + \int_{\widetilde{\Gamma}} \widetilde{\psi}(u)\ \mathrm{d}\mathcal{H}^1 + \int_{\Gamma^c} \psi^c(u)\ \mathrm{d}\mathcal{H}^1 + \theta\mu^s(\mathbb{R}^2).$$

Moreover, we define the functional $\mathcal{F}^{m,M} : \mathcal{A} \to [0, +\infty]$ as

$$\mathcal{F}^{m,M}(\Omega, v, \mu) := \begin{cases} \mathcal{G}(\Omega, v, \mu) & \text{if } (\Omega, v, \mu) \in \mathcal{A}(m, M), \\ +\infty & \text{else,} \end{cases}$$

where $\mathcal{A}$ and $\mathcal{A}(m, M)$ are defined in Definitions 15 and 17.

2.2 Main result

In this section, we state the main results achieved in [35]. First, we need to introduce the notion of convergence we will use.

Definition 25. We say that $(w_\varepsilon, v_\varepsilon, u_\varepsilon)_\varepsilon \subset \mathcal{A}_p$ converges to $(\Omega, v, \mu) \in \mathcal{A}$, and we write $(w_\varepsilon, v_\varepsilon, u_\varepsilon) \to (\Omega, v, \mu)$ as $\varepsilon \to 0$, if

(i) $w_\varepsilon \to \chi_\Omega$ in $L^1_{\text{loc}}(Q^+)$, where χ_Ω is the indicator function of Ω;

(ii) $v_\varepsilon \to v$ in $L^2_{\text{loc}}(Q)$;

(iii) $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ weak* in the sense of measures, where

$$\mu_\varepsilon := \frac{u_\varepsilon}{\sigma}\Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)\mathcal{L}^2 \llcorner Q^+.$$

Remark 10. Thanks to Proposition 1, the configuration space $\mathcal{A}_p$ is a metric space.

We now present the main contribution of this Chapter.

Theorem 14. $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}$, as $\varepsilon \to 0$, with respect to the topology in Definition 25.

We now investigate the class of configurations for which we have compactness.

Remark 11. We notice that the function ψ does not need any further hypotheses for the coerciveness, beside being a strictly positive Borel function.

Theorem 15. Let $(w_\varepsilon, v_\varepsilon, u_\varepsilon)_\varepsilon \subset \mathcal{A}_p(m, M)$ be a sequence such that

$$\sup_{\varepsilon>0} \mathcal{F}(w_\varepsilon, v_\varepsilon, u_\varepsilon) < \infty,$$

$$\sup_{\varepsilon>0} \int_{Q^+} |E(v_\varepsilon)|^2 \, d\mathbf{x} < \infty,$$

$$\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$$

with

$$\operatorname{supp} \mu \subset \Gamma.$$

Then, there exists $(\Omega, v, \mu) \in \mathcal{A}(m, M)$ such that $(w_\varepsilon, v_\varepsilon, u_\varepsilon) \to (\Omega, v, \mu)$, as $\varepsilon \to 0$.

The following theorem is similar to Theorem 14, but when the constraints (42) and (99) on the configurations are in force.

Theorem 16. $\mathcal{F}_\varepsilon^{m,M} \xrightarrow{\Gamma} \mathcal{F}^{m,M}$, as $\varepsilon \to 0$, with respect to the topology in Definition 25.

The proofs of Theorems 14 and 16 are a consequence of the following two steps.

Step 1 *Liminf inequality.* In Theorem 20 we will prove that for every $(\Omega, v, \mu) \in \mathcal{A}$ and for every $(w_\varepsilon, v_\varepsilon, u_\varepsilon)_\varepsilon \subset \mathcal{A}_p$, such that $(w_\varepsilon, v_\varepsilon, u_\varepsilon) \to (\Omega, v, \mu)$ as $\varepsilon \to 0$, we have

$$\mathcal{F}(\Omega, v, \mu) \leqslant \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(w_\varepsilon, v_\varepsilon, u_\varepsilon).$$

Step 2 *Limsup inequality for the constrained problem.* In Theorem 21 we prove that for every $(\Omega, v, \mu) \in \mathcal{A}(m, M)$, there is a sequence $(w_\varepsilon, v_\varepsilon, u_\varepsilon)_\varepsilon \subset \mathcal{A}_p(m, M)$ such that $(w_\varepsilon, v_\varepsilon, u_\varepsilon) \to (\Omega, v, \mu)$ as $\varepsilon \to 0$ and

$$\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(w_\varepsilon, v_\varepsilon, u_\varepsilon) \leqslant \mathcal{F}(\Omega, v, \mu).$$

The following theorem is a consequence of Theorems 4, 15 and 16 and states the convergence of the minimising sequences for $\mathcal{F}_\varepsilon$.

Theorem 17. We have that

$$\lim_{\varepsilon\to 0} \min_{\mathcal{A}_p(m,M)} \mathcal{F}_\varepsilon = \min_{\mathcal{A}(m,M)} \mathcal{F}.$$

We remark that the family $(\mathcal{F}_\varepsilon)_\varepsilon$ is equi-coercive and lower semi-continuous and therefore we have the existence of minimisers (see Theorem 4).

We would like to briefly give an idea of the proof of Theorems 20 and 21.
The main difficulty of the liminf inequality comes from the surface term. Indeed, for a given configuration $(\Omega, v, \mu) \in \mathcal{A}$, the set Γ might present a dense cut set. To get around it, we define

$$C_\xi \coloneqq \{\mathbf{x} = (x,y) \in \Gamma^c : h^-(x) - y < \xi\}.$$

It is possible to prove that $\Gamma^c \setminus C_\xi$ is a finite number of vertical segments. Since we can now separate those cuts and include each of them in a suitable rectangle R, we suppose that we only have one cut point x^c and we repeat the argument for each other one afterward. The energy around such a cut is given by

$$\liminf_{\varepsilon\to 0} \frac{1}{\sigma} \int_{Q^+\cap R} \left[\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\right]\psi(u_\varepsilon)\ \mathrm{d}\mathbf{x}.$$

From here we can separate the energy on the left and on the right of each single cut, namely we can split $R = R^\ell \cup R^\mathrm{r}$ in such a way that $R^\ell \cap R^\mathrm{r} = \{x^c\} \times [0, h^-(x^c)]$. Therefore, when we pass to the liminf, we get

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \frac{1}{\sigma} \int_{Q^+\cap R} &\left[\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\right]\psi(u_\varepsilon)\ \mathrm{d}\mathbf{x} \\
&\geqslant \liminf_{\varepsilon\to 0} \frac{1}{\sigma} \int_{Q^+\cap R^\ell} \left[\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\right]\psi(u_\varepsilon)\ \mathrm{d}\mathbf{x} \\
&\qquad + \liminf_{\varepsilon\to 0} \frac{1}{\sigma} \int_{Q^+\cap R^\mathrm{r}} \left[\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\right]\psi(u_\varepsilon)\ \mathrm{d}\mathbf{x}.
\end{aligned}$$

First, we prove that

$$\frac{u_\varepsilon}{\sigma}\Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)\mathcal{L}^2 \llcorner R^\ell \overset{*}{\rightharpoonup} f\mathcal{H}^1 \llcorner (\Gamma^c \cap R) + (\mu^s)^\ell,$$
$$\frac{u_\varepsilon}{\sigma}\Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)\mathcal{L}^2 \llcorner R^{\mathrm{r}} \overset{*}{\rightharpoonup} g\mathcal{H}^1 \llcorner (\Gamma^c \cap R) + (\mu^s)^{\mathrm{r}},$$

for some $f, g \in L^1(\Gamma^c \setminus C_\xi)$, as $\varepsilon \to 0$ such that

$$f + g = u_{|\Gamma^c \setminus C_\xi} \quad \text{and} \quad (\mu^s)^\ell + (\mu^s)^{\mathrm{r}} = \mu^s. \tag{100}$$

Therefore, if we take into account the rectangle on the left of the cut, we can prove that

$$\liminf_{\varepsilon\to 0} \frac{1}{\sigma}\int_{Q^+\cap R^\ell} \Big[\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big]\psi(u_\varepsilon)\,\mathrm{d}\mathbf{x}$$
$$\geqslant \int_{\partial^* R^\ell} \widetilde{\psi}(f)\,\mathrm{d}\mathcal{H}^1 + \theta(\mu^s)^\ell(\mathbb{R}^2).$$

We can conclude by summing up the contribution given by the term integrated on R^r and by considering the definition of ψ^c, given the fact that

$$\widetilde{\psi}(f) + \widetilde{\psi}(g) \geqslant \psi^c(u),$$

provided that $f + g = u$.

The limsup inequality is based on a diagonalisation argument. Namely, we can approximate a given configuration $(\Omega, v, \mu) \in \mathcal{A}(m, M)$, with a sequence $(\Omega_\varepsilon, v_\varepsilon, u_\varepsilon\mathcal{H}^1 \llcorner \Gamma_\varepsilon)$ such that Ω_ε is the subgraph of a $\mathcal{C}^\infty$ function and $u_\varepsilon \in L^1(\Gamma_\varepsilon)$ is grid- constant (see Definition 8 and Theorems 22 and 24).
One of the key ingredients of the proof of Theorem 24 is the *wriggling process* see in Chapter 1, first introduced in [18], largely used in [17] and later refined in [28].
In Theorem 25, we use a similar strategy to the one used in [9, Theorem 3.1], in which the phase-field approximating sequence $(w_\varepsilon)_\varepsilon$ is built

by making use of the *almost* optimal profile problem as explained in [55, Proposition 2], which is the solution of the following differential equation,

$$\begin{cases} \varepsilon^2 \left|\gamma_\varepsilon'(t)\right|^2 = P(\gamma_\varepsilon(t)) + \sqrt{\varepsilon} \quad t \in \mathbb{R}, \\ \gamma_\varepsilon(0) = 0, \\ \gamma_\varepsilon(1) = 1. \end{cases} \tag{101}$$

We can define the phase-field variable as

$$w_\varepsilon(\mathbf{x}) \coloneqq \gamma_\varepsilon\Big(\frac{d_\Omega(\mathbf{x})}{\varepsilon}\Big),$$

where γ_ε is the solution of (101) and

$$d_\Omega(\mathbf{x}) \coloneqq \operatorname{dist}(\mathbf{x}, \Omega) - \operatorname{dist}(\mathbf{x}, \mathbb{R}^2 \setminus \Omega),$$

is the signed distance from $\partial\Omega$. Now, once we make sure that all the constraints are satisfied, we follow a path inspired by the one contained in [9].

2.3 Compactness

In this section we give the proof of Theorem 15. For the reader's convenience, we recall the statement.

Theorem 18. Let $(w_\varepsilon, v_\varepsilon, u_\varepsilon)_\varepsilon \subset \mathcal{A}_p(m, M)$ be a sequence such that

$$\sup_{\varepsilon>0} \mathcal{F}_\varepsilon(w_\varepsilon, v_\varepsilon, u_\varepsilon) < \infty, \tag{102}$$

$$\sup_{\varepsilon>0} \int_{Q^+} \left|E(v_\varepsilon)\right|^2 \, \mathrm{d}\mathbf{x} < \infty, \tag{103}$$

$$\mu_\varepsilon \overset{*}{\rightharpoonup} \mu,$$

as $\varepsilon \to 0$, with

$$\operatorname{supp} \mu \subset \Gamma. \tag{104}$$

Then, there exists $(\Omega, v, \mu) \in \mathcal{A}(m, M)$ such that $(w_\varepsilon, v_\varepsilon, u_\varepsilon) \to (\Omega, v, \mu)$, as $\varepsilon \to 0$.

Proof. Step 1. The compactness of the phase-field variable w_ε follows a standard argument, which makes use of the first bound in (102), and can be found in [55]. Therefore, we have the existence of $w \in \mathrm{BV}(Q^+;\{0,1\})$ such that, up to a subsequence $w_\varepsilon \to \chi_\Omega$ in $L^1(Q^+)$. Moreover

$$\int_{Q^+} w_\varepsilon \, \mathrm{d}\mathbf{x} \to \int_{Q^+} \chi_\Omega(\mathbf{x}) \, \mathrm{d}\mathbf{x} = |\Omega| = M,$$

thus the mass constraint is preserved.
Step 2. From (104) we have that $\operatorname{supp}\mu \subset \Gamma$. From the fact that $\partial\Gamma \cap (a,b) = \emptyset$, we have, by standard properties of the weak* convergence that

$$\mu(Q^+) = \lim_{\varepsilon\to 0} \mu_\varepsilon(Q^+) = m. \tag{105}$$

Step 3. By (103), there exists a function $\overline{E} \in L^2(Q;\mathbb{R}^2)$ such that

$$E(v_\varepsilon) \rightharpoonup \overline{E}$$

in $L^2(Q;\mathbb{R}^2)$ as $\varepsilon \to 0$. By Korn's inequality, there exists $C > 0$ such that

$$\int_{Q^+} |\nabla v_\varepsilon|^2 \, \mathrm{d}\mathbf{x} \leqslant C \int_{Q^+} |E(v_\varepsilon)|^2 \, \mathrm{d}\mathbf{x} < \infty.$$

From that, again by compactness, we have the existence of a function $v \in H^1(Q;\mathbb{R}^2)$ such that, up to a subsequence (not relabelled), $v_\varepsilon \rightharpoonup v$ as $\varepsilon \to 0$. By uniqueness of the weak limit, $E(v_\varepsilon) \rightharpoonup \overline{E} = E(v)$.
By putting together the three above steps, we conclude the proof of the Theorem. □

Remark 12. In Theorem 15, hypothesis (104) is essential. Indeed, if we drop it we cannot ensure the validity of (105) and neither the fact that $(\mu_\varepsilon)_\varepsilon$ is converging to a Radon measure that is supported on Γ. This behaviour is due to the fact that the phase-field variable and the density one are completely independent, see Figure 2.1.

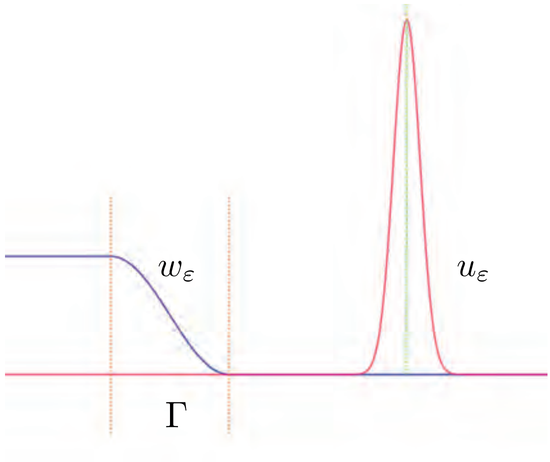

Figure 2.1: The phase-field variable converges to χ_Ω, whereas the u_ε, being independent from the other variables, can concentrate in an area away from Γ.

2.4 Liminf inequality

The liminf inequality relies on the strategy used in Theorem 9 and a corollary of the main result contained in [17]. In this paper, the authors work with a surface energy in a bounded domain in $\mathbb{R}^n$, with an adatom density at the interface. We recall such a result, with our notation. The proof is presented in [17, Theorem 3.10].

Theorem 19. Let $\Sigma \subset \mathbb{R}^2$ be an open set and consider the functional

$$\mathcal{E}_\varepsilon(w,u) := \frac{1}{\sigma}\int_\Sigma \left[\varepsilon\left|\nabla w\right|^2 + \frac{1}{\varepsilon}P(w)\right]\psi(u)\ \mathrm{d}\mathbf{x},$$

where P as in Definition 21 and, with an abuse of notation, $(w,u) \in \mathcal{A}_p$. Then, $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}$, as $\varepsilon \to 0$, where

$$\mathcal{E}(A,\mu) := \int_{\partial^* A\cap\Sigma} \widetilde{\psi}(u)\ \mathrm{d}\mathcal{H}^1 + \theta\mu^s(\mathbb{R}^2), \tag{106}$$

for $(A,\mu) \in \mathcal{A}$, with respect to the topology in Definition 25. Here θ is as in Definition 14 and μ^s is the singular part of μ with respect to $\mathcal{H}^1$.

Remark 13. We notice that if we consider an open subset B compactly contained in Σ and denote by $\mathcal{C}_b(B)$ the set of continuous and bounded functions on B, then, if we consider the convergence

$$\int_B \varphi \, \mathrm{d}\mu_\varepsilon \to \int_B \varphi \, \mathrm{d}\mu,$$

for every $\varphi \in \mathcal{C}_b(B)$, we have that the set of integration in the Γ-limit in (106) is $\partial^* A \cap \overline{B}$.

The main theorem of this section, namely the liminf inequality, is the following.

Theorem 20. Let $(\Omega, v, \mu) \in \mathcal{A}$. Then, for every $(w_\varepsilon, v_\varepsilon, u_\varepsilon)_\varepsilon \subset \mathcal{A}_p$, such that $(w_\varepsilon, v_\varepsilon, u_\varepsilon) \to (\Omega, v, \mu)$ as $\varepsilon \to 0$, we have

$$\mathcal{F}(\Omega, v, \mu) \leqslant \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(w_\varepsilon, v_\varepsilon, u_\varepsilon).$$

Proof. Step 1. Bulk term. Take any compact set $K \subset\subset \Omega$. We have

$$\begin{aligned}
\liminf_{\varepsilon \to 0} \int_{Q^+} & (w_\varepsilon + \eta_\varepsilon) W\big(E(v_\varepsilon) - E_0(y)\big) \, \mathrm{d}\mathbf{x} \\
& \geqslant \liminf_{\varepsilon \to 0} \int_K (w_\varepsilon + \eta_\varepsilon) W\big(E(v_\varepsilon) - E_0(y)\big) \, \mathrm{d}\mathbf{x} \\
& \geqslant \int_K W\big(E(v) - E_0(y)\big) \, \mathrm{d}\mathbf{x}, \qquad (107)
\end{aligned}$$

where in the last step we used the fact that W is convex and the fact that $\liminf_{\varepsilon \to 0} w_\varepsilon = 1$ on every $K \subset\subset \Omega$. Now we can consider an increasing sequence of compact sets $K_j \subset\subset \Omega$ and conclude with the use of the Dominated Convergence Theorem. Here we remark that, differently from Theorem 9, the v_ε's are defined on the entire Q and there is no need of the additional technicalities as in the mentioned Theorem, in which the displacement sequence was defined only in $H^1_{\mathrm{loc}}(\Omega_\varepsilon; \mathbb{R}^2)$.

Step 2. Surface term. We would like to separate $\widetilde{\Gamma}$ and Γ^c. On the regular part we apply the result contained in [17] together with an error that occurs when the cut part meet the regular part. On the cut part, we isolate each vertical cut, far enough from the regular part, and we look at the contribution given by the left and right contribution given by u_ε.

Fix $\xi > 0$ and consider the set

$$C_\xi \coloneqq \{\mathbf{x} = (x, y) \in \Gamma^c : h^-(x) - y < \xi\}.$$

By a standard measure theory argument, there is a sequence $(\xi_\gamma)_\gamma \subset \mathbb{R}$ such that $\xi_\gamma \to 0$, as $\gamma \to 0$ and

$$\mu(\Gamma \cap \partial C_{\xi_\gamma}) = 0, \tag{108}$$

for every $\gamma > 0$. As a consequence, from Lemma 2 and for every $\gamma > 0$, we have that $\Gamma^c \setminus C_{\xi_\gamma}$ consists of a finite number of vertical segments, whose projections on the x-axis correspond to the set $(x^i)_{i=1}^N$. Recalling the definition of Γ^c (see Definition 6), it holds that C_{ξ_γ} is monotonically converging to the empty set, as $\varepsilon \to 0$. Therefore, we get that

$$\mu(C_{\xi_\gamma}) \to 0, \qquad \mu(\Gamma^c \setminus C_{\xi_\gamma}) \to \mu(\Gamma^c), \tag{109}$$

as $\gamma \to 0$. Let $\delta = \delta(\varepsilon) > 0$ such that $\delta \to 0$ as $\varepsilon \to 0$ and we have $\delta < |x^i - x^j|$, for every $i, j = 1, \dots, N$. As we have a finite number of cuts, in order to simplify the notation, we do the following construction as we had only one cut point, and then we repeat it for each other one. Therefore, let $(x^c, h(x^c))$ be the only cut point of Γ. Consider the rectangle

$$R = R(\delta, \xi_\gamma) \coloneqq (x^c - \delta, x^c + \delta) \times (-\delta, h^-(x^c) - \xi_\gamma).$$

Up to further reducing δ, we can assume that $\widetilde{\Gamma} \cap R = \emptyset$. We can split R as

$$R^\ell = R^\ell(\delta, \xi_\gamma) \coloneqq (x^c - \delta, x^c) \times (-\delta, h^-(x^c) - \xi_\gamma),$$
$$R^{\mathrm{r}} = R^{\mathrm{r}}(\delta, \xi_\gamma) \coloneqq (x^c, x^c + \delta) \times (-\delta, h^-(x^c) - \xi_\gamma).$$

We remark that we need to consider rectangles that go below $\{y = 0\}$, as a cut might touch the y-axes and a singular measure (e.g. a Dirac delta)

might be present at the endpoint of such a cut.
Therefore, we get

$$
\begin{aligned}
\liminf_{\varepsilon\to 0} \frac{1}{\sigma} \int_{Q^+} \Big[\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big]\psi(u_\varepsilon)\ \mathrm{d}\mathbf{x} \geqslant \\
\liminf_{\varepsilon\to 0} \frac{1}{\sigma} \int_{Q^+\cap R} \Big[\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big]\psi(u_\varepsilon)\ \mathrm{d}\mathbf{x} \\
+ \liminf_{\varepsilon\to 0} \frac{1}{\sigma} \int_{Q^+\setminus R} \Big[\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big]\psi(u_\varepsilon)\ \mathrm{d}\mathbf{x}. \quad (110)
\end{aligned}
$$

Step 2.1 Cut part Γ^c*.* We deal now with the first term on the right-hand side of (110). We have

$$
\begin{aligned}
\liminf_{\varepsilon\to 0} \frac{1}{\sigma} \int_{Q^+\cap R} \Big[\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big]\psi(u_\varepsilon)\ \mathrm{d}\mathbf{x} \\
\geqslant \liminf_{\varepsilon\to 0} \frac{1}{\sigma} \int_{Q^+\cap R^\ell} \Big[\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big]\psi(u_\varepsilon)\ \mathrm{d}\mathbf{x} \\
+ \liminf_{\varepsilon\to 0} \frac{1}{\sigma} \int_{Q^+\cap R^{\mathrm{r}}} \Big[\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big]\psi(u_\varepsilon)\ \mathrm{d}\mathbf{x}. \quad (111)
\end{aligned}
$$

Now, we localise the weak* convergence of u_ε to μ. Namely, we prove

$$
\begin{aligned}
\frac{u_\varepsilon}{\sigma}\Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)\mathcal{L}^2 \llcorner R^\ell \overset{*}{\rightharpoonup} f\mathcal{H}^1\llcorner(\Gamma^c\cap R) + (\mu^s)^\ell, \\
\frac{u_\varepsilon}{\sigma}\Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)\mathcal{L}^2 \llcorner R^{\mathrm{r}} \overset{*}{\rightharpoonup} g\mathcal{H}^1\llcorner(\Gamma^c\cap R) + (\mu^s)^{\mathrm{r}},
\end{aligned}
$$

for some $f, g \in L^1(\Gamma^c \setminus C_{\xi_\gamma})$, as $\varepsilon \to 0$ such that

$$
f + g = u_{|\Gamma^c\setminus C_{\xi_\gamma}} \quad \text{and} \quad (\mu^s)^\ell + (\mu^s)^{\mathrm{r}} = \mu^s,
$$

where $(\mu^\ell)^s$ and $(\mu^{\mathrm{r}})^s$ are supported in $\Gamma^c \setminus C_{\xi_\gamma}$. Since the strategy is similar to the one proposed in Theorem 9, we only recall the main ideas. By the weak* covergence of u_ε to μ and by compactness we have that up to a subsequence

$$
\frac{u_\varepsilon}{\sigma}\Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)\mathcal{L}^2 \llcorner R^\ell \overset{*}{\rightharpoonup} \mu^\ell
$$

and

$$\frac{u_\varepsilon}{\sigma}\Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)\mathcal{L}^2 \llcorner R^{\mathrm{r}} \overset{*}{\rightharpoonup} \mu^{\mathrm{r}},$$

for some Radon measures μ^ℓ and μ^{r}. It is possible to prove that $\operatorname{supp}\mu^\ell \subset \Gamma^c \setminus C_{\xi_\gamma}$ and $\operatorname{supp}\mu^{\mathrm{r}} \subset \Gamma^c \setminus C_{\xi_\gamma}$ and, by the Radon-Nikodym decomposition, there are $f, g \in L^1(\Gamma^c \setminus C_{\xi_\gamma})$ such that

$$\mu^\ell = f\mathcal{H}^1 \llcorner (\Gamma^c \cap R) + (\mu^s)^\ell \quad \text{and} \quad \mu^{\mathrm{r}} = g\mathcal{H}^1 \llcorner (\Gamma^c \cap R) + (\mu^s)^{\mathrm{r}},$$

where $(\mu^\ell)^s$ and $(\mu^r)^s$ are singular measures with respect to $f\mathcal{H}^1 \llcorner (\Gamma^c \setminus C_{\xi_\gamma})$ and $g\mathcal{H}^1 \llcorner (\Gamma^c \setminus C_{\xi_\gamma})$ respectively. It is a straight computation to prove the fact that $\mu = \mu^\ell + \mu^{\mathrm{r}}$, which implies (100).

We now focus on the first term on the right-hand side of (111). Notice that since $w_\varepsilon \to \chi_{\Omega\cap R}$ in $L^1(Q^+ \cap R)$, we can apply Theorem 19 (see Remark 13 with $\Sigma = Q$ and $B = R^\ell$) and we get

$$\begin{aligned}\liminf_{\varepsilon\to 0} \frac{1}{\sigma}\int_{Q^+\cap R^\ell} \Big[\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P\big(w_\varepsilon\big)\Big]\psi(u_\varepsilon)\, \mathrm{d}\mathbf{x} \\ \geqslant \int_{\partial^* R^\ell} \widetilde{\psi}(f)\, \mathrm{d}\mathcal{H}^1 + \theta(\mu^s)^\ell(\Gamma^c \setminus C_{\xi_\gamma}).\end{aligned}$$

By applying the same argument to the second term on the right-hand side of (111), we get

$$\begin{aligned}\liminf_{\varepsilon\to 0} \frac{1}{\sigma}\int_{Q^+\cap R} \Big[\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P\big(w_\varepsilon\big)\Big]\psi(u_\varepsilon)\, \mathrm{d}\mathbf{x} \\ \geqslant \int_{\partial^* R^\ell} \widetilde{\psi}(f)\, \mathrm{d}\mathcal{H}^1 + \theta(\mu^s)^\ell(\Gamma^c \setminus C_{\xi_\gamma}) \\ + \int_{\partial^* R^{\mathrm{r}}} \widetilde{\psi}(g)\, \mathrm{d}\mathcal{H}^1 + \theta(\mu^s)^{\mathrm{r}}(\Gamma^c \setminus C_{\xi_\gamma}) \\ \geqslant \int_{\Gamma^c\setminus C_{\xi_\gamma}} \psi^c(u)\, \mathrm{d}\mathcal{H}^1 + \theta\mu^s(\widetilde{\Gamma} \setminus C_{\xi_\gamma}),\end{aligned} \tag{112}$$

where in the last step we used Definition 12 and (100).

Step 2.2 Regular part $\widetilde{\Gamma}$. We deal with the second term on the right-hand side of (110). We can see the remaining cut part C_{ξ_γ} in $Q^+ \setminus R$ as a singular measure with respect to $\mathcal{H}^1$, namely

$$u_\varepsilon \mathcal{L}^2(Q^+ \setminus R) \overset{*}{\rightharpoonup} \mu \llcorner (Q^+ \setminus R) = u\mathcal{H}^1 \llcorner \widetilde{\Gamma} + \mu^s(Q^+ \setminus R) + u\mathcal{H}^1 \llcorner C_{\xi_\gamma},$$

where we used (108). Now, we can apply Theorem 19 and since $\widetilde{\Gamma} \cap C_{\xi_\gamma} \subset Q^+ \setminus R$, we obtain

$$\begin{aligned}\liminf_{\varepsilon \to 0} \frac{1}{\sigma} \int_{Q^+ \setminus R} \Big[\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} P\big(w_\varepsilon\big)\Big] \psi(u_\varepsilon) \, \mathrm{d}\mathbf{x} \\ \geqslant \int_{\widetilde{\Gamma} \cup C_{\xi_\gamma}} \widetilde{\psi}(u) \, \mathrm{d}\mathcal{H}^1 \\ + \theta\Big(\mu^s(\widetilde{\Gamma} \cup C_{\xi_\gamma}) + \int_{C_{\xi_\gamma}} u \, \mathrm{d}\mathcal{H}^1\Big).\end{aligned} \tag{113}$$

Step 3. Conclusion. Using (107), (112) and (113) we get

$$\begin{aligned}&\int_\Omega W\big(E(v) - E_0(y)\big) \, \mathrm{d}\mathbf{x} + \int_{\Gamma^c \setminus C_{\xi_\gamma}} \psi^c(u) \, \mathrm{d}\mathcal{H}^1 + \theta \mu^s(\Gamma^c \setminus C_{\xi_\gamma}) \\ &\quad + \int_{\widetilde{\Gamma} \cup C_{\xi_\gamma}} \widetilde{\psi}(u) \, \mathrm{d}\mathcal{H}^1 + \theta\Big(\mu^s(\widetilde{\Gamma} \cup C_{\xi_\gamma}) + \int_{C_{\xi_\gamma}} u \, \mathrm{d}\mathcal{H}^1\Big) \\ &\leqslant \liminf_{\varepsilon \to 0} \int_{Q^+} (w_\varepsilon + \eta_\varepsilon) W\big(E(v_\varepsilon) - E_0(y)\big) \, \mathrm{d}\mathbf{x} \\ &\quad + \liminf_{\varepsilon \to 0} \frac{1}{\sigma} \int_{Q^+} \Big[\varepsilon |\nabla w_\varepsilon(|^2 + \frac{1}{\varepsilon} P\big(w_\varepsilon\big)\Big] \psi(u_\varepsilon) \, \mathrm{d}\mathbf{x} \\ &\leqslant \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(w_\varepsilon, v_\varepsilon, u_\varepsilon).\end{aligned} \tag{114}$$

Now, by letting $\xi_\gamma \to 0$, by (108) and since $C_{\xi_\gamma} \to \emptyset$, we get

$$\mu^s(\Gamma^c \setminus C_{\xi_\gamma}) \to \mu^s(\Gamma^c), \quad \mu^s(\widetilde{\Gamma} \cup C_{\xi_\gamma}) \to \mu^s(\widetilde{\Gamma}) \quad \text{and} \quad \int_{C_{\xi_\gamma}} u \ \mathrm{d}\mathcal{H}^1 \to 0. \tag{115}$$

In conclusion, from (114) and (115), we get the desired liminf inequality

$$\mathcal{F}(\Omega, v, u) \leqslant \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(w_\varepsilon, v_\varepsilon, u_\varepsilon).$$

□

2.5 Limsup inequality

The main result of this section is the mass-constrained limsup inequality. Its proof is not a direct application of the main result of [17, Theorem 3.10]. This is due to the interplay of the phase-field variable in the bulk and in the surface term. For that reason our approach is partially inspired by [9].

Theorem 21. Let $(\Omega, v, \mu) \in \mathcal{A}(m, M)$. Then, there exists $(w_\varepsilon, v_\varepsilon, u_\varepsilon)_\varepsilon \subset \mathcal{A}_p(m, M)$, such that $(w_\varepsilon, v_\varepsilon, u_\varepsilon) \to (\Omega, v, \mu)$ as $\varepsilon \to 0$, and

$$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(w_\varepsilon, v_\varepsilon, u_\varepsilon) \leqslant \mathcal{F}(\Omega, v, \mu).$$

Due to the work carried out in Chapter 1, we can reduce to prove the existence of a recovery sequence for a simpler case. We now see, step by step, the argument that allows us to reduce to the case in Theorem 25.

Remark 14. In what follows, a diagonal argument will be used to obtain the desired result. We note that this is possible as we are in the setting in which the Γ-convergence can be rephrased in terms of the liminf and limsup inequality.

We start with an approximation results proved in [28] and reported in Chapter 1.

Theorem 22. For every $(\Omega, v, \mu) \in \mathcal{A}(m, M)$, there exists a sequence $(\Omega_\varepsilon, v_\varepsilon, \mu_\varepsilon)_\varepsilon \subset \mathcal{A}_r(m, M)$, with $\mu_\varepsilon = u_\varepsilon \mathcal{H}^1 \llcorner \Gamma_\varepsilon$, where u_ε is grid-constant, such that $(\Omega_\varepsilon, v_\varepsilon, \mu_\varepsilon) \to (\Omega, v, \mu)$ as $\varepsilon \to 0$. Moreover,

$$\limsup_{\varepsilon \to 0} \mathcal{F}(\Omega_\varepsilon, v_\varepsilon, \mu_\varepsilon) \leqslant \mathcal{F}(\Omega, v, \mu).$$

Proof. The proof is given in Theorems 11 and 12. □

Therefore, Theorem 21 will be a consequence of the following result together with a diagonal argument.

Theorem 23. Let $(\Omega, v, \mu) \in \mathcal{A}_r(m, M)$ be such that $\mu = u\mathcal{H}^1 \llcorner \Gamma$ with $u \in L^1(\Gamma)$ grid constant. Then, there exists $(w_\varepsilon, v_\varepsilon, u_\varepsilon)_\varepsilon \subset A_p(m, M)$, such that $(w_\varepsilon, v_\varepsilon, u_\varepsilon) \to (\Omega, v, \mu)$ as $\varepsilon \to 0$, and

$$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(w_\varepsilon, v_\varepsilon, u_\varepsilon) \leqslant \int_\Omega W\big(E(v) - E_0(y)\big)\, \mathrm{d}\mathbf{x} + \int_\Gamma \widetilde{\psi}(u)\, \mathrm{d}\mathcal{H}^1.$$

Using the following approximation result proved in [28] and reported in Chapter 1, we can simplify even further our statement by considering the functional with ψ instead of $\widetilde{\psi}$ in the surface term.

Theorem 24. Let $(\Omega, v, \mu) \in \mathcal{A}_r(m, M)$. Then, there exists a sequence $(\Omega_\varepsilon, v_\varepsilon, \mu_\varepsilon)_\varepsilon \subset \mathcal{A}_r(m, M)$ such that $(\Omega_\varepsilon, v_\varepsilon, \mu_\varepsilon) \to (\Omega, v, \mu)$ and

$$\lim_{\varepsilon \to \infty} \mathcal{H}(\Omega_\varepsilon, v_\varepsilon, \mu_\varepsilon) = \int_\Omega W\big(E(v) - E_0(y)\big)\, \mathrm{d}\mathbf{x} + \int_\Gamma \widetilde{\psi}(u)\, \mathrm{d}\mathcal{H}^1,$$

where $\mathcal{H}$ is defined (18).

Proof. The proof is given in Theorem 13. □

Remark 15. From a standard result (see, for instance, [40, Theorem 5.32 and Remark 5.33]) we can assume in without loss of generality that ψ is convex.

Thus, the following Theorem, together with a diagonalisation argument, will give the limsup inequality.

Theorem 25. For every configuration $(\Omega, v, \mu) \in \mathcal{A}_r(m, M)$, there is a sequence $(w_\varepsilon, v_\varepsilon, u_\varepsilon)_\varepsilon \subset \mathcal{A}_p(m, M)$ such that $(w_\varepsilon, v_\varepsilon, u_\varepsilon) \to (\Omega, v, \mu)$ as $\varepsilon \to 0$. Moreover,

$$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(w_\varepsilon, v_\varepsilon, u_\varepsilon) \leqslant \mathcal{H}(\Omega, v, \mu).$$

Proof. Step 1. C^∞ approximation. Let $(\Omega, v, \mu) \in \mathcal{A}_r(m, M)$. We approximate h by convolution in order to get a $\mathcal{C}^\infty$ approximant g_ε such that the mass constraint is satisfied, for every $\varepsilon > 0$.
Consider a convolution kernel $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$ (namely, $\int_{\mathbb{R}} \rho \, \mathrm{d}x = 1, \rho \geqslant 0$ and $\operatorname{supp} \rho \subset [-1/2, 1/2]$). Let, for each $x \in \mathbb{R}$,

$$\rho_\varepsilon(x) \coloneqq \frac{1}{\varepsilon}\rho\Big(\frac{x}{\varepsilon}\Big)$$

and $g_\varepsilon : (a, b) \to \mathbb{R}$ be defined as

$$g_\varepsilon(x) \coloneqq h * \rho_\varepsilon(x).$$

Let Ω_ε be the subgraph of g_ε. By standard results, $\Gamma_\varepsilon \coloneqq \partial\Omega_\varepsilon \cap \big((a, b) \times \mathbb{R}\big)$ is a $\mathcal{C}^\infty$ curve and since $g_\varepsilon \to h$ uniformly as $\varepsilon \to 0$ we have

$$\mathbb{R}^2 \setminus \Omega_\varepsilon \xrightarrow{H} \mathbb{R}^2 \setminus \Omega. \tag{116}$$

In addition,

$$\lim_{\varepsilon \to 0} \mathcal{H}^1(\Gamma_\varepsilon) = \mathcal{H}^1(\Gamma), \tag{117}$$

Conisider a grid $(R^j)_{j=1}^N$ for which μ is grid-constant. From (*iii*) of Definition 7, the uniform convergence and (117), we can assume that there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ and for each $j = 1, \ldots, N$ we have

$$\Gamma \cap R^j \neq \emptyset \quad \text{and} \quad \Gamma_\varepsilon \cap R^j \neq \emptyset.$$

We also note that from the previous assumption that

$$\lim_{\varepsilon \to 0} \mathcal{H}^1(\Gamma_\varepsilon \cap R^j) = \mathcal{H}^1(\Gamma \cap R^j), \tag{118}$$

for each $j = 1, \ldots, N$.
We define $\overline{u}_\varepsilon : \Gamma_\varepsilon \to \mathbb{R}$ as

$$\overline{u}_{\varepsilon|\Gamma_\varepsilon \cap R^j} \coloneqq \frac{\mathcal{H}^1(\Gamma \cap R^j)}{\mathcal{H}^1(\Gamma_\varepsilon \cap R^j)} u^j. \tag{119}$$

By assumption on u we get that

$$\int_{\Gamma_\varepsilon} \overline{u}_\varepsilon \ \ \mathrm{d}\mathcal{H}^1 = \sum_{j=1}^{N} \overline{u}_\varepsilon \mathcal{H}^1(\Gamma_\varepsilon \cap R^j) = \sum_{j=1}^{N} u^j \mathcal{H}^1(\Gamma \cap R^j) = m.$$

By using standard properties of the convolution, we obtain

$$\int_a^b g_\varepsilon \ \mathrm{d}x = \int_a^b h * \rho_\varepsilon \ \mathrm{d}x = \Big(\int_a^b h(x) \ \mathrm{d}x \Big) \Big(\int_{\mathbb{R}} \rho_\varepsilon(x) \ \mathrm{d}x \Big) = M.$$

We are left to define a displacement sequence $\overline{v}_\varepsilon \in H^1(\Omega_\varepsilon; \mathbb{R}^2)$. We claim that for every $x \in (a, b)$ it holds

$$g_\varepsilon(x) \leqslant h(x) + \varepsilon L, \tag{120}$$

where ℓ is the Lipschitz constant of h. We have

$$\begin{aligned} g_\varepsilon(x) = h * \rho_\varepsilon(x) = & \int_{\mathbb{R}} h(t)\rho_\varepsilon(x-t) \ \mathrm{d}t \\ \leqslant & \int_a^b h(x)\rho_\varepsilon(x-t) \ \mathrm{d}t + \int_a^b |h(t) - h(x)| \, \rho_\varepsilon(x-t) \ \mathrm{d}t \\ \leqslant & \ h(x) + \ell\varepsilon. \end{aligned}$$

Thanks to (120), we have

$$\{(x, y - \varepsilon\ell) : (x, y) \in \Omega_\varepsilon\} \subset\subset \Omega$$

and thus, the function $v_\varepsilon : Q \to \mathbb{R}^2$, set as

$$v_\varepsilon(x, y) \coloneqq v(x, y - \varepsilon\ell),$$

is well defined in a neighbourhood of Ω_ε.
We conclude that $(\Omega_\varepsilon, v_\varepsilon, \mu_\varepsilon)_\varepsilon \subset \mathcal{A}_r(m, M)$, where $\mu_\varepsilon \coloneqq u_\varepsilon \mathcal{H}^1 \llcorner \Gamma_\varepsilon$, and that $(\Omega_\varepsilon, v_\varepsilon, u_\varepsilon) \to (\Omega, v, \mu)$ from (116) and the fact that v_ε is a translation of v and $||v_\varepsilon - v||_{H^1(Q;\mathbb{R}^2)} \to 0$ (see [13, Theorem 4.26]) as $\varepsilon \to 0$.
Now we prove that we also have the approximation of the energy. First, we have

$$\Big| \int_{\Omega_\varepsilon} W\big(E(v_\varepsilon) - E_0(y)\big) \ \mathrm{d}\mathbf{x} - \int_{\Omega} W\big(E(v) - E_0(y)\big) \ \mathrm{d}\mathbf{x} \Big|$$

$$\leqslant \int_{\Omega_\varepsilon} \left|W\big(E(v_\varepsilon) - E_0(y)\big) - W\big(E(v) - E_0(y)\big)\right| \,\mathrm{d}\mathbf{x}$$

$$+ \int_{\Omega_\varepsilon \setminus \Omega} W\big(E(v) - E_0(y)\big) \,\mathrm{d}\mathbf{x}. \tag{121}$$

The second term on the right-hand side of (121) goes to zero by Lebesgue Dominated Convergence Theorem. Regarding the first term, by a continuity argument for W (as it is a quadratic form), we obtain that for every $\lambda > 0$ there is a $\overline{\varepsilon} > 0$ such that for every $\varepsilon < \overline{\varepsilon}$ we have

$$\left|W\big(E(v_\varepsilon) - E_0(y)\big) - W\big(E(v) - E_0(y)\big)\right| \leqslant \lambda. \tag{122}$$

From (121) and by taking into account (122), we obtain

$$\Big|\int_{\Omega_\varepsilon} W\big(E(v_\varepsilon) - E_0(y)\big) \,\mathrm{d}\mathbf{x} - \int_{\Omega} W\big(E(v) - E_0(y)\big) \,\mathrm{d}\mathbf{x}\Big|$$

$$\leqslant \lambda\,|\Omega_\varepsilon| + \int_{\Omega_\varepsilon \setminus \Omega} W\big(E(v) - E_0(y)\big) \,\mathrm{d}\mathbf{x} \to 0, \tag{123}$$

as $\varepsilon \to 0$. Moreover, by (117) and (119), we have

$$\int_{\Gamma_\varepsilon} \psi(\overline{u}_\varepsilon) \,\mathrm{d}\mathcal{H}^1 = \sum_{j=1}^{N} \psi\Big(\frac{\mathcal{H}^1(\Gamma \cap R^j)}{\mathcal{H}^1(\Gamma_\varepsilon \cap R^j)} u^j\Big) \mathcal{H}^1(\Gamma_\varepsilon \cap R^j)$$

$$\to \int_{\Gamma} \psi(u) \;\mathrm{d}\mathcal{H}^1, \tag{124}$$

as $\varepsilon \to 0$, where we used the convexity of ψ and (118). From (123) and (124) we can conclude that

$$\lim_{\varepsilon \to 0} \mathcal{H}(\Omega_\varepsilon, v_\varepsilon, u_\varepsilon) = \mathcal{H}(\Omega, v, \mu).$$

Therefore, in the sequel we can assume h to be of class $\mathcal{C}^\infty$.

Step 2. Displacement and phase-field sequences. Given $\varepsilon > 0$, consider the almost optimal profile problem

$$\begin{cases} \varepsilon^2 \left|\gamma_\varepsilon'(t)\right|^2 = P(\gamma_\varepsilon(t)) + \sqrt{\varepsilon} \quad 0 \leqslant t \leqslant 1, \\ \gamma_\varepsilon(0) = 1 \\ \gamma_\varepsilon(1) = 0. \end{cases} \tag{125}$$

We have that (125) has an unique solution $\gamma_\varepsilon \in \mathcal{C}^1([0,1])$ and $0 \leqslant \gamma_\varepsilon \leqslant 1$. Moreover, we can extend γ_ε on $\mathbb{R}$ by setting $\gamma_\varepsilon = 1$ for $x < 0$ and $\gamma_\varepsilon = 0$ for $x > 1$. Note that, for every $\varepsilon > 0$,

$$|\gamma_\varepsilon'(t)| = \frac{1}{\varepsilon}\sqrt{P(\gamma_\varepsilon(t)) + \sqrt{\varepsilon}} < \frac{C}{\varepsilon}, \tag{126}$$

as the potential P is bounded in $[0,1]$.
From Step 1 we can assume that Ω is the subgraph of a smooth function. We define the phase-field approximating sequence $z_\varepsilon : Q^+ \to \mathbb{R}$ as

$$z_\varepsilon(\mathbf{x}) := \gamma_\varepsilon\Big(\frac{d_\Omega(\mathbf{x})}{\varepsilon}\Big).$$

We have that $z_\varepsilon \in H^1(Q^+)$, indeed from the regularity of Ω follows that $d_\Omega \in \mathcal{C}^2(\mathbb{R}^2)$, and therefore $\gamma_\varepsilon \in \mathcal{C}^1(\mathbb{R})$. We first prove that $z_\varepsilon \to \chi_\Omega$ in $L^1_{\mathrm{loc}}(Q)$, as $\varepsilon \to 0$. Take any compact $K \subset Q$. We have that

$$\begin{aligned}\int_{Q^+\cap K} |z_\varepsilon(\mathbf{x}) - \chi_\Omega(\mathbf{x})|\ \mathrm{d}\mathbf{x} &= \int_{\{\mathbf{x}\in Q^+\cap K:\ 0\leqslant |d_\Omega(\mathbf{x})|\leqslant\varepsilon\}} |z_\varepsilon(\mathbf{x}) - \chi_\Omega(\mathbf{x})|\ \mathrm{d}\mathbf{x} \\ &\leqslant |\{\mathbf{x} \in Q^+ \cap K :\ 0 \leqslant |d_\Omega(\mathbf{x})| \leqslant \varepsilon\}|. \end{aligned} \tag{127}$$

Using standard properties of the Minkowski content (see [1, Definition 2.100 and Theorem 2.104]) we have that

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} |\{\mathbf{x} \in Q^+ \cap K :\ 0 \leqslant |d_\Omega(\mathbf{x})| \leqslant \varepsilon\}| = \mathcal{H}^1(\partial\Omega \cap K), \tag{128}$$

as $\varepsilon \to 0$. Therefore, from (127) and (128), we can conclude that for every compact set K,

$$\lim_{\varepsilon\to 0} \int_{Q^+\cap K} |z_\varepsilon(\mathbf{x}) - \chi_\Omega(\mathbf{x})|\ \mathrm{d}\mathbf{x} = 0.$$

In order to get the mass constraint satisfied, we set

$$\alpha_\varepsilon := \frac{M}{\int_{Q^+} z_\varepsilon(\mathbf{x})\ \mathrm{d}\mathbf{x}}.$$

Note that, from similar computation to (127), we can deduce

$$\int_{Q^+} z_\varepsilon(\mathbf{x})\ \mathrm{d}\mathbf{x} \geqslant M. \tag{129}$$

Now, since

$$\int_{Q^+} |z_\varepsilon| \, \mathrm{d}\mathbf{x} \to M,$$

and from (129), we have

$$\alpha_\varepsilon \leqslant 1 \quad \text{and} \quad \lim_{\varepsilon\to 0} \alpha_\varepsilon = 1. \tag{130}$$

We define a rescaled phase-field variable $w_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ as

$$w_\varepsilon(x,y) \coloneqq z_\varepsilon(x, \alpha_\varepsilon y),$$

In such a way, for each $\varepsilon > 0$, w_ε satisfies

$$\int_{Q^+} w_\varepsilon(\mathbf{x}) \, \mathrm{d}\mathbf{x} = M,$$

and $w_\varepsilon \to \chi_\Omega$ in $L^1_{\mathrm{loc}}(Q)$, as $\varepsilon \to 0$. Moreover, for every $\varepsilon > 0$, $w_\varepsilon \in H^1(Q,\mathbb{R})$. Indeed, from (127), (130) and up to a change of variable, it is enough to show that, for every $\varepsilon \geqslant 0$, $z_\varepsilon \in H^1(Q,\mathbb{R})$. We have that, by using (128), we have $z_\varepsilon \in L^2(Q^+)$. For the gradient, we have the following estimate,

$$\int_{Q^+} |\nabla z_\varepsilon|^2 \, \mathrm{d}\mathbf{x} = \int_{\{\mathbf{x}\in Q^+ :\ 0 \leqslant |d_\Omega(\mathbf{x})| \leqslant \varepsilon\}} \left| \frac{\gamma_\varepsilon'(d_\Omega(\mathbf{x}))}{\varepsilon} \right|^2 \mathrm{d}\mathbf{x} \leqslant \frac{C^2}{\varepsilon} \mathcal{H}^1(\partial\Omega) < \infty,$$

where $C > 0$ is as in (126). Thus, we obtain that $z_\varepsilon \in H^1(Q,\mathbb{R})$, for every $\varepsilon > 0$, which implies $w_\varepsilon \in H^1(Q,\mathbb{R})$.
Let ℓ be the Lipschitz constant of h and assume $\ell \geqslant 1$. We claim that if $(x,y) \in Q \setminus \Omega$ is such that $d_\Omega(x,y) < \varepsilon$, then

$$(x, y - \ell\varepsilon) \in \Omega. \tag{131}$$

Indeed, take any point $(\bar{x}, h(\bar{x})) \in \Gamma$. We have

$$|h(\bar{x}) - h(x)| \leqslant \ell \, |\bar{x} - x| \, ,$$

from which we can infer, by adding and subtracting y and by using $\ell \geqslant 1$, that

$$y - h(x) \leqslant \ell(|\bar{x} - x| + |h(\bar{x}) - y|).$$

Now, if we take the infimum on both sides for $\bar{x} \in \Gamma$, we obtain

$$y - h(x) \leqslant \ell d_\Omega(x, y) \leqslant \ell\varepsilon,$$

for which (131) follows. The case in which $\ell < 1$ is easier and therefore omitted.
We define the approximating displacement sequence $v_\varepsilon : Q \to \mathbb{R}^2$ as

$$v_\varepsilon(x, y) := \begin{cases} v(x, y - \ell\varepsilon) w_\varepsilon(x, y - \ell\varepsilon) & \text{if } w_\varepsilon(x, y - \ell\varepsilon) > 0, \\ 0 & \text{else,} \end{cases}$$

Since $h \in \mathcal{C}^\infty(a, b)$, the definition of v_ε is well posed and $v_\varepsilon \in H^1(Q; \mathbb{R}^2)$. Indeed, for every compact set $K \subset Q$

$$\int_{Q \cap K} |v_\varepsilon|^2 \, \mathrm{d}\mathbf{x} \leqslant ||v||^2_{L^2(Q;\mathbb{R}^2)} \tag{132}$$

and therefore $v_\varepsilon \in L^2_{\mathrm{loc}}(Q; \mathbb{R}^2)$. Now, we set $\widetilde{w}_\varepsilon(x, y) := w_\varepsilon(x, y - \ell\varepsilon)$ and estimate

$$\begin{aligned} \int_{Q \cap K} |\nabla v_\varepsilon|^2 \, \mathrm{d}\mathbf{x} \leqslant \int_{Q \cap K} \Big(& |\nabla v_\varepsilon(x, y) \widetilde{w}_\varepsilon(x, y)|^2 \\ & + 2 \, |\nabla v_\varepsilon(x, y) \widetilde{w}_\varepsilon(x, y)| \, |v_\varepsilon(x, y) \otimes \nabla \widetilde{w}_\varepsilon(x, y)| \\ & + |v_\varepsilon(x, y) \otimes \nabla \widetilde{w}_\varepsilon(x, y)|^2 \Big) \, \mathrm{d}\mathbf{x}. \end{aligned} \tag{133}$$

To see that $v_\varepsilon \in H^1_{\mathrm{loc}}(Q; \mathbb{R}^2)$, taking into account (132), we prove that the right-hand side of (133) is bounded. The first term is easily estimated by the Sobolev norm of the gradient of v, indeed, set $Q \cap K_\varepsilon := Q \cap K + (0, \ell\varepsilon)$, we have

$$\begin{aligned} \int_{Q \cap K} |\nabla v_\varepsilon(x, y) \widetilde{w}_\varepsilon(x, y)|^2 \, \mathrm{d}\mathbf{x} &\leqslant \int_{Q \cap K} |\nabla v_\varepsilon(x, y)|^2 \, \mathrm{d}\mathbf{x} \\ &= \int_{Q \cap K_\varepsilon} |\nabla v|^2 \, \mathrm{d}\mathbf{x} \\ &\leqslant ||\nabla v||_{L^2(Q)} \, . \end{aligned}$$

We estimate the second one and then conclude for the entire right-hand side of (133). We have that

$$\begin{aligned}\int_{Q\cap K} |v_\varepsilon(x,y)\otimes\nabla\widetilde{w}_\varepsilon(x,y)|^2 \ \mathrm{d}\mathbf{x} &\leqslant \int_{Q\cap K} |v_\varepsilon(x,y)|^2\,|\nabla\widetilde{w}_\varepsilon(x,y)|^2 \ \mathrm{d}\mathbf{x}\\ &\leqslant \int_{Q\cap K_\varepsilon} |v(x,y)|^2 \left|\frac{\gamma_\varepsilon'(d_\Omega(x,\alpha_\varepsilon y))}{\varepsilon^2}\right|^2 \mathrm{d}\mathbf{x}\\ &= \frac{C^2\,||v||^2_{L^2(Q;\mathbb{R}^2)}}{\varepsilon^4},\end{aligned}$$

where we used (126) and (128). Therefore, for each ε, $v_\varepsilon\in H^1_{\mathrm{loc}}(Q;\mathbb{R}^2)$. By using similar computations, it is possible to show that $v_\varepsilon\to v$ in $L^2_{\mathrm{loc}}(Q;\mathbb{R}^2)$, as $\varepsilon\to 0$. Consider

$$E_\varepsilon(y) \coloneqq E_0(y-\ell\varepsilon) = \begin{cases} te_1\otimes e_1 & \text{if } y\geqslant \ell\varepsilon,\\ 0 & \text{if } y<\ell\varepsilon.\end{cases}$$

It follows, by definition, that

$$E(v_\varepsilon) = \begin{cases} E\big(v_\varepsilon(x,y)\big)\widetilde{w}_\varepsilon(x,y) + v_\varepsilon(x,y)\odot\nabla\widetilde{w}_\varepsilon(x,y) & \text{if } \widetilde{w}_\varepsilon(x,y)>0,\\ 0 & \text{else.}\end{cases}$$

The reason for which the following computation is made with the translation of w_ε, namely $\widetilde{w}_\varepsilon$, is in order to avoid to see the gradient of w_ε in the area where it equals 1, as it is made clear in Figure 2.2.
First, we set

$$\begin{aligned}\widetilde{A}_\varepsilon &\coloneqq \left\{\widetilde{w}_\varepsilon(x,y)=1,\ h(x)+\varepsilon<y<h(x)+\varepsilon(\ell-1)\right\},\\ A_\varepsilon &\coloneqq \left\{w_\varepsilon(x,y)=1,\ h(x)+\varepsilon(1-\ell)<y<h(x)+-\varepsilon\right\}.\end{aligned}$$

We compute

$$\begin{aligned}&\int_{Q^+}\big(w_\varepsilon(\mathbf{x})+\eta_\varepsilon\big)W\big(E(v_\varepsilon)-E_\varepsilon(y)\big)\ \mathrm{d}\mathbf{x}\\ &\qquad\qquad = (1+\eta_\varepsilon)\Big[\int_{\{w_\varepsilon>0\}} W\big(E(v_\varepsilon)-E_\varepsilon(y)\big)\ \mathrm{d}\mathbf{x}\end{aligned}$$

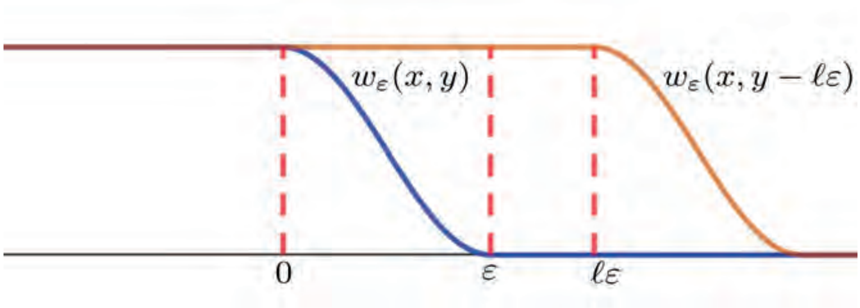

Figure 2.2: For each x, the function $\widetilde{w}_\varepsilon(x,y) = w_\varepsilon(x, y-\ell y)$ allows us to see e the potential term W, where $\nabla \widetilde{w}_\varepsilon = 0$.

$$+ \int_{\widetilde{A}_\varepsilon} W\big(E(v_\varepsilon) - E_\varepsilon(y)\big)\ \mathrm{d}\mathbf{x}\Big]$$

$$+ \eta_\varepsilon \int_{\{0<\widetilde{w}_\varepsilon<1\}} W\big(E(v_\varepsilon) - E_\varepsilon(y)\big)\ \mathrm{d}\mathbf{x}. \tag{134}$$

For the first term on the right-hand side of (134), we have

$$\begin{aligned} \int_{\{w_\varepsilon>0\}} W\big(E(v_\varepsilon) - E_\varepsilon(y)\big)\ \mathrm{d}\mathbf{x} &= \int_{\{w_\varepsilon(x,y+\ell\varepsilon)>0\}} W\big(E(v) - E_0(y)\big)\ \mathrm{d}\mathbf{x} \\ &\leqslant \int_\Omega W\big(E(v) - E_0(y)\big)\ \mathrm{d}\mathbf{x}, \end{aligned} \tag{135}$$

since $\{w_\varepsilon(x, y+\ell\varepsilon) > 0\} \subset \Omega$. Proceeding with the second term, we have

$$\int_{\widetilde{A}_\varepsilon} W\big(E(v_\varepsilon) - E_\varepsilon(y)\big)\ \mathrm{d}\mathbf{x} = \int_{A_\varepsilon} W\big(E(v) - E_0(y)\big)\ \mathrm{d}\mathbf{x}, \tag{136}$$

which goes to 0, as $\varepsilon \to 0$, by Dominated Convergence Theorem. Finally

$$\begin{aligned} &\eta_\varepsilon \int_{\{0<\widetilde{w}_\varepsilon<1\}} W\big(E(v_\varepsilon) - E_\varepsilon(y)\big)\ \mathrm{d}\mathbf{x} \\ &\qquad \leqslant K\eta_\varepsilon \int_{\{0<\widetilde{w}_\varepsilon<1\}} \Big(\big|E(v_\varepsilon)\widetilde{w}_\varepsilon - E_\varepsilon(y)\big|^2 \\ &\qquad\qquad + 2\big|E(v_\varepsilon)\widetilde{w}_\varepsilon - E_\varepsilon(y)\big|\, |v_\varepsilon(x,y)|\, |\nabla \widetilde{w}_\varepsilon(x,y)| \end{aligned}$$

$$+ |v_\varepsilon(x,y)|^2 \, |\nabla \widetilde{w}_\varepsilon(x,y)|^2 \Big) \, \mathrm{d}\mathbf{x}, \tag{137}$$

where $K > 0$ is the constant from the upper bound of the growth of W. We estimate the three terms in the sum on the right-hand side of (137). We have

$$\begin{aligned}
\eta_\varepsilon \int_{\{0<\widetilde{w}_\varepsilon<1\}} & \left| E\big(v_\varepsilon\big)\widetilde{w}_\varepsilon - E_\varepsilon(y) \right|^2 \, \mathrm{d}\mathbf{x} \\
&= \eta_\varepsilon \int_{\{0<w_\varepsilon<1\}} |E(v)w_\varepsilon - E_0(y)|^2 \, \mathrm{d}\mathbf{x} \\
&\leqslant \eta_\varepsilon \int_{\{0<w_\varepsilon<1\}} |E(v)w_\varepsilon|^2 + |E_0(y)|^2 + 2\,|E(v)w_\varepsilon|\,|E_0(y)| \, \mathrm{d}\mathbf{x} \\
&\leqslant \eta_\varepsilon \int_{\{0<w_\varepsilon<1\}} |E(v)|^2 \, \mathrm{d}\mathbf{x} + \int_{\{0<w_\varepsilon<1\}} |E_0(y)|^2 \, \mathrm{d}\mathbf{x} \\
&\quad + 2 \int_{\{0<w_\varepsilon<1\}} |E(v)|\,|E_0(y)| \, \mathrm{d}\mathbf{x}.
\end{aligned} \tag{138}$$

Now, all the terms on the right-hand of (138) are bounded, since $\nabla v \in L^2(\{0 < w_\varepsilon < 1\};\mathbb{R}^2) \subset L^1(\{0 < w_\varepsilon < 1\};\mathbb{R}^2)$ and since $|E_0(y)|$ is constant on $\{0 < w_\varepsilon < 1\}$. From that, we conclude that

$$\eta_\varepsilon \int_{\{0<\widetilde{w}_\varepsilon<1\}} \left| E\big(v_\varepsilon\big)\widetilde{w}_\varepsilon - E_\varepsilon(y) \right|^2 \, \mathrm{d}\mathbf{x} \to 0, \tag{139}$$

since $\eta_\varepsilon \to 0$, as $\varepsilon \to 0$.
For the second term on the right-hand side of (137), by taking into account (126), we have

$$\begin{aligned}
\eta_\varepsilon \int_{\{0<\widetilde{w}_\varepsilon<1\}} & |v_\varepsilon|^2 \, |\nabla \widetilde{w}_\varepsilon|^2 \, \mathrm{d}\mathbf{x} \\
&\leqslant C^2 \frac{\eta_\varepsilon}{\varepsilon^2} \int_{\{0<w_\varepsilon<1\}} |v|^2 \, \mathrm{d}\mathbf{x} \\
&\leqslant C^2 \frac{\eta_\varepsilon}{\varepsilon^2} \, ||v||_2 \left| \{\mathbf{x} \in Q^+ : \; 0 < d_\Omega(\mathbf{x}) \leqslant \varepsilon\} \right| \to 0,
\end{aligned} \tag{140}$$

as $\varepsilon \to 0$, indeed, by (128)

$$\frac{1}{\varepsilon} \left| \{\mathbf{x} \in Q^+ : \; 0 < d_\Omega(\mathbf{x}) \leqslant \varepsilon\} \right| \to \mathcal{H}^1(\Gamma)$$

and $\eta_\varepsilon/\varepsilon \to 0$ by assumption.
Finally, the third term on the right-hand side of (137) goes to 0 by a similar approach to the one used for (139) and (140).
In conclusion, from (134), (135), (139) and (140) we obtain

$$\limsup_{\varepsilon\to 0} \int_{Q^+} \big(w_\varepsilon(\mathbf{x}) + \eta_\varepsilon\big) W\big(E(v_\varepsilon) - E_\varepsilon(y)\big)\,\mathrm{d}\mathbf{x} \leqslant \int_\Omega W\big(E(v) - E_0(y)\big)\,\mathrm{d}\mathbf{x}. \tag{141}$$

Step 3. Density sequence. Since u is grid-constant, there exists a family of squares $\{R^j\}_{j=1}^N$, for which

$$u_{|R^j\cap\Gamma} = u^j \geqslant 0.$$

Note that for every $\mathbf{x} \in Q$ such that $|d_\Omega(\mathbf{x})| > \varepsilon$, we have $w_\varepsilon(\mathbf{x}) = 0$ or $w_\varepsilon(\mathbf{x}) = 1$. Up to further reducing ε, from Lemma 8, we can assume that, for every $j = 1, \ldots, N$,

$$\{\mathbf{x} \in \mathbb{R}^2 \ : \ |d_\Omega(\mathbf{x})| < \varepsilon\} \subset R^j,$$

so that if $\mathbf{x} \notin R^j$, for every $j = 1, \ldots, N$, we have

$$P(w_\varepsilon) = |\nabla w_\varepsilon|^2 = 0. \tag{142}$$

We set

$$p_\varepsilon^j := \frac{1}{\sigma} \int_{Q^+\cap R^j} \left(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} P(w_\varepsilon)\right)\mathrm{d}\mathbf{x}$$

On each R^j, we define the approximating density $u_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ as

$$u_{\varepsilon|R^j}(\mathbf{x}) := \begin{cases} \dfrac{\mathcal{H}^1(\Gamma\cap R^j)}{p_\varepsilon^j} u^j & \mathbf{x} \in R^j \\ 0 & \text{else} \end{cases}$$

and notice that

$$\lim_{\varepsilon\to 0} \frac{\mathcal{H}^1(\Gamma\cap R^j)}{p_\varepsilon^j} = 1, \tag{143}$$

for every $j = 1, \dots, N$. We note that if we set

$$\mu_\varepsilon := \frac{1}{\sigma}\Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)u_\varepsilon \mathcal{L}^2 \llcorner Q^+,$$

we have

$$\mu_\varepsilon(\mathbb{R}^2) = \frac{1}{\sigma}\int_{Q^+}\Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)u_\varepsilon \,\mathrm{d}\mathbf{x} = \sum_{j=1}^N u^j \mathcal{H}^1(\Gamma \cap R^j) = m.$$

Moreover, $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$, as $\varepsilon \to 0$, indeed, take any $\varphi \in \mathcal{C}_c(Q)$, we have

$$\begin{aligned}
\int_{Q^+}\varphi \,\mathrm{d}\mu_\varepsilon &= \frac{1}{\sigma}\int_{Q^+}\Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)u_\varepsilon\varphi \,\mathrm{d}\mathbf{x}\\
&= \sum_{j=1}^N u^j\Big(\frac{\mathcal{H}^1(\Gamma\cap R^j)}{p^j_\varepsilon} - 1\Big)\int_{Q^+\cap R^j}\Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)\varphi \,\mathrm{d}\mathbf{x}\\
&\qquad + \sum_{j=1}^N \frac{u^j}{\sigma}\int_{Q^+\cap R^j}\Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)\varphi \,\mathrm{d}\mathbf{x}\\
&\to \sum_{j=1}^N u^j \mathcal{H}^1(\Gamma\cap R^j),
\end{aligned}$$

as $\varepsilon \to 0$, where we used [17, Proposition 5.4, Lemma 6.2].
Now, we show that

$$\limsup_{\varepsilon\to 0}\int_{Q^+}\Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)\psi(u_\varepsilon)\,\mathrm{d}\mathbf{x} \leqslant \sigma\int_\Gamma \psi(u)\,\mathrm{d}\mathcal{H}^1.$$

First we have, by applying a change of variable and taking into account (130) and (142), that

$$\begin{aligned}
&\int_{Q^+}\Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)\psi(u_\varepsilon)\,\mathrm{d}\mathbf{x}\\
&\qquad \leqslant \sum_{j=1}^N \frac{1}{\alpha_\varepsilon}\int_{R^j}\Big(\varepsilon\,|\nabla z_\varepsilon|^2 + \frac{1}{\varepsilon}P(z_\varepsilon)\Big)\psi\Big(\frac{\mathcal{H}^1(\Gamma\cap R^j)}{p^j_\varepsilon}u^j\Big)\,\mathrm{d}\mathbf{x}. \qquad (144)
\end{aligned}$$

Morover, since ψ is continuous (as it is convex by assumption), for every $\lambda > 0$, there is $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$, we have that, by taking into account (143), that

$$\left|\psi\Big(\frac{\mathcal{H}^1(\Gamma \cap R^j)}{p_\varepsilon^j} u^j\Big) - \psi(u^j)\right| < \lambda.$$

Therefore, by adding and subtracting $\psi(u^j)$ and for $\varepsilon < \varepsilon_0$, from (144), we have

$$\begin{aligned}\int_{Q^+} \Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)\psi(u_\varepsilon)\ \mathrm{d}\mathbf{x} \\ \leqslant \sum_{j=1}^N \frac{\lambda}{\alpha_\varepsilon}\int_{R^j}\Big(\varepsilon\,|\nabla z_\varepsilon|^2 + \frac{1}{\varepsilon}P(z_\varepsilon)\Big)\ \mathrm{d}\mathbf{x} \\ + \sum_{j=1}^N \frac{\psi(u^j)}{\alpha_\varepsilon}\int_{R^j}\Big(\varepsilon\,|\nabla z_\varepsilon|^2 + \frac{1}{\varepsilon}P(z_\varepsilon)\Big)\ \mathrm{d}\mathbf{x}\end{aligned} \tag{145}$$

Finally, since the sequence $(z_\varepsilon)_\varepsilon$ makes use of the solution of problem (125), we can apply the same argument as in [55, Proposition 2] for the limsup inequality. By taking the limsup on both sides of (145) and then, by letting $\lambda \to 0$, we obtain

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\int_{Q^+}\Big(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}P(w_\varepsilon)\Big)\psi(u_\varepsilon)\ \mathrm{d}\mathbf{x} \\ &\leqslant \sigma\sum_{j=1}^N \psi(u^j)\mathcal{H}^1(\Gamma\cap R^j) \\ &= \sigma\int_\Gamma \psi(u)\ \mathrm{d}\mathcal{H}^1.\end{aligned} \tag{146}$$

Step 4. Conclusion. Using (141) and (146) we obtain that

$$\limsup_{\varepsilon\to 0}\mathcal{F}_\varepsilon(w_\varepsilon, v_\varepsilon, u_\varepsilon) \leqslant \mathcal{H}(\Omega, v, u),$$

which concludes the proof of Theorem 25. □

By putting together Theorems 25 and 20 we proved Theorem 14 and 16.

Chapter 3

Geometrically constrained walls in three dimensions

This Chapter shows the results investigated in [29] about geometrically constrained walls in a dumbbell-shaped domain in $\mathbb{R}^3$ (See Figure 3.1). The model features a sufficiently smooth potential which is minimal at the imposed values of the magnetisation in the bulk parts of the dumbbell, and a gradient term penalising transitions; the two are competing as soon as the values of the magnetisation in the bulks are not the same.

The study of the behaviour of the magnetisation in a dumbbell-shaped domain is relevant in micro- and nano-electronics applications, where the neck of the dumbbell models magnetic point contacts. We refer the reader to [24, 50, 51, 57, 63, 65] for an incomplete list of applications and contexts of relevance of geometrically constrained walls. If one imposes two different values of the magnetisation, one in each of the two macroscopic components, a transition is expected in the vicinity of the neck, as initially observed by Bruno in [14]: if the neck is small enough, so that the geometry of the material varies drastically when passing from one bulk to the other, it can play a crucial role in determining the location of the magnetic wall, by influencing the mere minimisation of the magnetic energy. Three scenarios are to be considered: the transition may happen either completely inside the neck, or partly inside and partly outside the neck, or completely outside the neck.

The body of literature on this problem counts many physical contribu-

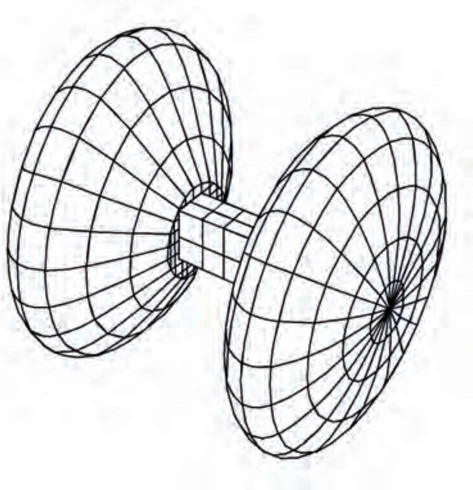

Figure 3.1: A pictorial representation of a typical domain of interest.

tions stemming from Bruno's work [14] and a few mathematical items, which we are going to briefly review to give a context to our novel results. In [14], Bruno considers the special geometry of a thin ($0 < h \ll 1$) three-dimensional wall $\Omega = S \times (-h, h)$, where S is a planar region with a dumbbell shape whose neck is located at the origin and the bulks are in $\{x < 0\}$ and $\{x > 0\}$. He assumes that the preferred directions of the magnetisation vector are $\mathbf{m} = (0, 0, \pm 1)$ and makes the Ansatz that it varies only in the y-z plane, as a function of the x-coordinate alone, namely

$$\mathbf{m}(\mathbf{x}) = (0, \cos(u(x)), \sin(u(x)))\,. \tag{147}$$

The energy that Bruno minimises is the one usually describing Bloch walls and turns out to be a functional of u alone, with two emerging length scales when imposing that $\mathbf{m} \approx (0, 0, \pm 1)$ in $\{\pm x > 0\}$: one driven by the shape S of the domain, the other one dictated by the physical parameters entering the expression of the energy. Despite Bruno's insightful intuition, the special form of the magnetisation has some limitations, some of which were removed (for instance, by allowing $\mathbf{m}$ to vary also in the x-z plane and considering fully three dimensional geometries) in [58].

Among the mathematical literature on this topic, we mention [3, 4, 5, 6, 21, 46, 47, 48, 49] as far as the PDE aspect is concerned, and

[16, 20, 19, 62] for variational approaches.

For the reader's convenience, we recall here the setting of the problem already mentioned in the introduction.
To model the geometries that are of interest in the applications, we will consider an infinitesimally small neck, whose size is determined by three parameters $\varepsilon, \delta, \eta > 0$:

$$N_\varepsilon := \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 : |x| \leqslant \varepsilon, |y| < \delta, |z| < \eta\}, \tag{148}$$

with the understanding that all three of them vanish when $\varepsilon \to 0$, that is $\delta = \delta(\varepsilon) \to 0$ and $\eta = \eta(\varepsilon) \to 0$, as $\varepsilon \to 0$. The full domain is described by

$$\Omega_\varepsilon := \Omega^\ell_\varepsilon \cup N_\varepsilon \cup \Omega^{\mathrm{r}}_\varepsilon, \tag{149}$$

where $\Omega^\ell_\varepsilon = \Omega^\ell - (\varepsilon, 0, 0)$ and $\Omega^{\mathrm{r}}_\varepsilon = \Omega^{\mathrm{r}} + (\varepsilon, 0, 0)$, for certain open sets $\Omega^\ell \subset \{x < 0\}$ and $\Omega^{\mathrm{r}} \subset \{x > 0\}$ such that $0 \in \partial\Omega^\ell \cap \partial\Omega^{\mathrm{r}}$. This geometry makes the x direction the preferred one, whereas the y- and z-direction can be interchanged upon a change of coordinates; this motivates the fact that we will use, throughout the work, the subscript ε alone as an indication of the smallness of the neck.

We are interested in understanding the asymptotic behaviour, as $\varepsilon \to 0$, of stable critical points (see Definition 27) of the energy

$$\mathcal{F}(u, \Omega_\varepsilon) := \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla u(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} + \int_{\Omega_\varepsilon} W(u(\mathbf{x})) \,\mathrm{d}\mathbf{x}\,, \tag{150}$$

defined for $u \in H^1(\Omega_\varepsilon)$, where $\mathrm{d}\mathbf{x} = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ and

$$\begin{cases} W \colon \mathbb{R} \to [0, +\infty) \text{ is a function of class } \mathcal{C}^2 \text{ such that} \\ W^{-1}(0) = \{\alpha, \beta\} \text{ for some } \alpha < \beta \\ \text{and } \lim\limits_{|t| \to +\infty} W(t) = +\infty. \end{cases}$$

In (150), the function u represents a suitable quantity related to the magnetisation field defined on Ω_ε and the potential W favours the values $u(\mathbf{x}) = \alpha$ and $u(\mathbf{x}) = \beta$ for the magnetisation, corresponding to those to be imposed in the bulks. Here, the competition between the potential and the gradient terms is significantly influenced by the geometry of Ω_ε.

The energy (150) was considered in [14] as a simplified model for studying the magnetisation inside a thin dumbbell domain under the assumption that the magnetic field is of the form

$$\mathbf{m}(\mathbf{x}) = (0, \cos(u(x)), \sin(u(x)))\,.$$

Despite this simplifying assumption, the mathematical analysis is rich enough to exhibit non-trivial behaviours of the magnetisation.

3.1 Main result

In this section we present the main achievement we reached in [29].

Set up of the problem

We study a mathematical model to characterise magnetic domain walls in a three-dimensional dumbbell-shaped domain The two bulks are modelled by two bounded, connected, open sets $\Omega^\ell, \Omega^{\mathrm{r}} \subset \mathbb{R}^3$ such that

(H1) the origin $(0,0,0)$ belongs to $\partial\Omega^\ell \cap \partial\Omega^{\mathrm{r}}$;
(H2) $\Omega^\ell \subset \{x < 0\}$ and $\Omega^{\mathrm{r}} \subset \{x > 0\}$;
(H3) there exists $r_0 > 0$ such that $(\partial\Omega^\ell) \cap B_{r_0}(0,0,0)$ and $(\partial\Omega^{\mathrm{r}}) \cap B_{r_0}(0,0,0)$ are contained in the plane $\{x = 0\}$, i.e., the bulks are flat and vertical near the origin, where the conjunction with the neck will be located.

Remark 16. We point out that assumption (H3) is made for mere convenience and it does not affect the generality of our results. While allowing the reader to focus on the main qualitative geometrical properties of the domain, it can be removed following the strategy outlined in [59].

We let $\varepsilon > 0$ and define the neck region as in (148), so that the dumbbell-shaped domain Ω_ε is defined as in (149), where $\Omega^\ell_\varepsilon = \Omega^\ell - (\varepsilon, 0, 0)$ and $\Omega^{\mathrm{r}}_\varepsilon = \Omega^{\mathrm{r}} + (\varepsilon, 0, 0)$. We notice that Ω_ε is a bounded, connected, open set with Lipschitz boundary.

We now give the relevant definitions of critical points and isolated local minimiser for the functional $\mathcal{F}(\cdot, \Omega_\varepsilon)$ introduced in (150).

Definition 26. We say that a function $u_\varepsilon \in H^1(\Omega_\varepsilon)$ is a *critical point of* $\mathcal{F}(\cdot, \Omega_\varepsilon)$ if it is a weak solution to the system

$$\begin{cases} \Delta u_\varepsilon = W'(u_\varepsilon) & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Definition 27. For $\varepsilon > 0$, let $u_\varepsilon \in H^1(\Omega_\varepsilon)$ be a critical point of $\mathcal{F}(\cdot, \Omega_\varepsilon)$. We say that the family $(u_\varepsilon)_\varepsilon$ is an *admissible family of nearly locally constant critical points* if

(a) there exists $\bar{\varepsilon} > 0$ such that $\sup\left\{\|u_\varepsilon\|_\infty : \varepsilon \in (0, \bar{\varepsilon}]\right\} =: \overline{M} < +\infty$;
(b) $\|u_\varepsilon - \alpha\|_{L^1(\Omega^\ell_\varepsilon)} \to 0$ and $\|u_\varepsilon - \beta\|_{L^1(\Omega^{\mathrm{r}}_\varepsilon)} \to 0$ with $\alpha < \beta$, as $\varepsilon \to 0$.

Moreover, we say that $(u_\varepsilon)_\varepsilon \subset H^1(\Omega_\varepsilon)$ is an *admissible family of local minimisers* if it satisfies, additionally,

(c) there exist $\varepsilon_0 > 0$ and $\theta_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$ we have

$$\mathcal{F}(v, \Omega_\varepsilon) \geqslant \mathcal{F}(u_\varepsilon, \Omega_\varepsilon),$$

for all $v \in H^1(\Omega_\varepsilon)$ such that $0 < \|v - u_\varepsilon\|_{L^1(\Omega_\varepsilon)} \leqslant \theta_0$.

Regarding the existence of minimisers, [52, Theorem 3.1] can easily be adapted to our setting.

Theorem 26. For $\varepsilon > 0$, let $u_{0,\varepsilon} \colon \Omega_\varepsilon \to \mathbb{R}$ be defined as

$$u_{0,\varepsilon}(\mathbf{x}) := \begin{cases} \alpha & \text{if } \mathbf{x} \in \Omega^\ell_\varepsilon\,, \\ \dfrac{\alpha + \beta}{2} & \text{if } \mathbf{x} \in N_\varepsilon\,, \\ \beta & \text{if } \mathbf{x} \in \Omega^{\mathrm{r}}_\varepsilon\,. \end{cases}$$

If $u_\varepsilon \in H^1(\Omega_\varepsilon)$ is such that $\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) \leqslant \mathcal{F}(v, \Omega_\varepsilon)$ for every $v \in B_\varepsilon$, where

$$B_\varepsilon := \{u \in H^1(\Omega_\varepsilon) : ||u - u_{0,\varepsilon}||_{L^2(\Omega_\varepsilon)} \leqslant d,\ \|u\|_{L^\infty(\Omega_\varepsilon)} < \infty\}, \tag{151}$$

with $d < \min\{|\alpha|\,|\Omega^\ell|^{1/2}, |\beta|\,|\Omega^{\mathrm{r}}|^{1/2}\}$, then the family $(u_\varepsilon)_\varepsilon$ is an admissible family of local minimisers according to Definition 27, and $\|u_\varepsilon - u_{0,\varepsilon}\|_{L^2(\Omega_\varepsilon)} \to 0$, as $\varepsilon \to 0$.

Unlike [52], we do not assume axial symmetry of the domain and this results in a richer variety of regimes. In particular, we find that some of these regimes admit sub-regimes, as was discovered for magnetic thin films in [59, 60]. We discuss all the possible cases in the next section.

Heuristics

In this section, we show how to heuristically guess where the main part of the energy concentrates, just by looking at the asymptotic relationships between the three geometric parameters $\varepsilon, \delta, \eta$.

First of all, we note that, given the privileged role of the parameter ε, it is trivial to see that the roles of δ and η can be interchanged upon switching the coordinate axes y and z. The regimes investigated in [52] corresponds to the cases where $\delta \sim \eta$. Therefore, here we limit ourselves to consider the other following regimes:

(*i*) *Super thin*: $\varepsilon \gg \delta \gg \eta$;

(*ii*) *Flat thin*: $\varepsilon \approx \delta \gg \eta$;

(*iii*) *Window thick*: $\delta \gg \eta \gg \varepsilon$;

(*iv*) *Narrow thick*: $\delta \gg \varepsilon \approx \eta$;

(*v*) *Letter-box*: $\delta \gg \varepsilon \gg \eta$.

We now want to guess where the transition will happen: completely inside, completely outside, or in both regions. To understand this, we reason as follows. First of all, we expect the main part of the energy to be the Dirichlet integral. Therefore, we consider two harmonic functions that play the role of competitors for the minimisation problem

$$\min\{\mathcal{F}(v, \Omega_\varepsilon)\ : v \in B_\varepsilon\};$$

one where the transition from α to β happens inside the neck, and the other one where it happens only outside (and the competitor is constant inside the neck). We then compare their energies (whose computations will be carried out in Section 3.2) to get a guess of where the transition will occur. The first harmonic function will be referred to as the *affine competitor*, and has energy of order

$$\text{Energy of the affine competitor } = \frac{\delta\eta}{\varepsilon}.$$

The second harmonic function will be referred to as the *elliptical competitor*, and has energy of order

$$\text{Energy of the elliptical competitor } = \frac{\delta}{|\ln(\eta/\delta)|}.$$

When one of the two energies is dominant with respect to the other, it is clear where we expect the transition to happen. In the case they are of the same order, we guess that the transition is both inside and outside the neck. This will be later confirmed by rigorous analysis. The comparison of the energies of the two harmonic competitors leads to the following heuristics:

(*i*) *Super thin neck*: In this case, we have

$$\frac{\delta\eta}{\varepsilon}\frac{|\ln(\eta/\delta)|}{\delta} = \frac{\delta}{\varepsilon}\Big(\frac{\eta}{\delta}\,|\ln(\eta/\delta)|\Big) \to 0,$$

as $\varepsilon \to 0$. Then we expect the transition to happen inside N_ε.

(*ii*) *Flat thin neck*: In this case, we obtain

$$\frac{\delta\eta}{\varepsilon}\frac{|\ln(\eta/\delta)|}{\delta} \approx \frac{\eta}{\delta}\,|\ln(\eta/\delta)| \to 0,$$

thus the transition is occurring inside N_ε.

(*iii*) *Window thick neck*: The comparison of the energies of the harmonic competitors gives

$$\frac{\delta\eta}{\varepsilon}\frac{|\ln(\eta/\delta)|}{\delta} = \frac{\eta}{\varepsilon}\,|\ln(\eta/\delta)| \to +\infty,$$

as $\varepsilon \to 0$. The transition is expected to happen entirely outside the neck.

(*iv*) *Narrow thick neck*: In this case, we have

$$\frac{\delta\eta}{\varepsilon}\frac{|\ln(\eta/\delta)|}{\delta} \approx |\ln(\eta/\delta)| \to +\infty,$$

as $\varepsilon \to 0$. Therefore, we expect the transition to happen outside the neck.

(*v*) *Letter-box neck*: In this case, we have the presence of sub-regimes. Indeed, the comparison of the orders of the energies of the harmonic competitors gives

$$\frac{\delta\eta}{\varepsilon}\frac{|\ln(\eta/\delta)|}{\delta} = \frac{\eta}{\varepsilon}\,|\ln(\eta/\delta)|\,,$$

whose asymptotic behaviour is not clear. Therefore, we have to consider the following sub-regimes:

(*a*) *Sub-critical letter box neck*, when

$$\frac{\delta\eta}{\varepsilon}\frac{|\ln(\eta/\delta)|}{\delta} \to 0,$$

as $\varepsilon \to 0$. In such a case, we expect the transition happens inside the neck.

(*b*) *Critical letter box neck*, when

$$\frac{\delta\eta}{\varepsilon}\frac{|\ln(\eta/\delta)|}{\delta} \to \ell \in (0, +\infty),$$

as $\varepsilon \to 0$. In this case, we expect the transition happens both inside and outside the neck.

(*c*) *Super critical letter box neck*, when

$$\frac{\delta\eta}{\varepsilon}\frac{|\ln(\eta/\delta)|}{\delta} \to +\infty,$$

as $\varepsilon \to 0$. Here, the transition is expected to happen outside the neck.

Main results

We study the full three-dimensional case of the problem with no symmetry assumption on the shape of the neck: it will be a parallelepiped as in (148) with all three dimensions independent from one another and all vanishing to zero as $\varepsilon \to 0$. When considering the mutual rate of convergence to zero of the three parameters $\varepsilon, \delta, \eta$, we single out five regimes that do not emerge in the analysis of Kohn and Slastikov, and for each of them we study the energy scaling. We notice that in our three-dimensional setting the scale-invariant Poincaré inequality is not always available in the various regimes. This makes our analysis different from the one proposed in [52]. The mismatch between the parameters δ and η leads to an inequality that is meaningless. This inequality ensures that, given an open, bounded and connected set $A \subset \mathbb{R}^3$, there exists a constant $C > 0$ such that

$$\left(\int_{\lambda A} \left| u\left(\frac{\mathbf{x}}{\lambda}\right) - \overline{u}_A \right|^6 \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{6}} \leqslant C \left(\int_{\lambda A} \left| \nabla u\left(\frac{\mathbf{x}}{\lambda}\right) \right|^2 \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}},$$

for all $\lambda > 0$, and all $u \in H^1(A)$. Here, $\overline{u}_A$ denotes the average of u in A. Note that the argument to get such inequality is the same to guess the conjugate exponent in the Gagliardo-Nierenberg inequality. Despite that, we are able to investigate the behaviour of local minimisers and the associated rescaled limiting problem, which possesses a variational structure in every regime.

Due to the peculiar geometry of our problem induced by the mismatch between η and δ, namely $\eta \ll \delta$, the cross section of the junction of the neck with the bulks is a rectangle with a very large aspect ratio; this allows us to find an ellipsoidal competitor carrying less energy than the spherical one proposed in the previous works. As it depends on $|\ln(\eta/\delta)|$, the energy scaling turns out to exhibit sub-regimes in some of these cases, as described in Section 3.1.

For all of the above-mentioned regimes, we identify where the transition happens. In particular, we rigorously justify the expectations derived from the above heuristics. We identify a proper rescaling of the independent variables that allow us to prove that such rescaled profile converges to a solution of a Dirichlet energy in a limiting domain with suitable boundary conditions. Only in the critical letter box regime, we need to assume convergence to a limiting profile, and we prove that the latter solves a minimisation problem (see Remark 17). In all cases, local minimisers will converge to a constant in the region where the transition does not happen.

We refer the reader to Section 3.3 for the precise statements and proofs of these results.

Remark 17. The reason why in the critical letter box regime we cannot prove compactness of a sequence of local minimisers, is the following. We do expect to see part of the transition inside the neck. Therefore, we rescale the local minimiser u_ε as $v_\varepsilon(x,y,z) := u_\varepsilon(\varepsilon x, \delta y, \eta z)$. In such a way, we get that

$$\frac{\varepsilon}{2\delta\eta}\int_{N_\varepsilon} |\nabla u_\varepsilon|^2 \;\mathrm{d}\mathbf{x} = \frac{1}{2}\int_{[-1,1]^3} \left((\partial_x v_\varepsilon)^2 + \frac{\varepsilon^2}{\delta^2}(\partial_y v_\varepsilon)^2 + \frac{\varepsilon^2}{\eta^2}(\partial_z v_\varepsilon)^2\right) \mathrm{d}\mathbf{x}.$$

The left-hand side is bounded thanks to the energy of the affine competitor. Unfortunately, since in this regime $\varepsilon \ll \delta$, we do not get a lower bound of the y-derivative of the function v_ε, even if we prove that each limit of the sequence $(v_\varepsilon)_\varepsilon$ will only depend on the first variable.

3.2 Competitors

The goal of this section is to build two harmonic competitors and to compute the order of their energies. For clarity, we present the affine and elliptic competitors separately. However, at the end of the section, they are mixed together in a more general way.

Affine competitor

Here we build the affine competitor inside the neck and we compute its energy. Let $A, B \in \mathbb{R}$, and define the affine function $\xi_\varepsilon \colon \mathbb{R}^3 \to \mathbb{R}$ as

$$\xi_\varepsilon(\mathbf{x}) \coloneqq \begin{cases} A & \text{if } \mathbf{x} \in \Omega_\varepsilon^\ell\,, \\ \dfrac{B-A}{2\varepsilon}x + \dfrac{B+A}{2} & \text{if } \mathbf{x} \in N_\varepsilon\,, \\ B & \text{if } \mathbf{x} \in \Omega_\varepsilon^{\mathrm{r}}\,. \end{cases} \tag{152}$$

Then, we have that

$$\frac{1}{2}\int_{N_\varepsilon} |\nabla \xi_\varepsilon|^2 \mathrm{d}\mathbf{x} = \frac{\delta\eta}{\varepsilon}(B-A)^2. \tag{153}$$

Elliptic competitor

In [52], the authors built a harmonic competitor by imposing boundary conditions on *half-spheres* centred at the edges of the neck. The choice of the spherical geometry was dictated by the fact that the authors required $\delta = \eta$. In our case, the geometry will be that of an ellipsoid, suggested by the fact that one of the parameters δ and η is larger than the other.

In order to define our competitor, we first need to introduce the so-called *prolate spheroidal coordinates*. Consider, for $a > 0$, the change of coordinates $(x, y, z) = \Psi(\mu, \nu, \varphi)$, given by

$$\begin{cases} x = a \sinh\mu \sin\nu \cos\varphi, \\ y = a\cosh\mu\cos\nu, \\ z = a\sinh\mu\sin\nu\sin\varphi, \end{cases} \tag{154}$$

where $(0, \pm a, 0)$ are the coordinates of the foci, $\varphi \in [0, 2\pi]$ is the polar angle, $\nu \in [0, \pi]$ is the azimuthal angle and $a, \mu > 0$. For $M > 0$, define the ellipsoid

$$E(a, M) \coloneqq \{\Psi(\mu, \nu, \varphi) : \mu < 2M\}. \tag{155}$$

Moreover, we need consider the left and the right halves of the set $E(a, M)$ translated at the edges of the neck. Namely, we consider the open sets

$$E_\varepsilon^\ell(a, M) \coloneqq (E(a, M) \cap \{x < 0\}) - (\varepsilon, 0, 0),$$

and

$$E^{\mathrm{r}}_{\varepsilon}(a, M) \coloneqq (E(a, M) \cap \{x > 0\}) + (\varepsilon, 0, 0).$$

Note that if $a \cosh(2M) < r_0$, where $r_0 > 0$ is given by assumption (H3), we get that $E^{\ell}_{\varepsilon}(a, M) \subset \Omega^{\ell}_{\varepsilon}$, and that $E^{\mathrm{r}}_{\varepsilon}(a, M) \subset \Omega^{\mathrm{r}}_{\varepsilon}$. For $0 < m < M$, we define the function $\xi_{\varepsilon} \colon \mathbb{R}^3 \to \mathbb{R}$ as

$$\xi_{\varepsilon}(x, y, z) \coloneqq \begin{cases} \alpha & \text{in } \Omega^{\ell}_{\varepsilon} \setminus E^{\ell}_{\varepsilon}(a, M)\,, \\ \dfrac{\alpha+\beta}{2} - h(x+\varepsilon, y, z) & \text{in } E^{\ell}_{\varepsilon}(a, M)\,, \\ \dfrac{\alpha+\beta}{2} & \text{in } N_{\varepsilon}\,, \\ \dfrac{\alpha+\beta}{2} + h(x-\varepsilon, y, z) & \text{in } E^{\mathrm{r}}_{\varepsilon}(a, M)\,, \\ \beta & \text{in } \Omega^{\mathrm{r}}_{\varepsilon} \setminus E^{\mathrm{r}}_{\varepsilon}(a, M)\,, \end{cases} \tag{156}$$

where $h \colon E(a, M) \setminus E(a, m) \to \mathbb{R}$ is the solution to the problem

$$\begin{cases} \Delta h = 0 & \text{in } E(a, M) \setminus E(a, m)\,, \\ h = \dfrac{\beta - \alpha}{2} & \text{on } \partial E(a, M)\,, \\ h = 0 & \text{on } \partial E(a, m)\,. \end{cases}$$

Now, our goal is to find the function h explicitly and to estimate, asymptotically, its Dirichlet energy. We look for the solution in the form $h = h(\mu)$. Then the Laplacian in prolate spheroidal coordinates is given by

$$\Delta h = \frac{1}{a^2(\sinh^2 \mu + \sin^2 \nu)} \left(h_{\mu\mu} + (\coth \mu) h_{\mu}\right) = 0,$$

or equivalently

$$(\sinh \mu h_{\mu})_{\mu} = 0.$$

It follows that

$$h_{\mu} = \frac{c}{\sinh \mu},$$

and thus

$$h(\mu) = c \ln |k \tanh(\mu/2)|. \tag{157}$$

Enforcing the boundary conditions

$$h(2M) = \frac{\beta - \alpha}{2} \quad \text{and} \quad h(2m) = 0$$

yields

$$k = \coth m \qquad \text{and} \qquad c = \frac{\beta - \alpha}{2 \ln \left(\dfrac{\tanh M}{\tanh m} \right)}. \tag{158}$$

Hence, we can write

$$h(\mu) = \frac{(\beta - \alpha)}{2 \ln \left(\dfrac{\tanh M}{\tanh m} \right)} \ln \left(\frac{\tanh \mu/2}{\tanh m} \right).$$

We are now in position to compute the Dirichlet energy of the function ξ_ε. Indeed, by using changing variables and fact that the Jacobian determinant of the transformation Ψ in (154) is $a^3 \sinh \mu \sin \nu (\sinh^2 \mu + \sin^2 \nu)$, we obtain

$$\begin{aligned}
\frac{1}{2} \int_{E(a,M) \setminus E(a,m)} |\nabla \xi_\varepsilon|^2 \, \mathrm{d}\mathbf{x} = & \frac{1}{2} \int_{E(a,M) \setminus E(a,m)} |\nabla h|^2 \, \mathrm{d}\mathbf{x} \\
= & c^2 a \int_0^\pi \int_0^{2\pi} \int_{2m}^{2M} \frac{\sin \nu}{\sinh \mu} \, \mathrm{d}\nu \mathrm{d}\varphi \mathrm{d}\mu \\
= & \frac{\pi a (\beta - \alpha)^2}{\ln \left(\dfrac{\tanh M}{\tanh m} \right)}.
\end{aligned} \tag{159}$$

In the above expression, there are still two choices that we can make: that of the parameters a and m. We now want to choose them in such a way that

$$(N_\varepsilon \cap \{x = \pm\varepsilon\})^\circ \subset \overline{E(a,m) \cap \{x = 0\}} \pm (\varepsilon, 0, 0), \tag{160}$$

where $(\cdot)^\circ$ denotes the internal part of a set.
To enforce (160), we note that (154) implies that, for all $(x, y, z) \in \partial E(a, m)$ such that $x = 0$, it holds

$$\frac{y^2}{a^2 \cosh^2 m} + \frac{z^2}{a^2 \sinh^2 m} = 1. \tag{161}$$

Therefore, choosing a and m to satisfy

$$a \sinh m = \sqrt{2}\eta, \qquad a \cosh m = \sqrt{2}\delta$$

guarantees the validity of (160).

We now want to get an asymptotic estimate of (159), taking into account the fact that all the regimes in this chapter consider the case in which $\eta \ll \delta$. By definition $a^2 = \delta^2 - \eta^2$, and then $a \approx \delta$ as $\varepsilon \to 0$. Moreover

$$\tanh m = \frac{\eta}{\delta}.$$

Observe that in our regimes, when $\delta \gg \eta$, then $m \ll 1$ and

$$\ln\left(\frac{\tanh M}{\tanh m}\right) = \ln \tanh M - \ln \tanh m \approx -\ln \tanh m = -\ln \frac{\eta}{\delta}, \tag{162}$$

for ε small enough. This, together with (159) implies that, for ε small enough,

$$\frac{1}{2}\int_{E(a,M)\setminus E(a,m)} |\nabla \xi_\varepsilon|^2 \, \mathrm{d}\mathbf{x} \approx \frac{\pi\delta(\beta-\alpha)^2}{|\ln \frac{\eta}{\delta}|}. \tag{163}$$

Finally, we note that the elliptic competitor just built gives a better upper bound on the energy of the minimiser u_ε than the one that could be obtained in [52], with a spherical harmonic function. Indeed, in the spherical harmonic case, they obtained an upper estimate with a term of order δ. Therefore, we can notice that

$$\frac{\delta}{|\ln(\eta/\delta)|} \ll \delta,$$

as $\varepsilon \to 0$. Thus, we obtained a competitor whose order of energy is asymptotically lower than the previous one. This is particularly relevant since such a competitor follows the geometry of our problem, in which the shape of the neck presents the y coordinate way smaller than the z coordinate, ruled by δ and η respectively.

Mixed competitor

The idea now, is to mix the affine competitor in the neck, together with the ellipsoidal just built. The purpose of this new competitor, is to describe whenever the transition happens simultaneously inside and outside the neck. Consider $A, B \in \mathbb{R}$ such that $\alpha \leqslant A \leqslant B \leqslant \beta$. Let $h\colon E^\ell(a,M) \setminus E^\ell(a,m) \to \mathbb{R}$ be the solution to

$$\begin{cases} \Delta w = 0 & \text{in } E^\ell(a,M) \setminus E^\ell(a,m), \\ w = \alpha & \text{on } \partial E^\ell(a,M), \\ w = A & \text{on } \partial E^\ell(a,m). \end{cases}$$

and $g\colon E^{\mathrm{r}}(a,M) \setminus E^{\mathrm{r}}(a,m) \to \mathbb{R}$ the solution to

$$\begin{cases} \Delta w = 0 & \text{in } E^{\mathrm{r}}(a,M) \setminus E^{\mathrm{r}}(a,m), \\ w = \beta & \text{on } \partial E^{\mathrm{r}}(a,M), \\ w = B & \text{on } \partial E^{\mathrm{r}}(a,m). \end{cases}$$

Recalling (156) and (154), we define the function $\xi_\varepsilon\colon \mathbb{R}^3 \to \mathbb{R}$ as

$$\xi_\varepsilon(\mathbf{x}) := \begin{cases} \alpha & \text{in } \Omega^\ell_\varepsilon \setminus E^\ell_\varepsilon(a,M), \\ h(x+\varepsilon,y,z) & \text{in } E^\ell_\varepsilon(a,M) \setminus E^\ell_\varepsilon(a,m), \\ A & \text{in } E^\ell_\varepsilon(a,m), \\ \dfrac{B-A}{2\varepsilon}x + \dfrac{B+A}{2} & \text{in } N_\varepsilon, \\ B & \text{in } E^{\mathrm{r}}_\varepsilon(a,m), \\ g(x-\varepsilon,y,z) & \text{in } E^{\mathrm{r}}_\varepsilon(a,M) \setminus E^{\mathrm{r}}_\varepsilon(a,m), \\ \beta & \text{in } \Omega^{\mathrm{r}}_\varepsilon \setminus E^{\mathrm{r}}_\varepsilon(a,M). \end{cases} \tag{164}$$

We now want to estimate, asymptotically, its energy. Using the same argument used to obtain (157), we can write the explicit solution of the problems above as

$$h(\mu) = c^\ell \ln|k^\ell \tanh(\mu/2)|, \qquad \text{and} \qquad g(\mu) = c^{\mathrm{r}} \ln|k^{\mathrm{r}} \tanh(\mu/2)|.$$

We can explicitly obtain $c^\ell, k^\ell, c^{\mathrm{r}}, k^{\mathrm{r}} \in \mathbb{R}$ by imposing the boundary conditions and we get

$$c^\ell = \frac{\alpha - A}{\ln\left(\dfrac{\tanh M}{\tanh m}\right)}, \qquad c^{\mathrm{r}} = \frac{\beta - B}{\ln\left(\dfrac{\tanh M}{\tanh m}\right)}$$

and

$$k^\ell = \frac{\exp\left(\ln\left|\dfrac{\tanh M}{\tanh m}\right|\dfrac{A}{\alpha - A}\right)}{\tanh M} \qquad k^{\mathrm{r}} = \frac{\exp\left(\ln\left|\dfrac{\tanh M}{\tanh m}\right|\dfrac{B}{\beta - B}\right)}{\tanh M}.$$

Let h_ε be the solution of the problem

$$\begin{cases} \alpha & \text{in } \Omega_\varepsilon^\ell \setminus E_\varepsilon^\ell(a, M), \\ h(x+\varepsilon, y, z) & \text{in } E_\varepsilon^\ell(a, M) \setminus E_\varepsilon^\ell(a, m), \\ A & \text{in } E_\varepsilon^\ell(a, m). \end{cases}$$

Arguing like in (159), we get that

$$\frac{1}{2}\int_{E_\varepsilon^\ell(a,M)\setminus E^\ell(a,m)} |\nabla h_\varepsilon|^2 \,\mathrm{d}\mathbf{x} = \frac{\pi a (A-\alpha)^2}{|\ln(\eta/\delta)|}. \tag{165}$$

Similarly, if g_ε is the solution of the problem

$$\begin{cases} B & \text{in } E_\varepsilon^{\mathrm{r}}(a, m), \\ g(x-\varepsilon, y, z) & \text{in } E_\varepsilon^{\mathrm{r}}(a, M) \setminus E_\varepsilon^{\mathrm{r}}(a, m), \\ \beta & \text{in } \Omega_\varepsilon^{\mathrm{r}} \setminus E_\varepsilon^{\mathrm{r}}(a, M), \end{cases}$$

we have

$$\frac{1}{2}\int_{E_\varepsilon^{\mathrm{r}}(a,M)\setminus E^{\mathrm{r}}(a,m)} |\nabla g_\varepsilon|^2 \ \mathrm{d}\mathbf{x} = \frac{\pi a (B-\beta)^2}{|\ln(\eta/\delta)|}, \tag{166}$$

as $\varepsilon \to 0$. Therefore, from (153), (165), and (166), we obtain

$$\mathcal{F}(\xi_\varepsilon, \Omega_\varepsilon) = \frac{\pi a}{|\ln(\eta/\delta)|}\left[(A-\alpha)^2 + (B-\beta)^2\right] + \frac{\delta\eta}{\varepsilon}(B-A)^2.$$

Since $\eta \ll \delta$, we have that $a \approx \delta$. Thus, for ε small enough, we can write

$$\mathcal{F}(\xi_\varepsilon, \Omega_\varepsilon) \approx \frac{\pi \delta}{|\ln(\eta/\delta)|}\left[(A-\alpha)^2 + (B-\beta)^2\right] + \frac{\delta\eta}{\varepsilon}(B-A)^2. \tag{167}$$

Now we compute the minimum of the right-hand of (167) under the constraint that $\alpha \leqslant A \leqslant B \leqslant \beta$. It is possible to see that a solution is in the interior of the admissible region, and thus the optimal A and B are given by the solution of the system

$$\begin{cases} \dfrac{\pi}{|\ln(\eta/\delta)|}(A-\alpha) - \dfrac{\eta}{\varepsilon}(B-A) = 0, \\ \dfrac{\pi}{|\ln(\eta/\delta)|}(B-\beta) + \dfrac{\eta}{\varepsilon}(B-A) = 0, \end{cases}$$

which are

$$A = \frac{\dfrac{\pi\alpha}{|\ln(\eta/\delta)|} + \dfrac{\eta}{\varepsilon}(\alpha+\beta)}{\dfrac{\pi}{|\ln(\eta/\delta)|} + \dfrac{2\eta}{\varepsilon}} \quad \text{and} \quad B = \frac{\dfrac{\pi\beta}{|\ln(\eta/\delta)|} + \dfrac{\eta}{\varepsilon}(\alpha+\beta)}{\dfrac{\pi}{|\ln(\eta/\delta)|} + \dfrac{2\eta}{\varepsilon}}. \tag{168}$$

The choice of A and B in (168) will be crucial in the various regimes when we will need to infer the boundary conditions of the rescaled profile at the edge of the neck. In conclusion, from (167), if u_ε is a local minimiser, we then have

$$\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) \leqslant \frac{\pi\delta}{|\ln(\eta/\delta)|}\left[(A-\alpha)^2 + (B-\beta)^2\right] + \frac{\delta\eta}{\varepsilon}(B-A)^2. \tag{169}$$

Finally, notice that the right-hand side of (169) has a clear separation between the energetic contribution of the competitor inside and outside the neck, as well as their orders of the energy.

3.3 Analysis of the problem in the several regimes

In this section we carry out the rigorous analysis of the asymptotic behaviour of the solution, obtaining information on its energy and its behaviour inside and close to the neck. To this aim, define

$$N := [-1,1]^3,$$

which is the neck N_ε rescaled to size of order 1, that is, under the change of coordinates $(x,y,z) \to (x/\varepsilon, y/\delta, z/\eta)$. In the following subsections, we will perform various rescalings and we will always denote by $\widetilde{\Omega}_\varepsilon$ the corresponding rescaled domains.
We also recall the definition of Hausdorff convergence of sets.

Definition 28. We say that a sequence of closed sets $(A_n)_n \subset \mathbb{R}^3$ *converges in the Hausdoff metric* to a closed set A if

$$\lim_{n\to\infty} \max\left\{\sup_{\mathbf{x}\in A} \mathrm{d}(\mathbf{x}, A_n),\ \sup_{\mathbf{y}\in A_n} \mathrm{d}(\mathbf{y}, A)\right\} = 0.$$

Here, $\mathrm{d}(\mathbf{x}, A)$ denotes the distance between the point $\mathbf{x} \in \mathbb{R}^3$ and the set A. We denote this convergence by $A_n \overset{\mathrm{H}}{\to} A$.

3.3.1 Super-thin neck

In this regime the parameters are ordered as $\varepsilon \gg \delta \gg \eta$. Namely, we have

$$\lim_{\varepsilon\to 0} \frac{\delta}{\varepsilon} = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{\eta}{\delta} = 0.$$

According to the heuristics in Section 3.1, we expect the transition to happen entirely inside the neck. If u_ε is a local minimiser of the functional (150), the convenient rescaling that works is

$$v_\varepsilon(x, y, z) := u_\varepsilon(\varepsilon x, \delta y, \eta z).$$

Define $\widetilde{\Omega}_\varepsilon$, $\widetilde{\Omega}^\ell_\varepsilon$, and $\widetilde{\Omega}^\mathrm{r}_\varepsilon$ as the rescaled domain Ω_ε, Ω^ℓ_ε, and $\Omega^\mathrm{r}_\varepsilon$, respectively. Note that, as $\varepsilon \to 0$,

$$\mathbb{R}^3 \setminus \widetilde{\Omega}_\varepsilon \overset{\mathrm{H}}{\to} \mathbb{R}^3 \setminus \Omega_\infty\,,$$

where

$$\Omega_\infty := \Omega^\ell_\infty \cup N \cup \Omega^\mathrm{r}_\infty,$$

with $\Omega^\ell_\infty := \{x < -1\}$ and $\Omega^\mathrm{r}_\infty := \{x > 1\}$.

Theorem 27. Let $(u_\varepsilon)_\varepsilon \subset H^1(\Omega_\varepsilon)$ be an admissible family of local minimisers as in Definition 27. Assume $\varepsilon \gg \delta \gg \eta$. Then,

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta\eta} \mathcal{F}(u_\varepsilon, \Omega_\varepsilon) = \lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta\eta} \mathcal{F}(u_\varepsilon, N_\varepsilon) = (\beta - \alpha)^2.$$

Moreover, for $\varepsilon > 0$ let $v_\varepsilon \colon \widetilde{\Omega}_\varepsilon \to \mathbb{R}$ be defined as

$$v_\varepsilon(x, y, z) := u_\varepsilon(\varepsilon x, \delta y, \eta z).$$

Then, the following hold:

(i) $v_\varepsilon \chi_{\widetilde{\Omega}^\ell_\varepsilon} - \alpha \chi_{\widetilde{\Omega}^\ell_\varepsilon} \to 0$ in $L^6(\mathbb{R}^3)$ and $v_\varepsilon \chi_{\widetilde{\Omega}^{\mathrm{r}}_\varepsilon} - \beta \chi_{\widetilde{\Omega}^{\mathrm{r}}_\varepsilon} \to 0$ in $L^6(\mathbb{R}^3)$, as $\varepsilon \to 0$;

(ii) There exists $\hat{v} \in H^1(N)$ such that $v_\varepsilon \rightharpoonup \hat{v}$ weakly in $H^1(N)$, as $\varepsilon \to 0$;

(iii) It holds that $\hat{v}(x,y,z) = v(x)$, where $v \in H^1(-1,1)$ is the unique minimiser of the variational problem

$$\min\left\{\frac{1}{2}\int_{-1}^{1} |v'|^2 \,\mathrm{d}x : \ v \in H^1(-1,1),\ v(-1) = \alpha,\ v(1) = \beta\right\}.$$

In particular, $v(x) = \dfrac{\beta-\alpha}{2}x + \dfrac{\alpha+\beta}{2}$.

Proof. Step 1: existence of $\hat{v}$. Note that, using assumption (H3), for $\varepsilon > 0$ sufficiently small, it holds that

$$2N \cap \Omega_\infty \subset \widetilde{\Omega}_\varepsilon.$$

The reason why we take $2N$ and not only N is because we need the boundary conditions to converge. We claim that

$$\sup_{\varepsilon>0} \|v_\varepsilon\|_{H^1(2N\cap\Omega_\infty)} < \infty.$$

First of all, using the fact that $\varepsilon \gg \delta \gg \eta$, we get that

$$\begin{aligned}
\|\nabla v_\varepsilon\|^2_{L^2(2N\cap\Omega_\infty)} =& \frac{\varepsilon}{\delta\eta}\int_{2N\cap\Omega_\infty}\left((\partial_x u_\varepsilon)^2 + \frac{\delta^2}{\varepsilon^2}(\partial_y u_\varepsilon)^2 + \frac{\eta^2}{\varepsilon^2}(\partial_z u_\varepsilon)^2\right)\mathrm{d}\mathbf{x} \\
\leqslant& \frac{\varepsilon}{\delta\eta}\mathcal{F}(u_\varepsilon,\Omega_\varepsilon) \leqslant C < \infty,
\end{aligned} \tag{170}$$

where the first step follows by using a change of variable, while the second one from (153) with $A = \alpha$ and $B = \beta$. Moreover,

$$\begin{aligned}
\int_{2N\cap\Omega_\infty} |v_\varepsilon|^2 \,\mathrm{d}\mathbf{x} = \frac{1}{\varepsilon\delta\eta}\int_{2N_\varepsilon\cap\Omega_\varepsilon} |u_\varepsilon|^2\,\mathrm{d}\mathbf{x} \leqslant& \frac{\left(\sup_{\mathbf{x}\in 2N_\varepsilon\cap\Omega_\varepsilon}|u_\varepsilon|^2\right)|2N_\varepsilon\cap\Omega_\varepsilon|}{\varepsilon\delta\eta} \\
<& +\infty.
\end{aligned}$$

Thus, we get that, up to a subsequence, v_ε converges weakly in $H^1(N)$ to a function $\hat{v} \in H^1(N)$. The independence of the subsequence will follow from Step 2, where we show that the limit is the *unique* solution to a variational problem.

Step 2: limiting problem and behavior inside the neck. We now want to characterize the function v as the unique solution to a variational problem. We do this in two steps: first, we identify a functional that will be minimized, and then we identify the boundary conditions.
We have that

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \frac{\varepsilon}{\delta\eta}\mathcal{F}(u_\varepsilon,\Omega_\varepsilon) &\geqslant \liminf_{\varepsilon\to 0} \frac{\varepsilon}{\delta\eta}\Big(\frac{1}{2}\int_{N_\varepsilon} |\nabla u_\varepsilon|^2 \,\mathrm{d}\mathbf{x} + \int_{N_\varepsilon} W(u_\varepsilon)\,\mathrm{d}\mathbf{x}\Big) \\
&= \liminf_{\varepsilon\to 0} \Big(\frac{1}{2}\int_N \Big((\partial_x v_\varepsilon)^2 + \frac{\varepsilon^2}{\delta^2}(\partial_y v_\varepsilon)^2 + \frac{\varepsilon^2}{\eta^2}(\partial_z v_\varepsilon)^2\Big)\,\mathrm{d}\mathbf{x} \\
&\qquad + \varepsilon^2 \int_N W(v_\varepsilon)\,\mathrm{d}\mathbf{x}\Big) \\
&\geqslant \liminf_{\varepsilon\to 0} \frac{1}{2}\int_N |\nabla v_\varepsilon|^2\,\mathrm{d}\mathbf{x} \geqslant \frac{1}{2}\int_N |\nabla v|^2\,\mathrm{d}\mathbf{x}, \qquad (171)
\end{aligned}$$

where in the last step we used the fact that $\varepsilon \gg \delta \gg \eta$. Notice that, from the bound (170) and the fact that $\varepsilon/\eta \to \infty$ and $\varepsilon/\delta \to \infty$, we necessarily have that v does not depend on y and z. Namely, v_ε converges to a function $\hat{v} \in H^1(N)$, of the form $\hat{v}(x,y,z) = v(x)$, where $v \in H^1(-1,1)$. Therefore, from (171), we can write

$$\begin{aligned}
\frac{1}{2}\int_N |\nabla v|^2\,\mathrm{d}\mathbf{x} &= 2\int_{-1}^{1} (\hat{v}')^2 \,\mathrm{d}x \\
&\geqslant 2\min\Big\{\int_{-1}^{1} (w')^2\,\mathrm{d}x :\ w \in H^1(-1,1),\ w(\pm 1) = \hat{v}(\pm 1)\Big\} \\
&= \big(\hat{v}(1) - \hat{v}(-1)\big)^2, \qquad (172)
\end{aligned}$$

where last step follows by an explicit minimisation.

Now we claim that $\hat{v}(-1) = \alpha$ and $\hat{v}(1) = \beta$. We prove the former, since the latter follows by using a similar argument. The idea (introduced in [52]) is to use the scale-invariant Poincaré inequality

$$\Big(\int_{\Omega_\varepsilon^\ell} |u_\varepsilon - \bar{u}_\varepsilon|^6 \, \mathrm{d}\mathbf{x}\Big)^{\frac{1}{6}} \leqslant C\Big(\int_{\Omega_\varepsilon^\ell} |\nabla u_\varepsilon|^2 \, \mathrm{d}\mathbf{x}\Big)^{\frac{1}{2}}, \tag{173}$$

where $C > 0$ and $\bar{u}_\varepsilon$ is the average on Ω_ε^ℓ of u_ε. Using a change of variable, we estimate (by neglecting the potential term as in (171)) the right-hand side of (173) as

$$\begin{aligned}\varepsilon\delta\eta \int_{\widetilde{\Omega}_\varepsilon^\ell} |v_\varepsilon - \bar{v}_\varepsilon|^6 \, \mathrm{d}\mathbf{x} &\leqslant C\Big(\frac{\delta\eta}{\varepsilon}\int_{\widetilde{\Omega}_\varepsilon^\ell} (\partial_x v_\varepsilon)^2 + \frac{\varepsilon^2}{\delta^2}(\partial_y v_\varepsilon)^2 + \frac{\varepsilon^2}{\eta^2}(\partial_z v_\varepsilon)^2 \, \mathrm{d}\mathbf{x}\Big)^3 \\ &\leqslant C\Big(\frac{\delta\eta}{\varepsilon}\Big)^3 \left(\frac{\varepsilon}{\delta\eta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon)\right)^3.\end{aligned}$$

Now, using the fact that $\delta\eta/\varepsilon^2 \to 0$ as $\varepsilon \to 0$ together with (170), we get that

$$\int_{\widetilde{\Omega}_\varepsilon^\ell} |v_\varepsilon - \bar{v}_\varepsilon|^6 \, \mathrm{d}\mathbf{x} \leqslant C\Big(\frac{\delta\eta}{\varepsilon^2}\Big)^2 \left(\frac{\varepsilon}{\delta\eta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon)\right)^3 \to 0, \tag{174}$$

as $\varepsilon \to 0$. Therefore, $v_\varepsilon \chi_{\widetilde{\Omega}_\varepsilon^\ell} - \bar{v}_\varepsilon \chi_{\widetilde{\Omega}_\varepsilon^\ell} \to 0$ in $L^6(\mathbb{R}^3)$, as $\varepsilon \to 0$. We now claim that $\bar{v}_\varepsilon \chi_{\widetilde{\Omega}_\varepsilon^\ell} - \alpha \chi_{\widetilde{\Omega}_\varepsilon^\ell} \to 0$ in $L^6(\mathbb{R}^3)$, as $\varepsilon \to 0$. Indeed, by definition of local minimiser we have that

$$||u_\varepsilon - \alpha||_{L^1(\Omega_\varepsilon^\ell)} \to 0,$$

as $\varepsilon \to 0$. Therefore, $\bar{u}_\varepsilon \chi_{\Omega_\varepsilon^\ell} - \alpha \chi_{\Omega_\varepsilon^\ell} \to 0$ in $L^6(\mathbb{R}^3)$, which yields that $\bar{v}_\varepsilon \chi_{\widetilde{\Omega}_\varepsilon^\ell} - \alpha \chi_{\widetilde{\Omega}_\varepsilon^\ell} \to 0$ in $L^6(\mathbb{R}^3)$. Thus, since $v_\varepsilon \to \hat{v}$ strongly in $L^2(2N \cap \Omega_\infty)$ as $\varepsilon \to 0$, we get that $\hat{v}(-1) = \alpha$.

Step 3: asymptotic behaviour of the energy. From (171) and (172) we can conclude that

$$\liminf_{\varepsilon \to 0} \frac{\varepsilon}{\delta\eta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) \geqslant (\beta - \alpha)^2. \tag{175}$$

On the other hand, denoting by ξ_ε the affine competitor in (152), we have that

$$\limsup_{\varepsilon \to 0} \frac{\varepsilon}{\delta\eta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) \leqslant \limsup_{\varepsilon \to 0} \frac{\varepsilon}{\delta\eta}\mathcal{F}(\xi_\varepsilon, N_\varepsilon) = (\beta - \alpha)^2. \tag{176}$$

Thus, from (175), and (176), we get

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta\eta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) = (\beta - \alpha)^2.$$

In particular, we get that all inequalities in (171) are equalities, proving that

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta\eta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) = \lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta\eta}\mathcal{F}(u_\varepsilon, N_\varepsilon).$$

This concludes the proof. □

3.3.2 Flat-thin neck

In this regime the parameters are ordered as $\varepsilon \approx \delta \gg \eta$. Namely, we have

$$\lim_{\varepsilon\to 0} \frac{\delta}{\varepsilon} = m \in (0, +\infty) \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{\eta}{\varepsilon} = 0.$$

In this case, the behaviour of an admissible family of local minimisers is similar to the super-thin regime and strategy of the proof is similar to that of Theorem 27. Therefore, we only highlight the main differences. Without loss of generality, we assume $m = 1$. Since we expect the transition to happen entirely inside the neck, we would like to use a rescaling for which the neck N_ε transforms in $N := [-1, 1]^3$. Given a local minimizer u_ε of the functional (150), the convenient rescaling that works is

$$v_\varepsilon(x, y, z) := u_\varepsilon(\varepsilon x, \varepsilon y, \eta z).$$

If we rescale in this way, the limiting domain becomes

$$\Omega_\infty = \Omega^\ell_\infty \cup N \cup \Omega^{\mathrm{r}}_\infty,$$

where $\Omega^\ell_\infty = \{x < -1\}$ and $\Omega^{\mathrm{r}}_\infty = \{x > 1\}$.

Theorem 28. Let $(u_\varepsilon)_\varepsilon \subset H^1(\Omega_\varepsilon)$ be an admissible family of local minimisers as in Definition 27. Assume $\varepsilon \approx \delta \gg \eta$. Then,

$$\lim_{\varepsilon\to 0} \frac{1}{\eta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) = \lim_{\varepsilon\to 0} \frac{1}{\eta}\mathcal{F}(u_\varepsilon, N_\varepsilon) = (\beta - \alpha)^2.$$

Moreover, for $\varepsilon > 0$ let $v_\varepsilon \colon \widetilde{\Omega}_\varepsilon \to \mathbb{R}$ be defined as

$$v_\varepsilon(x, y, z) := u_\varepsilon(\varepsilon x, \varepsilon y, \eta z).$$

Then, the following hold:

(i) $v_\varepsilon \chi_{\widetilde{\Omega}_\varepsilon^\ell} - \alpha \chi_{\widetilde{\Omega}_\varepsilon^\ell} \to 0$ in $L^6(\mathbb{R}^3)$ and $v_\varepsilon \chi_{\widetilde{\Omega}_\varepsilon^{\mathrm{r}}} - \beta \chi_{\widetilde{\Omega}_\varepsilon^{\mathrm{r}}} \to 0$ in $L^6(\mathbb{R}^3)$ as $\varepsilon \to 0$;

(ii) There exists a function $\hat{v} \in H^1(N)$ such that $v_\varepsilon \rightharpoonup \hat{v}$ weakly in $H^1(N)$ as $\varepsilon \to 0$;

(iii) It holds that $\hat{v}(x, y, z) = v(x)$, where $v \in H^1(-1, 1)$ is the unique minimiser of the variational problem

$$\min \left\{ \frac{1}{2} \int_{-1}^{1} |v'|^2 \,\mathrm{d}x : \; v \in H^1(-1, 1), \; v(-1) = \alpha, \; v(1) = \beta \right\}.$$

In particular, $v(x) = \dfrac{\beta - \alpha}{2} x + \dfrac{\alpha + \beta}{2}$.

Proof. In the same way as in Theorem 27, we can prove that

$$\sup_{\varepsilon > 0} \|\nabla v_\varepsilon\|_{L^2(N)}^2 \leqslant \frac{\varepsilon}{\delta \eta} F(u_\varepsilon, \Omega_\varepsilon) \leqslant C < \infty. \tag{177}$$

and

$$\sup_{\varepsilon > 0} \|v_\varepsilon\|_{L^2(N)} < C.$$

Therefore, by compactness there exists $v \in H^1(N)$ such that, up to a subsequence, $v_\varepsilon \rightharpoonup v$ in $H^1(N)$. The independence of the subsequence will follow from the fact that the limit is the *unique* solution to a variational problem.
Thus, we can write

$$\liminf_{\varepsilon \to 0} \frac{1}{\eta} \mathcal{F}(u_\varepsilon, N_\varepsilon) = \liminf_{\varepsilon \to 0} \left(\frac{1}{2} \int_N \left((\partial_x v_\varepsilon)^2 + (\partial_y v_\varepsilon)^2 + \frac{\varepsilon^2}{\eta^2} (\partial_z v_\varepsilon)^2 \right) \mathrm{d}\mathbf{x} \right.$$
$$\left. + \varepsilon^2 \eta \int_N W(v_\varepsilon) \,\mathrm{d}\mathbf{x} \right)$$

$$\begin{aligned}
&\geqslant \int_{[-1,1]^2} \left((\partial_x \hat{v})^2 + (\partial_y \hat{v})^2\right) \mathrm{d}x\mathrm{d}y \\
&\geqslant \min\left\{ \int_{[-1,1]^2} \left((\partial_x w)^2 + (\partial_y w)^2\right) \mathrm{d}x\mathrm{d}y : \ w \in H^1([-1,1]^2), \right. \\
&\qquad\qquad \left. w(\pm 1, y) = \hat{v}(\pm 1, y), \ \forall y \in [-1,1] \right\}, \qquad (178)
\end{aligned}$$

where in the previous to last step we used (177) and the fact that $\varepsilon/\eta \to +\infty$. Then we necessarily have that v does not depend on z. Namely, v_ε converges to a function $\hat{v} \in H^1(N)$, of the form $\hat{v}(x,y,z) = v(x,y)$, where $v \in H^1([-1,1]^2)$.
Now, would like to show that the boundary conditions $\hat{v}(\pm 1, y)$ are independent from y. This is done by acting similarly as in (173) and (174). Indeed by using the scale-invariant Poincaré inequality (173), we get

$$\int_{\Omega_\varepsilon^\ell} |v_\varepsilon - \bar{v}_\varepsilon|^6 \,\mathrm{d}\mathbf{x} \leqslant C \left(\frac{\eta}{\varepsilon}\right)^2 \left(\frac{1}{\eta}\mathcal{F}(v_\varepsilon, \Omega_\varepsilon)\right)^3.$$

From (177) and the fact that $\eta/\varepsilon \to 0$ as $\varepsilon \to 0$ we can conclude that $v = \alpha$ on Ω_∞^ℓ and $v = \beta$ on Ω_∞^r, independently on y. Therefore, from (178) we get

$$\begin{aligned}
&\liminf_{\varepsilon \to 0} \frac{1}{\eta} \mathcal{F}(u_\varepsilon, N_\varepsilon) \\
&\qquad \geqslant \min\left\{ \int_{[-1,1]^2} \left((\partial_x w)^2 + (\partial_y w)^2\right) \mathrm{d}x\mathrm{d}y : \ w \in H^1([-1,1]^2), \right. \\
&\qquad\qquad \left. w(-1, y) = \alpha, \ w(1,y) = \beta, \ \forall y \in [-1,1] \right\} \\
&\qquad \geqslant \min\left\{ \int_{[-1,1]^2} (\partial_x w)^2 \ \mathrm{d}x\mathrm{d}y : \ w \in H^1([-1,1]^2), \right. \\
&\qquad\qquad \left. w(-1, y) = \alpha, \ w(1,y) = \beta, \ \forall y \in [-1,1] \right\} \\
&\qquad = (\beta - \alpha)^2, \qquad (179)
\end{aligned}$$

where the last step follows by an explicit computation. On the other hand, denoting by ξ_ε the affine competitor in (152), we have that

$$\limsup_{\varepsilon\to 0} \frac{1}{\eta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) \leqslant \limsup_{\varepsilon\to 0} \frac{1}{\eta}\mathcal{F}(\xi_\varepsilon, N_\varepsilon) = (\beta-\alpha)^2. \tag{180}$$

Finally, using (179) (180), we get

$$\lim_{\varepsilon\to 0} \frac{1}{\eta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) = \lim_{\varepsilon\to 0} \frac{1}{\eta}\mathcal{F}(u_\varepsilon, N_\varepsilon) = (\beta-\alpha)^2.$$

This concludes the proof. □

Interlude: convergence of Neumann problems

In this short interlude, we recall a convergence result for solutions to elliptic problems with Neumann boundary conditions that will be crucial to carry out the analysis of the asymptotic behaviour of the rescaled profiles outside the neck. The result is the following. For a proof, we refer to [59, Proposition 6.2] (see also [22, 32]).

Theorem 29. Let $(\Omega_\varepsilon)_\varepsilon \subset \mathbb{R}^3$ be a sequence of open sets such that, as $\varepsilon \to 0$,

$$\chi_{\Omega_\varepsilon} \to \chi_{\Omega_\infty} \quad \text{in } L^1(\mathbb{R}^3) \qquad \text{and} \qquad \mathbb{R}^3\setminus\Omega_\varepsilon \xrightarrow{\mathrm{H}} \mathbb{R}^3\setminus\Omega_\infty \quad \text{locally},$$

for some open set Ω_∞. Let $p > 2$, and let $(f_\varepsilon)_\varepsilon \subset L^p_{\mathrm{loc}}(\mathbb{R}^3)$ be such that, as $\varepsilon \to 0$,

$$f_\varepsilon\chi_{\Omega_\varepsilon} \to f_\infty\chi_{\Omega_\infty} \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^3),$$

for some $f_\infty \in L^p_{\mathrm{loc}}(\mathbb{R}^3)$. Let $(u_\varepsilon)_\varepsilon \subset W^{2,p}(\Omega_\varepsilon)$ be the weak solution to

$$\begin{cases} \triangle u_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Assume that $(u_\varepsilon\chi_{\Omega_\varepsilon})_\varepsilon$ is locally equi-bounded in $L^\infty(\Omega_\varepsilon)$. Then, up to a subsequence,

$$v_\varepsilon\chi_{\widetilde{\Omega}_\varepsilon} \to \hat{v}\chi_{\Omega_\infty} \quad \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^3) \text{ for all } q \in [1,\infty),$$

as $\varepsilon \to 0$, and

$$\nabla v_\varepsilon \chi_{\Omega_\varepsilon} \to \nabla \hat{v} \chi_{\Omega_\infty} \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3),$$

as $\varepsilon \to 0$, where $\hat{v} \in W^{2,p}(\Omega_\infty)$ is the weak solution to

$$\begin{cases} \triangle \hat{u} = f_\infty & \text{in } \Omega_\infty, \\ \partial_\nu \hat{u} = 0 & \text{on } \partial\Omega_\infty. \end{cases}$$

Moreover, $u_\varepsilon \to \hat{u}$ in $W^{2,p}_{\text{loc}}(\Omega_\infty)$, as $\varepsilon \to 0$.

Remark 18. The reason why in the above result the convergence of the complements of the open sets Ω_ε is required, and the fact that the limiting set has to be open, is in order to ensure that at each point of the limiting set there is only one side where the limiting set is. For instance, we want to avoid situations of the type

$$\Omega_\varepsilon := \{(\cos\theta, \sin\theta) : \theta \in (0, 2\pi - \varepsilon)\},$$

or of the type

$$\Omega_\varepsilon := (0,1)^2 \cup ([1,2) \times (-\varepsilon, \varepsilon)).$$

In both cases, the limiting set has part of the topological boundary that creates troubles in defining the limiting PDE.

3.3.3 Window thick regime

Here we consider the scaling of the energy in the window thick regime $\delta \gg \eta \gg \varepsilon$, namely where

$$\lim_{\varepsilon \to 0} \frac{\eta}{\delta} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{\eta}{\varepsilon} = +\infty.$$

Since the ellipsoidal competitor outside the neck provides an energy whose order is lower than the energy of the affine competitor in the neck, we expect the transition happening outside the neck. If u_ε is a local minimiser of the functional (150), the convenient rescaling that allows us to both see a nice limiting space, and to use the scale-invariant Poincaré inequality (173) is

$$v_\varepsilon(x, y, z) := u_\varepsilon(\delta x, \delta y, \delta z).$$

Using this rescaling, the limiting domain becomes

$$\widetilde{\Omega}_\infty = \{x < 0\} \cup (\{0\} \times [-1,1] \times \{0\}) \cup \{x > 0\}.$$

However, note that

$$\mathbb{R}^3 \setminus \widetilde{\Omega}_\varepsilon \xrightarrow{\mathrm{H}} \mathbb{R}^3 \setminus \Omega_\infty \,,$$

where

$$\widetilde{\Omega}_\varepsilon := \frac{1}{\delta}\Omega_\varepsilon \,, \qquad \Omega_\infty := \{x < 0\} \cup \{x > 0\}.$$

We are now in position to prove the main result of this section.

Theorem 30. Let $(u_\varepsilon)_\varepsilon \subset H^1(\Omega_\varepsilon)$ be an admissible family of local minimisers as in Definition 27. Assume that $\delta \gg \eta \gg \varepsilon$. Then,

$$\lim_{\varepsilon\to 0} \frac{|\ln(\eta/\delta)|}{\delta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) = \lim_{\varepsilon\to 0} \frac{|\ln(\eta/\delta)|}{\delta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon \setminus N_\varepsilon) = \pi(\beta-\alpha)^2.$$

Moreover, for $\varepsilon > 0$ let $v_\varepsilon \colon \widetilde{\Omega}_\varepsilon \to \mathbb{R}$ be defined as

$$v_\varepsilon(x,y,z) := u_\varepsilon(\delta x, \delta y, \delta z).$$

Then, there exists $R > 0$ such that the following statements hold:

(i) $v_\varepsilon \to \frac{\alpha+\beta}{2}$ uniformly on B_R, as $\varepsilon \to 0$;

(ii) There exists $\hat{v} \in H^1_{\mathrm{loc}}(\Omega_\infty)$ such that $v_\varepsilon \chi_{\widetilde{\Omega}_\varepsilon} \to \hat{v}\chi_{\Omega_\infty}$ strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$, as $\varepsilon \to 0$;

(iii) The function $\hat{v}$ is the unique minimiser of the variational problem

$$\min\left\{\frac{1}{2}\int_{\Omega_\infty} |\nabla v|^2 \,\mathrm{d}x : v \in \mathcal{A}\right\},$$

where

$$\mathcal{A} := \Big\{v \in H^1_{\mathrm{loc}}(\Omega_\infty),\ v - \alpha\chi_{\Omega^\ell_\infty} - \beta\chi_{\Omega^{\mathrm{r}}_\infty} \in L^6(\Omega_\infty), \\ v = \frac{\alpha+\beta}{2} \text{ on } B_R \cap \Omega_\infty\Big\}, \quad (181)$$

and $\Omega^\ell_\infty := \{x < 0\}$, and $\Omega^{\mathrm{r}}_\infty := \{x > 0\}$.

Proof. Step 1: lower bound of the energy outside the neck. We claim that there exist $m_1, m_2 \in \mathbb{R}$ such that

$$\liminf_{\varepsilon \to 0} \frac{|\ln(\eta/\delta)|}{\delta} \mathcal{F}(u_\varepsilon, \Omega_\varepsilon) \geqslant 2\pi\big[(m_1 - \alpha)^2 + (m_2 - \beta)^2\big]. \tag{182}$$

The idea is to bound from below the energy of u_ε with that of an harmonic function defined in the region between two suitable ellipsoids, one on the left and one of the right of the neck. For $a, s > 0$, consider the ellipsoid

$$E(a,s) \coloneqq \{\Psi(\mu,\nu,\varphi) : \mu < s\},$$

where Ψ denotes the prolate ellipsoidal coordinates defined in (154). We define

$$\begin{aligned} E^\ell_\varepsilon(a,s) &\coloneqq (E(a,s) \cap \{x < 0\}) + (0,0,-\varepsilon), \\ E^{\mathrm{r}}_\varepsilon(a,s) &\coloneqq (E(a,s) \cap \{x > 0\}) + (0,0,\varepsilon). \end{aligned}$$

We claim that it is possible to find $m_1, m_2 \in \mathbb{R}$ satisfying the following property. Fix a and M such that $E(a, M) \subset B_{2R}$, and define

$$a_\varepsilon \coloneqq \delta a, \qquad M_\varepsilon \coloneqq \delta M.$$

Then, for any $\gamma, \mu > 0$, there exists $\varepsilon_0 > 0$ such that

$$m_1 - \mu \leqslant u_\varepsilon \leqslant m_1 + \mu \qquad \text{on } \partial E^\ell(a_\varepsilon, M_\varepsilon), \tag{183}$$

$$m_2 - \mu \leqslant u_\varepsilon \leqslant m_2 + \mu \qquad \text{on } \partial E^{\mathrm{r}}(a_\varepsilon, M_\varepsilon), \tag{184}$$

$$\alpha - \gamma \leqslant u_\epsilon \leqslant \alpha + \gamma \qquad \text{on } \partial E^\ell(a_\varepsilon, \rho), \tag{185}$$

$$\beta - \gamma \leqslant u_\varepsilon \leqslant \beta + \gamma \qquad \text{on } \partial E^{\mathrm{r}}(a_\varepsilon, \rho), \tag{186}$$

and

$$\begin{aligned} N_\varepsilon \cap \{x = -\varepsilon\} &\subset E^\ell(a_\varepsilon, M_\varepsilon) \cap \{x = -\varepsilon\}, \\ N_\varepsilon \cap \{x = \varepsilon\} &\subset E^{\mathrm{r}}(a_\varepsilon, M_\varepsilon) \cap \{x = \varepsilon\}. \end{aligned} \tag{187}$$

for all $\varepsilon < \varepsilon_0$. The claims (183) and (184) will be proved in Step 1.1, while (185) and (186) in Step 1.2.

We now show how to conclude. Fix $\mu, \gamma > 0$. Then, for $\varepsilon < \varepsilon_0$, we have that

$$
\begin{aligned}
\mathcal{F}(u_\varepsilon, \Omega_\varepsilon \setminus N_\varepsilon) \geqslant \frac{1}{2} \int_{E^\ell(a_\varepsilon,\rho)\setminus E^\ell(a_\varepsilon,M_\varepsilon)} & |\nabla u_\varepsilon|^2 \, \mathrm{d}\mathbf{x} \\
+ \frac{1}{2} \int_{E^{\mathrm{r}}(a_\varepsilon,\rho)\setminus E^{\mathrm{r}}(a_\varepsilon,M_\varepsilon)} & |\nabla u_\varepsilon|^2 \, \mathrm{d}\mathbf{x} \\
\geqslant \inf \Big\{ \frac{1}{2} \int_{E^\ell(a_\varepsilon,\rho)\setminus E^\ell(a_\varepsilon,M_\varepsilon)} & |\nabla v|^2 \, \mathrm{d}\mathbf{x} : \ v \in H^1(E^\ell(a_\varepsilon,\rho) \setminus E^\ell(a_\varepsilon, M_\varepsilon)), \\
& v \leqslant m_1 + \mu \text{ on } \partial E^\ell(a_\varepsilon, M_\varepsilon), \ v \geqslant \alpha - \gamma \text{ on } \partial E^\ell(a_\varepsilon, \rho) \Big\} \\
+ \inf \Big\{ \frac{1}{2} \int_{E^{\mathrm{r}}(a_\varepsilon,\rho)\setminus E^{\mathrm{r}}(a_\varepsilon,M_\varepsilon)} & |\nabla v|^2 \, \mathrm{d}\mathbf{x} : \ v \in H^1(E^{\mathrm{r}}(a_\varepsilon,\rho) \setminus E^{\mathrm{r}}(a_\varepsilon, M_\varepsilon)), \\
& v \leqslant m_2 + \mu \text{ on } \partial E^{\mathrm{r}}(a_\varepsilon, M_\varepsilon), \ v \geqslant \beta - \gamma \text{ on } \partial E^{\mathrm{r}}(a_\varepsilon, \rho) \Big\} \\
= \inf \Big\{ \frac{1}{2} \int_{E^\ell(a_\varepsilon,\rho)\setminus E^\ell(a_\varepsilon,M_\varepsilon)} & |\nabla v|^2 \, \mathrm{d}\mathbf{x} : \ v \in H^1(E(a_\varepsilon,\rho)^\ell \setminus E^\ell(a_\varepsilon, M_\varepsilon)), \\
& v = m_1 + \mu \text{ on } \partial E^\ell(a_\varepsilon, M_\varepsilon), \ v = \alpha - \gamma \text{ on } \partial E^\ell(a_\varepsilon, \rho) \Big\} \\
+ \inf \Big\{ \frac{1}{2} \int_{E^{\mathrm{r}}(a_\varepsilon,\rho)\setminus E^{\mathrm{r}}(a_\varepsilon,M_\varepsilon)} & |\nabla v|^2 \, \mathrm{d}\mathbf{x} : \ v \in H^1(E(a_\varepsilon,\rho)^{\mathrm{r}} \setminus E^{\mathrm{r}}(a_\varepsilon, M_\varepsilon)), \\
& v = m_2 + \mu \text{ on } \partial E^{\mathrm{r}}(a_\varepsilon, M_\varepsilon), \ v = \beta - \gamma \text{ on } \partial E^{\mathrm{r}}(a_\varepsilon, \rho) \Big\} \\
=: L_\varepsilon + R_\varepsilon. &
\end{aligned}
\tag{188}
$$

We now want to compute L_ε and R_ε. We show the argument for L_ε. The result for R_ε will follow by using the same reasoning. Arguing as in Section 3.2, we get that the solution to the problem defining L_ε is given, in prolate coordinates, by

$$
w(\mu) = c \ln |k \tanh(\mu/2)| ,
$$

where (see (158))

$$k = \coth(M_\varepsilon), \qquad c = \frac{m_1 + \mu - \alpha + \gamma}{\ln\left(\dfrac{\tanh(\rho)}{\tanh(M_\varepsilon)}\right)}.$$

In particular, (see (159)), we get that

$$L_\varepsilon = \frac{\pi a_\varepsilon (m_1 + \mu - \alpha + \gamma)^2}{\ln\left(\dfrac{\tanh(\rho)}{\tanh(M_\varepsilon)}\right)}. \tag{189}$$

Now, we want to understand the asymptotic behaviour of L_ε. First, we want to compute the asymptotic behaviour of the denominator on the right-hand side of (189). Let M_ε, and a_ε be such that

$$a_\varepsilon \sinh M_\varepsilon = 2\eta, \qquad a_\varepsilon \cosh M_\varepsilon = 2\delta.$$

In such a way, we have that (187) holds. Note that on $\partial E^\ell(a_\varepsilon, M_\varepsilon) \cap \{x = -\varepsilon\}$ we have

$$\frac{y^2}{a^2 \cosh^2(M\delta)} + \frac{z^2}{a^2 \sinh^2(M\delta)} = 1,$$

which yields that

$$\tanh M_\varepsilon = \frac{\eta}{\delta}, \qquad a_\varepsilon^2 = 4\delta^2 - 4\eta^2 \approx 4\delta^2. \tag{190}$$

As consequence, we get the following asymptotic estimate

$$\frac{1}{a_\varepsilon} \ln\left(\frac{\tanh(\rho)}{\tanh(M_{\delta\eta})}\right) = \frac{1}{a_\varepsilon}\big(\ln\tanh\rho - \ln\tanh M_{\delta\eta}\big) \approx \frac{|\ln(\delta/\eta)|}{2\delta}. \tag{191}$$

Therefore, from (189), (191), since $\gamma_\varepsilon^\ell \to 0$ and by arbitrariness of γ, we get

$$\liminf_{\varepsilon\to 0} \frac{|\ln(\delta/\eta)|}{\delta} L_\varepsilon \geqslant 2\pi(m_1 - \alpha)^2.$$

In a similar way, we obtain that

$$\liminf_{\varepsilon\to 0} \frac{|\ln(\delta/\eta)|}{\delta} R_\varepsilon \geqslant 2\pi(m_2 - \beta)^2.$$

This proves the claim.

Step 1.1: boundary conditions on the internal ellipsoid. Here, we want to prove the validity of (183) and of (184). We want to understand the limiting behaviour of v_ε. The idea is to obtain such information by looking at the limit of the PDE satisfied by the limit of the sequence $(v_\varepsilon)_\varepsilon$. First, we notice that since u_ε is a critical point of the energy F_ε, we have that u_ε satisfies the Euler-Lagrange equation

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla \varphi \, \mathrm{d}\mathbf{x} - \int_{\Omega_\varepsilon} W'(u_\varepsilon)\varphi \, \mathrm{d}\mathbf{x} = 0,$$

for all $\varphi \in H^1(\Omega_\varepsilon)$. We claim that

$$\int_{\widetilde{\Omega}_\varepsilon} \nabla v_\varepsilon \cdot \nabla \psi \, \mathrm{d}\mathbf{x} = \delta^2 \int_{\widetilde{\Omega}_\varepsilon} W'(v_\varepsilon)\psi \, \mathrm{d}\mathbf{x}, \tag{192}$$

for every $\psi \in H^1(\widetilde{\Omega}_\varepsilon)$. Indeed, fix $\varphi \in H^1(\Omega_\varepsilon)$. Using the change of variable $(\delta x', \delta y', \delta z') = (x, y, z)$, we get

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla \varphi \, \mathrm{d}\mathbf{x} = \delta^3 \int_{\widetilde{\Omega}_\varepsilon} \frac{1}{\delta^2} \nabla v_\varepsilon \cdot \nabla \psi \, \mathrm{d}\mathbf{x}', \tag{193}$$

where $\psi(x, y, z) \coloneqq \varphi(\delta x, \delta y, \delta z)$. Moreover,

$$\int_{\Omega_\varepsilon} W'(u_\varepsilon)\varphi \, \mathrm{d}\mathbf{x} = \delta^3 \int_{\widetilde{\Omega}_\varepsilon} W'(v_\varepsilon)\psi \, \mathrm{d}\mathbf{x}'.$$

This proves that v_ε is a weak solution to

$$\begin{cases} \Delta v_\varepsilon = \delta^2 W'(v_\varepsilon) & \text{in } \widetilde{\Omega}_\varepsilon \,, \\ \dfrac{\partial v_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\widetilde{\Omega}_\varepsilon \,, \end{cases}$$

as desired.

Now we want to obtain the limiting equation. Since, by assumption, $(u_\varepsilon)_\varepsilon$ is uniformly bounded in $L^\infty(\Omega_\varepsilon)$, the sequence $f_\varepsilon \coloneqq \delta^2 W'(v_\varepsilon)\chi_{\widetilde{\Omega}_\varepsilon}$ converges strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^3)$ to $f \coloneqq 0$, for all $p \geqslant 1$. Moreover,

$$\mathbb{R}^3 \setminus \widetilde{\Omega}_\varepsilon \xrightarrow{\mathrm{H}} \mathbb{R}^3 \setminus \Omega_\infty \,, \qquad \text{as } \varepsilon \to 0.$$

Thus, using Theorem 29, we get that there exists $\hat{v} \in W^{2,p}(\Omega_\infty)$, such that, up to a subsequence, it holds

$$v_\varepsilon \chi_{\widetilde{\Omega}_\varepsilon} \to \hat{v}\chi_{\Omega_\infty} \quad \text{in } L^q_{\text{loc}}(\mathbb{R}^3) \text{ for all } q \in [1,\infty),$$

and

$$\nabla v_\varepsilon \chi_{\widetilde{\Omega}_\varepsilon} \to \nabla \hat{v}\chi_{\Omega_\infty} \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^3;\mathbb{R}^3),$$

as $\varepsilon \to 0$. This proves (ii). In particular, if we fix $R > 1$, we get that

$$\int_{\Omega_\infty \cap B_{2R}} |\nabla \hat{v}|^2 \, \mathrm{d}\mathbf{x} = \lim_{\varepsilon \to 0} \int_{\widetilde{\Omega}_\varepsilon \cap B_{2R}} |\nabla v_\varepsilon|^2 \, \mathrm{d}\mathbf{x} = 0, \tag{194}$$

where we used (192) with v_ε as a test function, together with the fact that $\|W'(v_\varepsilon)v_\varepsilon\|_{L^\infty(\widetilde{\Omega}_\varepsilon}$ is uniformly bounded in ε. Therefore, recalling that Ω_∞ has two disjoint connected components, we get that there exist $m_1, m_2 \in \mathbb{R}$ such that

$$v_\varepsilon \to m_1 \quad \text{locally uniformly in } \Omega_\infty \cap B_{2R} \cap \{x < 0\},$$

and

$$v_\varepsilon \to m_2 \quad \text{locally uniformly in } \Omega_\infty \cap B_{2R} \cap \{x > 0\},$$

as $\varepsilon \to 0$, Moreover, for any choice of a and M such that $E(a,M) \subset B_{2R}$, we get that

$$v_\varepsilon \to m_1 \quad \text{locally uniformly in } E^\ell(a,M)$$

and

$$v_\varepsilon \to m_2 \quad \text{locally uniformly in } E^{\mathrm{r}}(a,M),$$

as $\varepsilon \to 0$, Going back to the original coordinates gives us the desired result. *Step 1.2: boundary conditions on the external ellipsoid.* Note that by assumption, we get that u_ε is a weak solution to

$$\begin{cases} \Delta u_\varepsilon = W'(u_\varepsilon) & \text{in } \widetilde{\Omega}_\varepsilon\,, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\widetilde{\Omega}_\varepsilon\,. \end{cases}$$

Moreover, $W'(u_\varepsilon)\chi_{\Omega_\varepsilon} \in L^p_{\text{loc}}(\mathbb{R}^3)$ for all $p \in [1,\infty]$, since by assumption, $(u_\varepsilon)_\varepsilon$ is uniformly bounded in $L^\infty(\Omega_\varepsilon)$. Therefore, arguing as in the previous step, and using Sobolev embeddings, we get that $u_\varepsilon \chi_{\Omega^\ell_\varepsilon}$ converges uniformly to α and $u_\varepsilon \chi_{\Omega_\varepsilon^{\mathrm{r}}}$ converges uniformly to β. In particular, let $r_0 > 0$ be given by assumption (H3). Fix $0 < \rho < r_0$ and define

$$E^\ell_{\varepsilon,\rho} := E_\rho \cap \{x < -\varepsilon\} \qquad \text{and} \qquad E^r_{\varepsilon,\rho} := E_\rho \cap \{x > \varepsilon\}.$$

Then, from the above argument, we get that

$$\alpha - \gamma^\ell_\varepsilon < u_\varepsilon < \alpha + \gamma^\ell_\varepsilon \quad \text{on } \partial E^\ell_{\varepsilon,\rho}\,,$$
$$\beta - \gamma^{\mathrm{r}}_\varepsilon < u_\varepsilon < \beta + \gamma^{\mathrm{r}}_\varepsilon \quad \text{on } \partial E^{\mathrm{r}}_{\varepsilon,\rho}\,,$$

as $\varepsilon \to 0$, where

$$\gamma^\ell_\varepsilon := ||u_\varepsilon - \alpha||_{L^\infty(E^\ell_{\varepsilon,\rho})} \to 0 \qquad \text{and} \qquad \gamma^r_\varepsilon := ||u_\varepsilon - \beta||_{L^\infty(E^r_{\varepsilon,\rho})} \to 0.$$

Step 2: energy estimate outside the neck. Consider the function $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$f(s,t) := 2\pi[(s-\alpha)^2 + (t-\beta)^2].$$

Then, the minimum of f over the set $\alpha \leqslant s \leqslant t \leqslant \beta$ is defined as

$$f\left(\frac{\beta+\alpha}{2}, \frac{\beta+\alpha}{2}\right) = \pi(\beta-\alpha)^2.$$

Thus, from (182), we obtain that

$$\liminf_{\varepsilon \to 0} \frac{|\ln(\eta/\delta)|}{\delta} \mathcal{F}(u_\varepsilon, \Omega_\varepsilon \setminus N_\varepsilon) \geqslant \pi(\beta-\alpha)^2.$$

Let ξ_ε be the function defined in (156). Then, by (158), we get that

$$\limsup_{\varepsilon \to 0} \frac{|\ln(\eta/\delta)|}{\delta} \mathcal{F}(\xi_\varepsilon, \Omega_\varepsilon \setminus N_\varepsilon) = \pi(\beta-\alpha)^2.$$

Thus, from we obtain that

$$\lim_{\varepsilon \to 0} \frac{|\ln(\delta/\eta)|}{\delta} \mathcal{F}(u_\varepsilon, \Omega_\varepsilon \setminus N_\varepsilon) = \pi(\beta-\alpha)^2.$$

In particular, this yields that $m_1 = m_2$, which proves (i). Moreover, by noticing that all the inequalities in (188) are equalities, we get that

$$\lim_{\varepsilon \to 0} \frac{|\ln(\eta/\delta)|}{\delta} \mathcal{F}(u_\varepsilon, \Omega_\varepsilon) = \lim_{\varepsilon \to 0} \frac{|\ln(\eta/\delta)|}{\delta} \mathcal{F}(u_\varepsilon, \Omega_\varepsilon \setminus N_\varepsilon).$$

Step 3: limiting problem. First, we prove that $\hat{v}$ is an admissible competitor for the problem in (iii). From (i), we know that

$$\hat{v} = \frac{\alpha+\beta}{2}, \qquad \text{on } B_R \cap \Omega_\infty .$$

We now prove that it satisfies also the boundary conditions at infinity. Using the scale-invariant Poincaré inequality (173), we get that

$$\|v_\varepsilon - \overline{v}_\varepsilon\|_{L^6(\widetilde{\Omega}_\varepsilon)} \leqslant C\|\nabla v_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)} ,$$

which, together with the fact that $\overline{v}_\varepsilon \to \alpha\chi_{\Omega^\ell_\infty} + \beta\chi_{\Omega^{\mathrm{r}}_\infty} \in L^6(\Omega_\infty)$, as $\varepsilon \to 0$, yields that $\hat{v}$ is an admissible competitor for the problem in (iii).

Finally, we prove that $\hat{v}$ solves the minimisation problem in (iii). The argument is similar to that of Step 3 of the proof of [52, Theorem 4.1]. Fix $M > |\alpha|, |\beta|$. We can assume, without loss of generality, that every function $\varphi \in \mathcal{A}$ is such that $\|\varphi\|_{L^\infty(\Omega_\infty)} \leqslant M$. Indeed, given $\varphi \in \mathcal{A}$, by considering the truncation $\widetilde{\varphi} := (\varphi \wedge M) \vee (-M)$ we get that $\widetilde{\varphi} \in \mathcal{A}$ and

$$\frac{1}{2}\int_{\Omega_\infty} |\nabla\widetilde{\varphi}|^2 \,\mathrm{d}\mathbf{x} \leqslant \frac{1}{2}\int_{\Omega_\infty} |\nabla\varphi|^2 \,\mathrm{d}\mathbf{x}.$$

Thus, let us take $\varphi \in \mathcal{A}$ with $\|\varphi\|_{L^\infty(\Omega_\infty)} \leqslant M$. Define the function $\varphi_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ as

$$\varphi_\varepsilon(x,y,z) := \varphi\left(\frac{x}{\delta}, \frac{y}{\delta}, \frac{z}{\delta}\right).$$

Then, there exist constants $C, \widetilde{C} > 0$, such that, for all $\varepsilon > 0$ it holds

$$\|\varphi_\varepsilon - u_{0,\varepsilon}\|_{L^2(\Omega_\varepsilon)} \leqslant C\|\varphi_\varepsilon - u_{0,\varepsilon}\|_{L^6(\Omega_\varepsilon)} = \widetilde{C}\delta^{\frac{1}{2}}.$$

Therefore, for ε sufficiently small, we get that φ_ε is an admissible competitor for the minimisation problem solved by u_ε. Thus,

$$\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) \leqslant \mathcal{F}(\varphi_\varepsilon, \Omega_\varepsilon). \tag{195}$$

Note that

$$\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) = \frac{\delta}{2}\int_{\widetilde{\Omega}_\varepsilon} |\nabla v_\varepsilon|^2 \,\mathrm{d}\mathbf{x} + \int_{\Omega_\varepsilon} W(u_\varepsilon)\,\mathrm{d}\mathbf{x},$$

and (recall that $\widetilde{\Omega}_\varepsilon \subset \Omega_\infty$)

$$\mathcal{F}(\varphi_\varepsilon, \Omega_\varepsilon) = \frac{\delta}{2} \int_{\widetilde{\Omega}_\varepsilon} |\nabla \varphi|^2 \, \mathrm{d}\mathbf{x} + \int_{\Omega_\varepsilon} W(\varphi_\varepsilon) \, \mathrm{d}\mathbf{x}.$$

Thus, taking the liminf on both sides of (195), and using the fact that $\varphi_\varepsilon, u_\varepsilon$ converges in L^2 to zeros of W, we get

$$\frac{1}{2} \int_{\Omega_\infty} |\nabla \hat{v}|^2 \, \mathrm{d}\mathbf{x} \leqslant \liminf_{\varepsilon \to 0} \mathcal{F}(u_\varepsilon, \Omega_\varepsilon) \leqslant \liminf_{\varepsilon \to 0} \mathcal{F}(\varphi_\varepsilon, \Omega_\varepsilon) = \frac{1}{2} \int_{\Omega_\infty} |\nabla \varphi|^2 \, \mathrm{d}\mathbf{x},$$

proving that that $\hat{v}$ solves the claimed minimisation problem. This concludes the proof. □

Remark 19. We highlight that, from Step 1.1 of the proof (see (194)), it follows that the transition happens outside any ball of radius δ around the neck.

3.3.4 Narrow thick regime

Here we consider the scaling of the energy in the narrow thick regime $\delta \gg \varepsilon \approx \eta$, namely when

$$\lim_{\varepsilon \to 0} \frac{\eta}{\delta} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{\eta}{\varepsilon} = l \in (0, +\infty).$$

Since the ellipsoidal competitor outside the neck provides an energy whose order is lower than the energy of the affine competitor in the neck, we expect the transition happening outside the neck. Without loss of generality, we assume $l = 1$. Denoting by u_ε a local minimiser of the functional (150), the convenient rescaling that allows us to both see a nice limiting space, and to use the rescaled Poincaré inequality is

$$v_\varepsilon(x, y, z) := u_\varepsilon(\delta x, \delta y, \delta z).$$

Using this rescaling, the limiting domain becomes

$$\widetilde{\Omega}_\infty = \{x < 0\} \cup (\{0\} \times [-1, 1] \times \{0\}) \cup \{x > 0\}.$$

However, note that, as $\varepsilon \to 0$,

$$\mathbb{R}^3 \setminus \widetilde{\Omega}_\varepsilon \xrightarrow{\mathrm{H}} \mathbb{R}^3 \setminus \Omega_\infty \,,$$

where

$$\widetilde{\Omega}_\varepsilon := \frac{1}{\delta}\Omega_\varepsilon, \qquad \Omega_\infty = \{x < 0\} \cup \{x > 0\}.$$

The same argument used in the proof of Theorem 30 yields the following result, therefore we omit the proof.

Theorem 31. Let $(u_\varepsilon)_\varepsilon \subset H^1(\Omega_\varepsilon)$ be an admissible family of local minimisers as in Definition 27. Assume that $\delta \gg \varepsilon \approx \eta$. Then,

$$\lim_{\varepsilon\to 0} \frac{|\ln(\eta/\delta)|}{\delta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) = \lim_{\varepsilon\to 0} \frac{|\ln(\eta/\delta)|}{\delta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon \setminus N_\varepsilon) = \pi(\beta-\alpha)^2.$$

Moreover, for $\varepsilon > 0$ let $v_\varepsilon \colon \widetilde{\Omega}_\varepsilon \to \mathbb{R}$ be defined as

$$v_\varepsilon(x, y, z) := u_\varepsilon(\delta x, \delta y, \delta z).$$

Then, there exists $R > 0$ such that the following hold:

(i) $v_\varepsilon \to \frac{\alpha+\beta}{2}$ uniformly on B_R, as $\varepsilon \to 0$;

(ii) There exists $\hat{v} \in H^1_{\mathrm{loc}}(\Omega_\infty)$ such that $v_\varepsilon\chi_{\widetilde{\Omega}_\varepsilon} \to \hat{v}\chi_{\Omega_\infty}$ strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$, as $\varepsilon \to 0$;

(iii) The function $\hat{v}$ is the unique minimiser of the variational problem

$$\min\left\{\frac{1}{2}\int_{\Omega_\infty} |\nabla v|^2 \,\mathrm{d}x : v \in \mathcal{A}\right\},$$

where $\mathcal{A}$ is defined in (181).

3.3.5 Letter-box regime

We now consider the regime in which $\delta \gg \varepsilon \gg \eta$, namely when

$$\lim_{\varepsilon\to 0} \frac{\delta}{\varepsilon} = +\infty \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{\eta}{\delta} = 0.$$

In this regime, the transition will happen all inside, all outside, or everywhere, depending on the parameter

$$\ell := \lim_{\varepsilon\to 0} \frac{\delta\eta}{\varepsilon}\frac{|\ln \eta/\delta|}{\delta}. \tag{196}$$

In particular, we will prove that if $\ell \in (0, +\infty)$, then the transition happens everywhere, while if $\ell = 0$ then transition occurs all inside and if $\ell = +\infty$ all outside.

Critical letter box regime

This sub-regime, corresponds to the case $\ell \in (0, +\infty)$. We capture the transition in the bulk by applying a similar argument to the one in Theorem 30, in which around the neck, the rescaled profile v_ε converges to the average of the two phases, namely $(\alpha + \beta)/2$. In the critical letter box regime we have instead that the rescaled profile converges to different constants m_1 on $\{x < 0\}$ and m_2 on $\{x > 0\}$. Once we have this information, we can understand how to describe the transition in the neck, by taking into account the fact that we know the boundary conditions. Then, a similar technique used in Theorem 27 applies.

Theorem 32. Let $(u_\varepsilon)_\varepsilon \subset H^1(\Omega_\varepsilon)$ be an admissible family of local minimisers as in Definition 27. Assume that $\delta \gg \varepsilon \gg \eta$, and that

$$\ell := \lim_{\varepsilon \to 0} \frac{\delta\eta}{\varepsilon} \frac{|\ln(\eta/\delta)|}{\delta} \in (0, +\infty).$$

Then,

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta\eta} \mathcal{F}(u_\varepsilon, \Omega_\varepsilon) = \frac{|\ln(\eta/\delta)|}{\delta\ell} \mathcal{F}(u_\varepsilon, \Omega_\varepsilon) = \frac{\pi(\beta - \alpha)^2}{\pi + \ell}. \tag{197}$$

In particular

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta\eta} \mathcal{F}(u_\varepsilon, N_\varepsilon) = \frac{\pi^2(\beta - \alpha)^2}{(\pi + \ell)^2},$$

and

$$\lim_{\varepsilon \to 0} \frac{|\ln(\eta/\delta)|}{\delta} \mathcal{F}(u_\varepsilon, \Omega_\varepsilon \setminus N_\varepsilon) = \frac{\pi\ell^2(\beta - \alpha)^2}{(\pi + \ell)^2}.$$

Moreover:

(*i*) Consider the rescaled profile $w_\varepsilon : \overline{\Omega}_\varepsilon \to \mathbb{R}$ defined as

$$w_\varepsilon(x, y, z) := u_\varepsilon(\varepsilon x, \delta y, \eta z), \tag{198}$$

where $\overline{\Omega}_\varepsilon$ is the rescaled domain of Ω_ε. Let us assume that $w_\varepsilon \rightharpoonup \hat{w}$ in $H^1(N)$, for some $\hat{w} \in H^1(N)$. Then, $\hat{w}(x, y, z) = w(x)$ where $w \in H^1([-1, 1])$ is the unique minimiser of the variational problem

$$\min \left\{ \frac{1}{2} \int_{-1}^{1} |v'|^2 \, dx : \ v \in H^1(-1, 1), \ v(-1) = \frac{\pi\alpha + \left(\frac{\alpha + \beta}{2}\right)\ell}{\pi + \ell},\right.$$

$$v(1) = \frac{\pi\beta + \left(\frac{\alpha+\beta}{2}\right)\ell}{\pi+\ell}\Bigg\};$$

(*ii*) Let

$$\widetilde{\Omega}_\varepsilon := \frac{1}{\delta}\Omega_\varepsilon, \qquad \Omega_\infty := \{x<0\} \cup \{x>0\}.$$

Define the rescaled profile $v_\varepsilon \in H^1(\widetilde{\Omega}_\varepsilon)$ as

$$v_\varepsilon(x,y,z) := u_\varepsilon(\delta x, \delta y, \delta z). \tag{199}$$

Then, there esists $\hat{v} \in H^1(\Omega_\infty)$ such that $v_\varepsilon \chi_{\widetilde{\Omega}_\varepsilon} \to \hat{v}\chi_{\Omega_\infty}$ strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$ as $\varepsilon \to 0$, where $\hat{v}$ is the solution of the minimisation problem

$$\min\left\{\frac{1}{2}\int_{\Omega_\infty} |\nabla v|^2 \,\mathrm{d}x : v \in \mathcal{B}\right\},$$

where,

$$\begin{aligned}\mathcal{B} := \Bigg\{ & v \in H^1(\Omega_\infty),\ v - \alpha\chi_{\{x<0\}} - \beta\chi_{\{x>0\}} \in L^6(\Omega_\infty),\\ & v = \frac{\pi\alpha + \left(\frac{\alpha+\beta}{2}\right)\ell}{\pi+\ell} \text{ on } B_M \cap \{x<0\},\\ & v = \frac{\pi\beta + \left(\frac{\alpha+\beta}{2}\right)\ell}{\pi+\ell} \text{ on } B_M \cap \{x>0\}\Bigg\},\end{aligned}$$

for some $M \geqslant 2$.

Proof. First of all, note that, using (169) and (196), for any given constants $A, B \in \mathbb{R}$ with $A \leqslant B$, we have

$$\begin{aligned}\frac{|\ln(\eta/\delta)|}{\delta}\mathcal{F}(u,\Omega_\varepsilon) \leqslant \pi\big[(A-\alpha)^2 + (B-\beta)^2\big]&\\ + \frac{\delta\eta}{\varepsilon}\frac{|\ln(\eta/\delta)|}{\delta}(B-A)^2.&\end{aligned} \tag{200}$$

Therefore, for every $\lambda > 0$ there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ we have

$$\frac{|\ln(\eta/\delta)|}{\delta}\mathcal{F}(u,\Omega_\varepsilon) \leqslant \pi\big[(A-\alpha)^2 + (B-\beta)^2\big] + (\ell+\lambda)(B-A)^2. \quad (201)$$

Step 1: lower bound of the energy in the bulk. The same strategy used in Theorem 30, in which we obtained the boundary conditions at the edge of the neck, applies.
Consider the rescaling $v_\varepsilon \in H^1(\widetilde{\Omega}_\varepsilon)$ defined in (199) and the limiting domain Ω_∞. By following the strategy in Step 1 of Theorem 30 we obtain that there is $R > 0$ and $\hat{v} \in H^1(\Omega_\infty)$ such that $v_\varepsilon \chi_{\widetilde{\Omega}_\varepsilon} \to \hat{v}\chi_{\Omega_\infty}$ in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$ and that $\hat{v}$ is constant on each connected component of $(\{x < 0\} \cup \{x > 0\}) \cap B_R$. In other words, there are $m_1, m_2 \in \mathbb{R}$ such that

$$\hat{v}_{|B_R} = \begin{cases} m_1 & \text{if} \quad x < 0, \\ m_2 & \text{if} \quad x > 0. \end{cases}$$

Therefore, we can show that the following lower bound estimate holds

$$\liminf_{\varepsilon\to 0} \frac{|\ln(\eta/\delta)|}{\delta}\mathcal{F}(u_\varepsilon,\Omega_\varepsilon) \geqslant 2\pi\big[(m_1-\alpha)^2 + (m_2-\beta)^2\big]. \quad (202)$$

Step 2: lower bound of the energy in the neck. From the previous step, we infer that for any $\mu > 0$, there exists $\varepsilon_1 > 0$ such that, for every $\varepsilon < \varepsilon_1$,

$$m_1 - \mu \leqslant u_\varepsilon \leqslant m_1 + \mu \qquad \text{on } \partial E^\ell(a_\varepsilon, M_\varepsilon),$$

$$m_2 - \mu \leqslant u_\varepsilon \leqslant m_2 + \mu \qquad \text{on } \partial E^{\mathrm{r}}(a_\varepsilon, M_\varepsilon),$$

for a suitable ellipsoid $E(a_\varepsilon, M_\varepsilon)$. From that, we obtain a lower bound of the energy in the neck. Indeed,

$$\frac{\varepsilon}{2\delta\eta}\int_{N_\varepsilon} |\nabla u_\varepsilon|^2 \,\mathrm{d}\mathbf{x} \geqslant \inf\Big\{\frac{1}{2}\int_{N_\varepsilon} |\nabla v|^2 \,\mathrm{d}\mathbf{x} : \ v \in H^1(N_\varepsilon),$$
$$v \geqslant m_1 - \mu \text{ on } \{x = -\varepsilon\} \text{ and } v \leqslant m_2 + \mu \text{ on } \{x = \varepsilon\}\Big\}$$

$$
\begin{aligned}
&\geqslant \inf\Big\{\frac{1}{2}\int_{N_\varepsilon} |\nabla v|^2 \,\mathrm{d}\mathbf{x} : \; v \in H^1(N_\varepsilon), \\
&\qquad\qquad v = m_1 - \mu \text{ on } \{x=-\varepsilon\} \text{ and } v = m_2+\mu \text{ on } \{x=\varepsilon\}\Big\} \\
&= \big(m_1 - m_2 - 2\mu\big)^2,
\end{aligned} \tag{203}
$$

where in the last step we used the fact that the minimiser of the above minimisation problem is given by the affine function.

Step 3: limit of the energy. By putting together (202), (203) and making use of (196), we obtain

$$
\begin{aligned}
\liminf_{\varepsilon\to 0} \frac{|\ln(\eta/\delta)|}{2\delta}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \,\mathrm{d}\mathbf{x} &\geqslant \liminf_{\varepsilon\to 0} \frac{|\ln(\eta/\delta)|}{2\delta}\int_{\Omega_\varepsilon^\ell \cup \Omega_\varepsilon^{\mathrm{r}}} |\nabla u_\varepsilon|^2 \,\mathrm{d}\mathbf{x} \\
&\quad + \liminf_{\varepsilon\to 0} \frac{|\ln(\eta/\delta)|}{2\delta}\int_{N_\varepsilon} |\nabla u_\varepsilon|^2 \,\mathrm{d}\mathbf{x} \\
&\geqslant 2\pi\big[(m_1-\alpha)^2 + (m_2-\beta)^2\big] \\
&\quad + \ell(m_1-m_2-2\mu)^2.
\end{aligned} \tag{204}
$$

On the other hand, from (201), we have

$$
\begin{aligned}
2\pi\big[(m_1-\alpha)^2 + (m_2-\beta)^2\big] &+ (\ell+\lambda)\big(m_1-m_2\big)^2 \\
&\geqslant \limsup_{\varepsilon\to 0} \frac{|\ln(\eta/\delta)|}{2\delta}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

By letting $\mu, \lambda \to 0$ in the above two inequalities, we obtain

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \frac{|\ln(\eta/\delta)|}{2\delta}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \,\mathrm{d}\mathbf{x} &= 2\pi\big[(m_1-\alpha)^2 + (m_2-\beta)^2\big] \\
&\quad + \ell(m_1-m_2)^2.
\end{aligned} \tag{205}
$$

The right-hand side is minimised for

$$
m_1 = \frac{\pi\alpha + \Big(\dfrac{\alpha+\beta}{2}\Big)\ell}{\pi+\ell} \quad \text{and} \quad m_2 = \frac{\pi\beta + \Big(\dfrac{\alpha+\beta}{2}\Big)\ell}{\pi+\ell}, \tag{206}
$$

which gives

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta\eta} \mathcal{F}(u_\varepsilon, \Omega_\varepsilon) = \frac{\pi(\beta-\alpha)^2}{\pi+\ell}.$$

In particular, by noticing that all the inequalities in (202), (203), and (204) are equalities, we get

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta\eta} \mathcal{F}(u_\varepsilon, N_\varepsilon) = \frac{\pi^2(\beta-\alpha)^2}{(\pi+\ell)^2},$$

and

$$\lim_{\varepsilon\to 0} \frac{|\ln(\eta/\delta)|}{\delta} \mathcal{F}(u_\varepsilon, \Omega_\varepsilon \setminus N_\varepsilon) = \frac{\pi\ell^2(\beta-\alpha)^2}{(\pi+\ell)^2}.$$

Step 4: limiting problems. Now we investigate the variational problem that the rescalings v_ε, defined in (199), in the bulk and w_ε, defined in (198), in the neck satisfy asymptotically.

Step 4.1: limiting problem in the neck. By acting like Step 1.1 of Theorem 30, let $R > 0$ be such that, [as $\varepsilon \to 0$,]

$$\begin{aligned} u_\varepsilon \to m_1 \quad &\text{uniformly on } B_{\delta R} \cap \{x \leqslant -\varepsilon\}, \\ u_\varepsilon \to m_2 \quad &\text{uniformly on } B_{\delta R} \cap \{x \geqslant \varepsilon\}, \end{aligned}$$

with m_1, m_2 defined in (206). In particular, u_ε has asymptotic boundary conditions ad the edge of the neck m_1 and m_2 respectively. Using the fact that $\delta \gg \varepsilon \gg \eta$, it follows that, if we consider the rescaling (198),

$$\begin{aligned} w_\varepsilon \to m_1 \quad &\text{uniformly on } \{x = -1\} \times [-1,1]^2, \\ w_\varepsilon \to m_2 \quad &\text{uniformly on } \{x = 1\} \times [-1,1]^2, \end{aligned} \tag{207}$$

as $\varepsilon \to 0$, which gives us the asymptotic boundary conditions at the edge of the neck satisfied by the limiting profile. By assumption, there exists $\hat{w} \in H^1(N)$ such that $w_\varepsilon \rightharpoonup \hat{w}$ in $H^1(N)$ as $\varepsilon \to 0$ and, from (207), $\hat{w}$ is an admissible competitor for the variational problem in (i). Moreover,

$$(m_1 - m_2)^2 \geqslant \frac{\varepsilon}{2\delta\eta} \liminf_{\varepsilon\to 0} \int_{N_\varepsilon} |\nabla u_\varepsilon|^2 \, \mathrm{d}\mathbf{x}$$

$$= \liminf_{\varepsilon \to 0} \frac{1}{2} \int_N \Big((\partial_x w_\varepsilon)^2 + \frac{\varepsilon^2}{\delta^2} (\partial_y w_\varepsilon)^2 + \frac{\varepsilon^2}{\eta^2} (\partial_z w_\varepsilon)^2 \Big) \, \mathrm{d}\mathbf{x}. \tag{208}$$

Since $\varepsilon/\eta \to \infty$, we have that $\hat{w}$ do not depend on the variable z and since we know that $w_\varepsilon \rightharpoonup \hat{w}$ we have, from (208), that

$$\begin{aligned}
(m_1 - m_2)^2 &\geqslant \liminf_{\varepsilon \to 0} \frac{1}{2} \int_N (\partial_x w_\varepsilon)^2 \, \mathrm{d}\mathbf{x} = \int_{[-1,1]^2} (\partial_x \hat{w}(x,y))^2 \, \mathrm{d}x \mathrm{d}y \\
&= \int_{-1}^{1} \int_{-1}^{1} (\partial_x \hat{w}(x,y))^2 \, \mathrm{d}x \mathrm{d}y \geqslant \frac{1}{2} \int_{-1}^{1} \left| \int_{-1}^{1} \partial_x \hat{w}(x,y) \, \mathrm{d}x \right|^2 \mathrm{d}y \\
&= \frac{1}{2} \int_{-1}^{1} \big| \hat{w}(1,y) - \hat{w}(-1,y) \big|^2 \mathrm{d}y \\
&= (m_1 - m_2)^2,
\end{aligned}$$

where in the second inequality we used Jensen inequality. Therefore, we conclude that $\hat{w}(x,y,z) = w(x)$ for $w \in H^1([-1,1])$ and w solves the variational problem in (i).

Step 4.2: limiting problem. We use a similar argument to the one employed in Step 3 of Theorem 30 applies. More specifically, $\hat{v}$ is admissible competitor for the problem in (ii) and

$$\hat{v}_{|B_R \cap \Omega_\infty} = \begin{cases} \dfrac{\pi\alpha + \Big(\dfrac{\alpha+\beta}{2}\Big)\ell}{\pi + \ell} & \text{on} \quad \{x < 0\}, \\[2ex] \dfrac{\pi\beta + \Big(\dfrac{\alpha+\beta}{2}\Big)\ell}{\pi + \ell} & \text{on} \quad \{x > 0\}. \end{cases}$$

By using the rescaled Poincaré inequality, we get that

$$\|v_\varepsilon - \overline{v}_\varepsilon\|_{L^6(\widetilde{\Omega}_\varepsilon)} \leqslant C \|\nabla v_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)}.$$

In analogy with Step 2 of Theorem 27 we get $\overline{v}_\varepsilon \to \alpha \chi_{\Omega_\infty^\ell} + \beta \chi_{\Omega_\infty^{\mathrm{r}}} \in L^6(\Omega_\infty)$, as $\varepsilon \to 0$. Finally, by applying the last part of Step 3 of Theorem 30, we have that $\hat{v}$ solves the variational problem in (ii). $\square$

Remark 20. Note that in this case, we do not have compactness of the rescaled profile w_ε inside the neck. This is due to the fact that the chosen rescaling that allows us to see the neck at scale one does not give a uniform bound on the gradient of the rescaled profile (in particular, the derivative with respect to the variable y cannot be bounded).

We now investigate the remaining two sub-regimes.

Super-critical Letter-box regime

In this sub-regime, we have

$$\ell = \lim_{\varepsilon\to 0} \frac{\delta\eta}{\varepsilon}\frac{|\ln(\eta/\delta)|}{\delta} = +\infty. \tag{209}$$

In this case, we recover the same result as in Theorem 30.

Theorem 33. Let $(u_\varepsilon)_\varepsilon \subset H^1(\Omega_\varepsilon)$ be an admissible family of local minimisers as in Definition 27. Assume that $\delta \gg \varepsilon \gg \eta$ and that $\ell = +\infty$. Then,

$$\lim_{\varepsilon\to 0} \frac{|\ln(\eta/\delta)|}{\delta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) = \lim_{\varepsilon\to 0} \frac{|\ln(\eta/\delta)|}{\delta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon \setminus N_\varepsilon) = \pi(\beta-\alpha)^2.$$

Moreover, let

$$\widetilde{\Omega}_\varepsilon := \frac{1}{\delta}\Omega_\varepsilon, \qquad \Omega_\infty := \{x<0\} \cup \{x>0\}.$$

Define the rescaled profile $v_\varepsilon \colon \widetilde{\Omega}_\varepsilon \to \mathbb{R}$ be defined as

$$v_\varepsilon(x,y,z) := u_\varepsilon(\delta x, \delta y, \delta z).$$

Then, there exists $R>0$ such that the following hold:

(i) $v_\varepsilon \to \frac{\alpha+\beta}{2}$ uniformly on B_R, as $\varepsilon \to 0$;

(ii) There exists $\hat{v} \in H^1_{\mathrm{loc}}(\Omega_\infty)$ such that $v_\varepsilon \chi_{\widetilde{\Omega}_\varepsilon} \to \hat{v}\chi_{\Omega_\infty}$ strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$, as $\varepsilon \to 0$;

(iii) The function $\hat{v}$ is the unique minimizer of the variational problem

$$\min\left\{\frac{1}{2}\int_{\Omega_\infty} |\nabla v|^2 \,\mathrm{d}x : v \in \mathcal{A}\right\},$$

where $\mathcal{A}$ is defined in (181).

Proof. The proof follows the ones for Theorems 32 and 30.

Step 1: bound of the energy. For any given constants $A, B > 0$ with $A \leqslant B$, we have

$$\frac{|\ln(\eta/\delta)|}{\delta}\mathcal{F}(u_\varepsilon) \leqslant \pi\big[(A-\alpha)^2 + (B-\beta)^2\big] + \frac{\delta\eta}{\varepsilon}\frac{|\ln(\eta/\delta)|}{\delta}(B-A)^2.$$

Since

$$\lim_{\varepsilon\to 0}\frac{\delta\eta}{\varepsilon}\frac{|\ln(\eta/\delta)|}{\delta} = +\infty,$$

the only way to get that the right-hand side of the above inequality is bounded uniformly in ε, is to choose $A = B$. This gives the estimate

$$\frac{|\ln(\eta/\delta)|}{\delta}\mathcal{F}(u_\varepsilon) \leqslant \pi[(A-\alpha)^2 + (A-\beta)^2].$$

Step 2. Lower bound of the energy. From (205), we get

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\frac{|\ln(\eta/\delta)|}{2\delta}\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2\,\mathrm{d}\mathbf{x} &\geqslant \liminf_{\varepsilon\to 0}\frac{|\ln(\eta/\delta)|}{2\delta}\int_{\Omega_\varepsilon^\ell\cup\Omega_\varepsilon^{\mathrm{r}}}|\nabla u_\varepsilon|^2\,\mathrm{d}\mathbf{x} \\
&\quad + \liminf_{\varepsilon\to 0}\frac{|\ln(\eta/\delta)|}{2\delta}\int_{N_\varepsilon}|\nabla u_\varepsilon|^2\,\mathrm{d}\mathbf{x} \\
&\geqslant \liminf_{\varepsilon\to 0}\Big[2\pi\big[(m_1-\alpha)^2 + (m_2-\beta)^2\big] \\
&\quad + \frac{|\ln(\eta/\delta)|}{\delta}\frac{\delta\eta}{\varepsilon}(m_1-m_2)^2\Big].
\end{aligned}$$

Therefore, as in Theorem 30, we have an optimality condition on m_1 and m_2, which, together with (209), leads to

$$A = m_1 = m_2 = \frac{\alpha+\beta}{2}.$$

Therefore

$$\lim_{\varepsilon\to 0}\frac{|\ln(\eta/\delta)|}{\delta}\mathcal{F}(u_\varepsilon,\Omega_\varepsilon) = \lim_{\varepsilon\to 0}\frac{|\ln(\eta/\delta)|}{\delta}\mathcal{F}(u_\varepsilon,\Omega_\varepsilon\setminus N_\varepsilon) = \pi(\beta-\alpha)^2.$$

The rest of the proof is identical to the one in Theorems 30 and 32 and we obtain the desired result. □

Sub-critical Letter-box regime

In this sub-regime, we have

$$\ell = \lim_{\varepsilon\to 0} \frac{\delta\eta}{\varepsilon}\frac{|\ln(\eta/\delta)|}{\delta} = 0. \tag{210}$$

In this case, we recover the same result as in Theorem 27.

Theorem 34. Let $(u_\varepsilon)_\varepsilon \subset H^1(\Omega_\varepsilon)$ be an admissible family of local minimisers as in Definition 27. Assume that $\delta \gg \varepsilon \gg \eta$ and that $\ell = 0$. Then,

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta\eta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) = \lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta\eta}\mathcal{F}(u_\varepsilon, N_\varepsilon) = (\beta-\alpha)^2.$$

Define the rescaled profile $v_\varepsilon \colon \overline{\Omega}_\varepsilon \to \mathbb{R}$ as

$$v_\varepsilon(x,y,z) \coloneqq u_\varepsilon(\varepsilon x, \delta y, \eta z),$$

where $\overline{\Omega}_\varepsilon$ is the rescaled domain of Ω_ε. Then the following hold:

(i) $v_\varepsilon \chi_{\widetilde{\Omega}^\ell_\varepsilon} - \alpha\chi_{\widetilde{\Omega}^\ell_\varepsilon} \to 0$ in $L^6(\mathbb{R}^3)$ and $v_\varepsilon \chi_{\widetilde{\Omega}^{\mathrm{r}}_\varepsilon} - \beta\chi_{\widetilde{\Omega}^{\mathrm{r}}_\varepsilon} \to 0$ in $L^6(\mathbb{R}^3)$, as $\varepsilon \to 0$;

(ii) There exists $\hat{v} \in H^1(N)$ such that $v_\varepsilon \rightharpoonup \hat{v}$ weakly in $H^1(N)$, as $\varepsilon \to 0$;

(iii) It holds that $\hat{v}(x,y,z) = v(x)$, where $v \in H^1(-1,1)$ is the unique minimizer of the variational problem

$$\min\left\{\frac{1}{2}\int_{-1}^{1} |v'|^2 \,\mathrm{d}x : \ v \in H^1(-1,1),\ v(-1) = \alpha,\ v(1) = \beta\right\},$$

In particular, $v(x) = \dfrac{\beta-\alpha}{2}x + \dfrac{\alpha+\beta}{2}$.

Proof. The proof is an adaptation of Theorem 33 and we remark only the differences. Regarding the bound of the energy, we have that for any given constants $A, B > 0$ with $A \leqslant B$, we have

$$\frac{\varepsilon}{\delta\eta}\mathcal{F}(u_\varepsilon, \Omega_\varepsilon) \leqslant \pi\frac{\varepsilon}{\delta\eta}\frac{\delta}{|\ln(\eta/\delta)|}\left[(A-\alpha)^2 + (B-\beta)^2\right] + (B-A)^2.$$

Since (210) holds, we obtain a bound for the energy which is compatible with a transition inside the neck by choosing $A = \alpha$ and $B = \beta$.

By using the same techniques as in Theorems (30) and (32) we obtain the desired result. $\square$

About the Author

Gabriele Fissore was born in Savigliano, Italy. He obtained his scientific diploma from *Liceo Arimondi* in Savigliano in 2015. His academic journey in mathematics began at the University of Torino, Italy.
In 2018, he earned his Bachelor's degree under the supervision of Professor Michele Rossi, with a thesis titled *Cellular Homology and Borsuk-Ulam's Theorem*. He subsequently completed his Master's degree in 2021, under the supervision of Professor Alberto Boscaggin, with a thesis titled *Global Bifurcation for a Prescribed Mean Curvature Equation*.
Just after obtaining the degree, Gabriele was awarded a PhD position at Radboud University in Nijmegen, the Netherlands, under the mentorship of Professor Riccardo Cristoferi.

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Summary

Materials science is a field that incorporates many branches of science, such as physics, chemistry, engineering, and, last but not least, mathematics. This dissertation specifically focuses on crystal growth and magnetism. We develop variational models to describe these two phenomena.

The first line of research concerns *thin films*, whose relevance and production have been growing exponentially due to the increasing demand from electronic devices and many other applications. Among the countless ways a crystal can form, we investigate the case in which a crystalline material is deposited layer by layer onto a fixed crystalline substrate. If the atoms of the substrate at the interface occupy the natural lattice positions of the thin film, such crystal growth is called *epitaxial*.

The second topic of this research concerns *magnetic domain walls*, which are regions where the magnetisation of a material transitions from one orientation to another. In particular, we study changes in magnetisation in domains with extreme geometries, such as dumbbell-shaped domains. The analysis of magnetisation behaviour in such geometries is relevant in micro- and nano-electronics applications, where the neck of the dumbbell serves as a model for magnetic contact points.

Samenvatting

Materiaalkunde is een vakgebied dat vele takken van de wetenschap omvat, zoals natuurkunde, scheikunde, engineering en niet te vergeten wiskunde. Dit proefschrift focust zich op kristalgroei en magnetisme. We ontwikkelen variationele modellen die deze twee fenomenen beschrijven.

Het eerste deel van dit proefschrift betreft *dunne films*, waarvan het belang en de productie exponentieel zijn toegenomen vanwege de hoge vraag van elektronische apparaten en vele andere toepassingen. Onder de talloze manieren waarop een kristal kan ontstaan, onderzoeken we het geval waarbij een kristallijn materiaal laag voor laag wordt afgezet op een vast kristallijn substraat. Als de atomen van het substraat op het grensvlak de natuurlijke roosterposities van de dunne film innemen, wordt een dergelijke kristalgroei *epitaxiaal* genoemd.

Het tweede onderwerp van het onderzoek betreft *magnetische domeinwanden.* Dat zijn gebieden waarin de magnetisatie van een materiaal verandert van de ene waarde naar een andere. We bestuderen in het bijzonder de verandering van magnetisatie in het geval dat het domein een extreme meetkundige structuur heeft, zoals die van een halter. De analyse van magnetisatie in zulke meetkundige structuren is relevant voor micro- en nano-elektronische toepassingen, waarbij de nek van de halster als een model voor magnetische puntcontacten dient.

Research data management

This thesis research has been carried out under the institute research data management policy of IMAPP, Radboud University. No data has been produced or analysed in this project.

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